Gabor frames and topology-based strategies for astronomical images

Darian M. Onchis, Pedro Real and Gilbert-Rainer Gillich

November 5, 2010

Abstract

The topic of this paper is the description of procedures to remove the fringes from complicated astronomical data sets, where the interference pattern is not regular, combining topological and Gabor-based analysis. A first method is given and it involves two successive steps: first we perform an identification of the fringes using Gabor frames and latter we filter the connected components using computational topology methods. This is a new approach trying to get the benefit of filtering the harmonic representation by means of topological methods.

1 Introduction

Fringes due to the interference of the light sources are common artifacts in digital imaging. Most of the fringes exhibit a relatively regular pattern and therefore their removal using Fourier-based techniques was satisfactory even in some complicated cases [1, 7]. The typical features are locally plane waves of approximatively fixed wavelength but slowly changing directions. Windowing Fourier-transform methods considering all possible orientations are well suited to remove those fringes by first identify and locate them. But for regular fringes pattern, given their relatively simple geometric structure and their low frequency content, even a low pass filtering in the Fourier domain is already of much help. Another problem is concerning with fringes of irregular patterns that can be found sometimes in astronomical recordings (Figure 1). For this complicated fringes most of the methods involving harmonic filtering are expected to fail due to the impossibility of globally localizing the low pass frequency content. Therefore, we are proposing here a new method involving both the Gabor transform for obtaining the localized frequency spectrum and computational topology to subsequently identify and remove the connected fringe components. The structure of this contribution is as follows: In the first sections we introduce the necessary results concerning Gabor frames, then we describe the problem of irregular fringes. The main section continues with the description of the proposed algorithm for fringe removal. In the last section experiments using the proposed method are presented and conclusions are drawn.

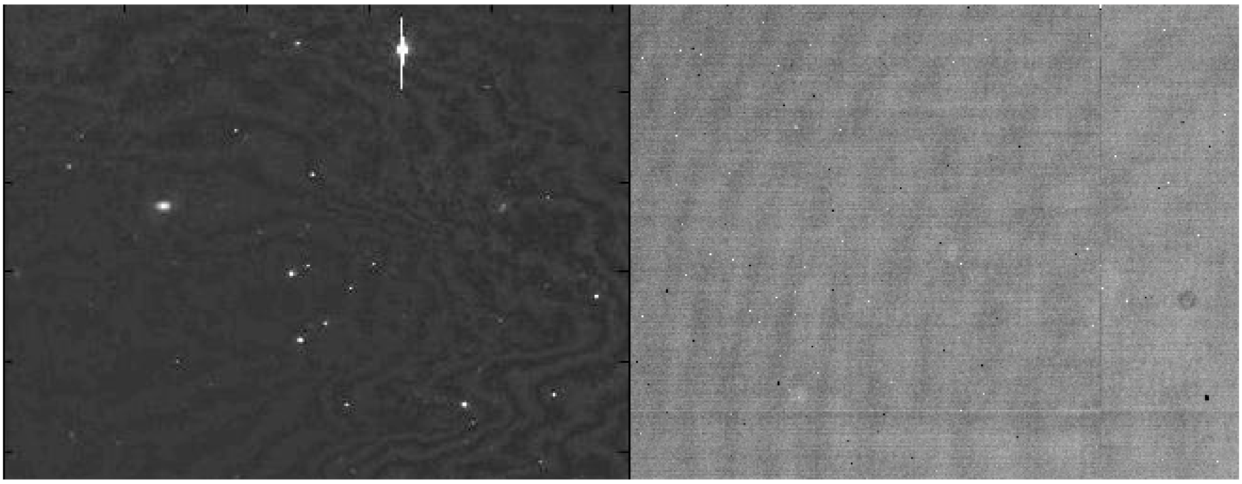

Figure 1: Regular (right) and irregular(left) fringes

2 Gabor-based representations

We start by defining the Fourier transform of an integrable function by $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)e^{-2\pi it\omega}dt$. The translation operator T_x and the modulation operator M_ω are given by

$$T_x f(t) = f(t-x), \qquad M_\omega f(t) = e^{2\pi i\omega t}f(t), \quad x,\omega \in \mathbb{R}^d. \tag{1}$$

Combined together they give rise to the so-called time-frequency shift $\pi(\lambda)$:

$$\pi(\lambda) = M_\omega T_x, \quad \lambda = (x,\omega) \in \mathbb{R}^{2d}. \tag{2}$$

Note that

$$\pi(\lambda_2)\pi(\lambda_1) = e^{2\pi i(x_1\omega_2 - x_2\omega_1)}\pi(\lambda_1)\pi(\lambda_2)$$

for $\lambda_1 = (x_1,\omega_1), \lambda_2 = (x_2,\omega_2) \in \mathbb{R}^{2d}$.

