Distributed boundary tracking using alpha and Delaunay-Čech shapes

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Abstract We demonstrate real time tracking of systematic failures in sensor networks, using distributed computation of the $\alpha-$shape derived from the network. More generally, our work may be applied to tracking the boundary of any time varying object, whose data is captured in the form of a point cloud. We also demonstrate the use of a new geometric object called the Delaunay-Čech shape, which is geometrically more appropriate than an $\alpha-$shape for some cases.
For a given point set S in a plane, we develop a distributed algorithm to compute the $\alpha-$shape of S. $\alpha-$shapes are well known geometric objects which generalize the idea of a convex hull, and provide a good definition for the shape of S. We assume that the distances between pairs of points which are closer than a certain distance $r > 0$ are provided, and we show constructively that this information is sufficient to compute the alpha shapes for a range of parameters, where the range depends on r.

1 Introduction

Many applications call for detecting and tracking the boundary of a dynamically changing space of interest [4][2]. We would expect any algorithm performing the task to include the following important properties: 1) the boundary output is geometrically close to the actual boundary, and 2) the interior of the boundary is topologically faithful to the original space. It is often the case that we are only given random samples from the space. We may then reconstruct the space by first placing balls of a certain radius around these points, and then by taking the union of these balls.

We start with the assumption that the union of the balls described above is a good approximation to the space of interest. Note that in some cases, this is by design. For example, in the case of systematic failures in sensor networks [2], the failure in the nodes is caused by a spatially propagating phenomenon, and our aim is to track its boundary. In this case, we construct a space by taking the union of balls of radius $r_c/2$ around each node, where r_c is its radius of communication. The radius of communication is the distance within which two nodes can communicate with each other.

The problem may also be viewed as one of computing the boundary of a set of points, provided with some geometric information. Given the pair-wise distances of nodes within a neighborhood, the above decision may be locally made by constructing an associated $\alpha-$shape. the $\alpha-$shape introduced in [5] gives a generalization of the convex hull of S, and an intuitive definition for the shape of points.

When there is a sufficient density of nodes, computing local coordinates is accurate (probabilistically), and distributed algorithms exist for computing modified versions of Delaunay triangulation [1, 8]. In this case, we define a certain *Delaunay-Čech triangulation*, which contains

an alpha complex, and which we show to be homotopy equivalent. For boundary tracking-based applications, the boundary of Delaunay-Čech triangulation will serve as a better geometric approximation to the boundary, while preserving the topological features.
Our contributions are:

- Given the distances between pairs of nodes whenever they are closer than $r_c > 0$, we develop an algorithm to compute the $\alpha-$shape for a range of parameters, where this range depends on r_c.
- We introduce the Delaunay-Čech triangulation, defined in Section 2.2, and show that it is homotopy equivalent to the alpha complex.

2 Preliminaries

2.1 Alpha complex and $\alpha-$shape

Consider a set of nodes $V \subset \mathbb{R}^2$, and a parameter r. Let V_i be the voronoi cell associated with node $v_i \in V$ in the voronoi decomposition of V. Define an alpha cell ($\alpha-$cell) of v_i as $\alpha(v_i, r) = V_i \cap B(v_i, r)$ where $B(v_i, r/2)$ is the closed ball of radius $r/2$ around v_i. The alpha complex, A_r (we are assuming V is implied in this notation), is defined as the nerve complex of the alpha cells, i.e., $(v_0, v_1, \ldots, v_k)$ spans a $k-$simplex in A_r if $\bigcap_i \alpha(v_i) \neq \varnothing$. Since the alpha cells are convex, the nerve theorem [9, 7] implies that the alpha complex has the same homotopy type as the union of the alpha cells, which in turn is equal to the union of the balls $B(v_i, r/2)$.

Given a set of nodes $V \subset \mathbb{R}^2$ †, and a parameter $r > 0$, the alpha shape, ∂A_r, is a 1-dimensional complex which generalizes the convex hull of V. To simplify the notation, we use (v_i, v_j) to denote an edge in a graph, a 1-simplex in a complex or the underlying line segment. A 1-simplex (v_i, v_j) belongs to ∂A_r if and only if a circle of radius $r/2$ passing through v_i and v_j does not contain any other node inside it. By "inside" a circle, we mean the interior of the ball to which this circle is a boundary. We say that such a circle satisfies the "$\alpha-$*condition*". ∂A_r also contains all the nodes $\{v_j\}$ such that a circle of radius r passing through v_j satisfies the $\alpha-$condition.

For a 2-dimensional simplicial complex K, we define the boundary of K to be the union of all the 1-simplices (along with their faces), where each is a face of at most one $2-$simplex, and all $0-$simplices which are not faces of any simplex in K. The alpha shape ∂A_r is the boundary of the alpha complex A_r[6].