In order to apply the Gabor theory to discrete finite images, we need to use frames and lattices. Frames can be seen as the generalized basis system in a Hilbert space $\boldsymbol{H}$, i.e. a family that allows every element $f \in \boldsymbol{H}$ to have an expansion of the form

$$f = SS^{-1}f = \sum_{i\in I}\langle S^{-1}f, g_i\rangle g_i = \sum_{i\in I}\langle f, S^{-1}g_i\rangle g_i$$

where S states for the bounded and positive (hence invertible) **frame operator**:

$$Sf = \sum_{i\in I}\langle f, g_i\rangle g_i \tag{3}$$

The family $(\gamma_i)_{i\in I} = (S^{-1}g_i)_{i\in I}$ is again a frame and is called a **canonical dual** frame. It is not the only dual frame, i.e. a frame $(d_i)_{i\in I}$ that allows the expansion of f as follow:

$$f = \sum_{i\in I}\langle f, g_i\rangle d_i = \sum_{i\in I}\langle f, d_i\rangle g_i \tag{4}$$

but the canonical dual frame $(\gamma_i)_{i\in I}$ provides the coefficients $(< f, g_i >)_{i\in I}$ with minimal l^2 norm ([9]).

A time-frequency lattice Λ is a discrete subgroup with compact quotient of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ $(= \mathbb{R}^d \times \hat{\mathbb{R}^{2d}})$, where by $\hat{\mathbb{R}^{2d}}$ we are referring to the same $\mathbb{R}^d$, but on frequency side. Its redundancy $|\Lambda|$ is the reciprocal value of the measure of a fundamental domain for the quotient $\mathbb{R}^{2d}/\Lambda$.

For a lattice Λ in $\mathbb{R}^{2d}$ and a so-called *Gabor atom* $g \in \boldsymbol{L}^2$ we define the associated Gabor family by

$$\mathscr{G}(g,\Lambda) = \{\pi(\lambda)g\}_{\lambda\in\Lambda}\,.$$

If $\mathscr{G}(g,\Lambda)$ is a frame for $\boldsymbol{L}^2$, we call it a *Gabor frame*. Since Λ has a group structure, the frame operator

$$Sf = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$$

has the property that it commutes with all time-frequency shifts of the form $\pi(\lambda)$ for $\lambda \in \Lambda$. Therefore, the canonical dual frame of $\mathscr{G}(g,\Lambda)$ is simply given by $\mathscr{G}(h,\Lambda)$ with $h = S^{-1}g$. The fact, that a canonical dual frame of a Gabor frame is again a Gabor frame, i.e., generated by a single function, is the key property in many applications. It reduces computational issues to solving the linear system $Sh = g$, where S is a positive definite operator.

A special and widely studied case are separable lattices of the form $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ for some positive lattice parameters α and β, whose redundancy is simply $(\alpha\beta)^{-1}$. The prototype of a function generating Gabor frames for such separable lattices is the Gaussian

$$\psi(x) = e^{-\pi x^2 \sigma^2}\,. \tag{5}$$

for some real $\sigma > 0$. The Gaussian generates a Gabor frame if and only if $\alpha\beta < 1$ [9]. We emphasize that for $\alpha\beta = 1$ the Gaussian generates a *unstable* generating system for $\boldsymbol{L}^2(\mathbb{R}^d)$, i.e., the resulting Gabor family is complete but coefficient sequences must not be bounded. In this context we mention a central result, the so-called *density theorem* and refer to [9] for detailed discussions. An elegant elementary proof of the density theorem has been provided by Janssen [9].

Theorem 1. *Assume that $\mathscr{G}(g,\alpha,\beta)$ is a frame. Then, $\alpha\beta \le 1$. Moreover, $\mathscr{G}(g,\alpha,\beta)$ is a Riesz basis for $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if $\alpha\beta = 1$.*

In his seminal paper Gabor chose the integer lattice $a = b = 1$ in $\mathbb{R}^2$ and used the Gaussian in order to define a Gabor system with maximal time-frequency localization. However, as mentioned above, this system is no longer stable though complete, and, indeed, the celebrated Balian-Low Theorem [9] states that good time-frequency localization and Gabor Riesz bases are not compatible:

Theorem 2. (Balian-Low) *If $\mathscr{G}(g,1,1)$ constitutes a Riesz basis for $\boldsymbol{L}^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}^2} |g(t)|^2 t^2 dt \int_{\mathbb{R}^2} |\hat{g}(\omega)|^2 \omega^2 d\omega = \infty\,.$$

The Balian-Low Theorem reveals a form of uncertainty principle and has inspired fundamental research, see [9] and references therein.

For practical image processing purposes, we need to consider only the (algebraic part) of the problem in the setting of finite (discrete) images. Therefore, we consider images of size $d \times d$, which evolve on the additive Abelian group $\mathscr{G} = \mathbb{Z}^d \times \mathbb{Z}^d$ and the image space is isomorphic to the corresponding space of complex numbers. The Gabor system $\mathscr{G}(g, \Lambda)$ consists in this situation of discrete finite time-frequency shifts of a complex windows g.

3 Algorithm for fringe removal

Compared to the 1D case, the Fourier transform for images has a second degree of freedom and this is the orientation. Instead of seeing ω_1 and ω_2 as the components of a 2D frequency, one might prefer to see ω as the parameter for the pattern density and define an orientation similar to the argument of complex numbers.

The 2D Fourier transform of a function is considering all possible pattern densities and all their orientations, and analyzing the amount of the corresponding 2D frequency among the variations of the function values. When we talk about low or high 2D frequencies, we mean low and high values of ω and include all orientations.

The term frequency as oscillations per time is not valid anymore for signals with more than one dimension. Images do not have a temporal domain, but two spatial domains, making a 2D frequency a description for oscillations per area. Instead of talking about the time-frequency analysis of images, one could rather use terms like position-frequency analysis or space-wave number analysis. Mathematically, time-frequency analysis happens in arbitrary dimensions anyway.

As a localized and sampled 2D windowed-Fourier transform, the Gabor representation, using discrete frames will give us at each point of the lattice a two dimensional spectral representation of the image ordered either with modulation or with translation priority.

Using this localized time-frequency representation the harmonic filtering in phase-space will identify the pattern in the phase arguments. But for an efficient computation of the irregular fringe pattern, we need to combine the harmonic methods with the topology-based tools in order to enhance the information we have at a specific position in the image. The resulting algorithm can be generically written as:

- Choose the window size and a lattice
- Perform a 2D Gabor transform with modulation priority. This will give us the shape and the positions of the fringes using the corresponding building blocks (frame elements)
- Do a threshholding to remove the low frequencies.
- Perform the Gabor synthesis using the dual frame.
- Identify and remove the existent connected components.

The algorithm is applying a Gabor filtering who searches for all directions. Once we have the directions, we search for connected components within the analyzed image containing this type of noise. For the algorithm to work, we have to partition the image in small portions (blocks) with only one direction. Therefore, we need apply the Gabor transform of an appropriate size to find the direction of the pattern and to follow the connected components.

Results of the de-fringing algorithm applied to irregular fringe patterns are presented in Figure 2.

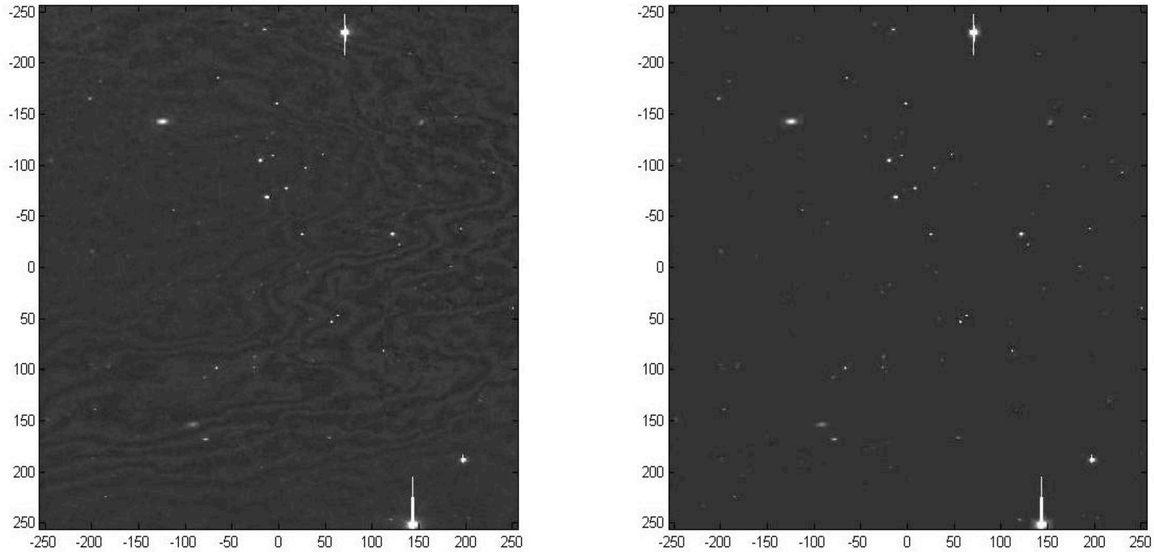

Figure 2: Original image with fringes (left) and the image after de-fringing (right)

4 Topological filtering

For the algorithm from the previous section, we need a method to search for black connected components in a binary 2D digital image. There are plenty of literature (sequence scanning, seed filling algorithms,...) for analyzing connected components in $2D$ discrete images. The elementary automatic sequence-scan method we use here for extracting the basic topology of the binary image implicitly represent the image as a cell complex and techniques for generating acyclic graphs that measure the degree of connectivity of this complex. In fact, this technique can be generalized to higher dimension and grey-level digital image segmentation making use of the computational algebraic topological results about homology (topological information related to n-dimensional holes the object has) in terms of special "spanning" trees located in the cell complex representation of the image [4–6]. Our method for binary 2D digital images allows to identify connected components and to determine a set of topological features (Betti numbers, Euler characteristic, ...) of the black part of the image. With some slightly modifications and working the pixels in a seed filling way, the

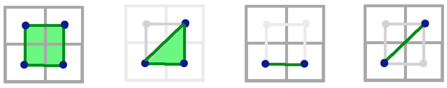

Figure 3: Cell structures

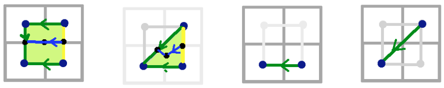

Figure 4: Tree cells structures

method is also able to compute at no extra time cost, geometrical features (diameter, curvature, skew, ,...) derived from the geometry of their skeleton or their boundary curves. It is intended that in a future such topology-based techniques could be correctly integrated in a Gabor filtering method, in order to improve algorithms for recognition of fringes and other astronomical patterns.

The whole procedure is to take every pixel in an image as a node, the nodes to the same connected area are put together to build a directed tree (called the *vertex tree*) then the different connected area are represented by the different trees. If we would be interested in the holes and genus of each connected component, we must build a second acyclic semi-directed graph (called the *edge forest*). The vertex and edge forest are embedded in a cell complex structure. In Figure 3 , it appear the different cell structures formed on the unit square $\mathcal{S}_8$ (up to rotation) formed by four 8-adjacent neighbors nodes. A node is represented by its coordinates (x, y), its color (black or white) and its *size*. The size value always is 1 except if the node is the root node of the vertex tree of the connected area, which in that case is the size of such area. From left to right, and up to down, scan every pixel of the image in horizontal line. When a pixel is reached, build the corresponding cell structure and two acyclic graphs (vertex and edge forest of the cell) on it. The vertex forest has as vertices the nodes of the images and as edges, some edges of the cell structure. The edge forest has as vertices the barycenters of the rest of the edges we do not consider in the vertex forest and all faces of the cell complex. Each edge in this forest connects a edge barycenter with its neighbor face barycenter. The vertex and edge trees for the four different configuration of black nodes (up to rotation) in $\mathcal{S}_8$ are given in Figure 4. In each pixel-step we coherently glue the previous global graph information with that of the current cell structure, giving at the end the vertex and edge forest associated to the foreground processed at that time. After a single-scan of the image, the vertex forest determines the different connected component we have. The trees in the edge forest whose leaves are border pixels of the foreground determines the different holes each connected component of the foreground has.

Some visual examples of application of this technique are showed below. In Figure 5, on the right and middle, it can be seen a zoom of an image in which the cell structure for the foreground, the vertex forest (in red), edge forest (in yellow) and holes (in blue) are displayed. In the left side,

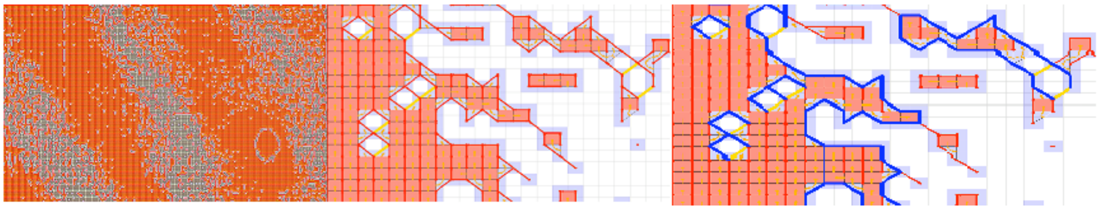

Figure 5: Topology applications

Figure 5 presents an application of the topological filtering to an astronomical image.

Using this topological filtering, we were able to properly identify the fringes pattern at the level of the whole image as a continuum by connecting the components from the layer hidden by the high energy of the stars as in the image below:

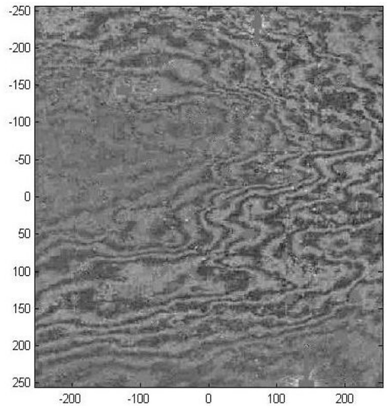

Figure 6: Topological reconstructed fringe pattern

5 Acknowledgement

The research has been partially funded by the Sectoral Operational Programme Human Resources Development 2007-2013 of the Romanian Ministry of Labor, Family and Social Protection through the Financial Agreement POSDRU/89/1.5/S/62557.

6 Conclusion

In a near future, we plan to design a topological filtering algorithm for 2D grey-level for astronomical images, that generalize the one from the Section 3. Also, we are aiming to produce methods for removing the fringes without correlating the noise or the sky background with the celestial objects.

References

[1] Kai Li and Bing Pan. Frequency-guided windowed Fourier ridges technique for automatic demodulation of a single closed fringe pattern. *Appl. Opt.*, 49(1):56–60, January 2010.

[2] Munther Gdeisat, Abdulbasit Abid, David Burton, Michael Lalor, Francis Lilley, Chris Moore, and Mohammed Qudeisat. Spatial and temporal carrier fringe pattern demodulation using the one-dimensional continuous wavelet transform. Recent progress, challenges, and suggested developments. *Optics and Laser in Engineering*, (47):1348–1361, 2009.

[3] Munther Gdeisat, David Burton, and Michael Lalor. Spatial carrier fringe pattern demodulation by use of a two-dimensional continuous wavelet transform. *Appl. Opt.*, 45(34):8722–8732, 2006.

[4] H.Molina-Abril, P. Real. Advanced Homological information on $3D$ Digital volumes. SSPR 2008, LNCS 5342,361-371,2008.

[5] Molina-Abril, H. and Real, P., Cell AT-models for digital volumes, LNCS 5534, 314–323, 2009.

[6] Molina-Abril, H. and Real, P.,Homological Computation Using Spanning Trees, CIARP2009, 272–278, 2009.

[7] Kemao Qian. Windowed Fourier Transform for Fringe Pattern Analysis: Addendum,. *Appl. Opt.*, 43,(17,):3472–3473,, June 2004.

[8] Jingang Zhong and Jiawen Weng. Dilating Gabor transform for the fringe analysis of 3-D shape measurement. *Opt. Eng.*, 43(4):895–899, 2004.

[9] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.

Authors address:
Darian Onchis, Gilbert-Rainer Gillich
Eftimie Murgu University
Traian Vuia Square 1-4
320085, Resita, ROMANIA

Pedro Real
University of Seville
Av. Reina Mercedes s/n
41012 Seville, SPAIN.