2.2 Delaunay-Čech Shape

For a set of nodes $V \subset \mathbb{R}^2$ and a parameter $r > 0$, define the geometric graph $G_r = (V, E)$ to be the set of vertices (V) and edges (E), where $e = (v_i, v_j)$ is in E if the distance between v_i and v_j is less than or equal to r. Let $\check{C}(V, r)$ denote the Čech complex with parameter r (the nerve complex of the set of balls $\{B(v_i, r/2)\}$) and let $DT(V)$ be the Delaunay triangulation of V. We define the Delaunay-Čech complex $D\check{C}_r$ with parameter r as $D\check{C}_r = DT(V) \cap \check{C}(V, r)$. We prove in our manuscript [3], $D\check{C}_r$ is homotopy equivalent to A_r. We call the boundary of $D\check{C}_r$, denoted by $\partial D\check{C}_r$ the Delaunay-Čech shape.

3 Demonstrations

The distributed algorithm for computing the α-shape as described in our manuscript [3] is given in Table 1, and the angles referred to are shown in figures to the right of the table. An example of an $\alpha-$shape for a set of points is shown in Figure 1. The figure illustrates that $\alpha-$shape is

†The alpha shape is generally defined for points in $\mathbb{R}^k$ for any dimension k.

a topologically faithful approximation to the boundary of the union of balls around each point (shown as shaded region). ***We will demonstrate the real-time tracking of a time-varying failure in a sensor network, by computing the $\alpha-$shape at each time point using the algorithm in Table 1.***

We prove in our manuscript [3], that the Delaunay-Čech complex $D\check{C}_r$ defined in Section 2.2 is topologically equivalent to the $\alpha-$complex A_r, and therefore, its boundary is also topologically faithful to the space of interest. Further as illustrated in Figure 2, the boundary of $D\check{C}_r$ is geometrically a better approximation to the space of interest, compared to the $\alpha-$shape. ***We demonstrate the advantage of the using the boundary of $D\check{C}_r$ over the $\alpha-$shape in tracking time-varying failures***.

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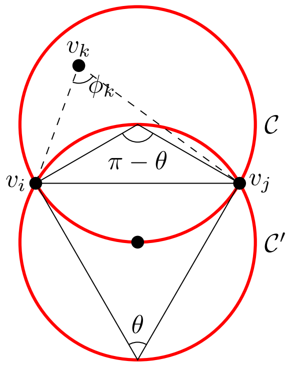
computing the $\alpha-$shape

At each edge $e = (v_i, v_j)$ in G,
  compute θ
  for each $v_k \in \mathcal{N}_i \cap \mathcal{N}_j$
    compute ϕ_k
    if $\phi_k > \pi - \theta$
      $e \notin \partial A$, terminate
    if $\phi_k \leq \theta$,
      continue to next node
    if $\theta < \phi \leq \pi - \theta$
      is v_k the first node satisfying this condition?
       assign v_k to $\mathcal{C}$
      else
       compute β
       if $\beta = |\angle v_k v_i v_j - \angle v_l v_i v_j|$
        continue to next node
       else
        $e \notin \partial A$, terminate
  $e \in \partial A$



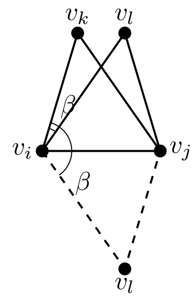


Table 1: Algorithm for computing the $\alpha-$shape. Note that all the computations require only local information. The angles denoted as illustrated in the figures on the right

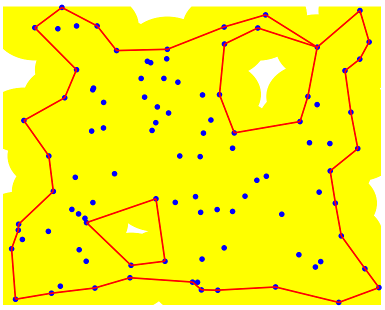
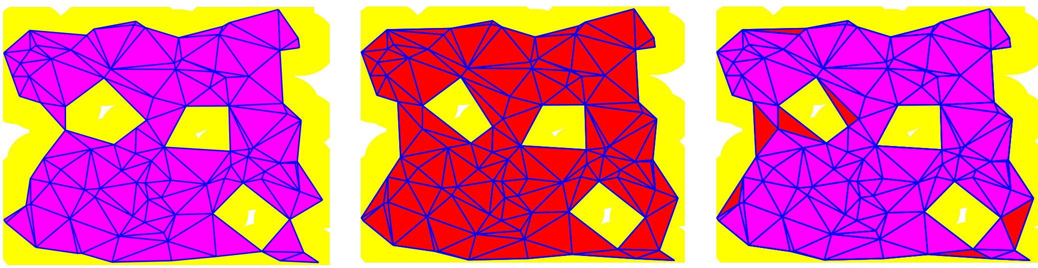


Figure 1: $\alpha-$shape with parameter $r_c/2$ for a set of points in $\mathbb{R}^2$ computed using algorithm in Table 1. The shaded region is the union of balls of radius $r_c/2$ centered at each point.



(a) $A_{r_c/2}(V)$

(b) $D\check{C}_{r_c/2}(V)$

(c) $A_{r_c/2}(V)$ super-imposed over $D\check{C}_{r_c/2}(V)$

Figure 2: Figure shows the homotopy equivalence between $A_{r_c/2}(V)$ and $D\check{C}_{r_c/2}(V)$. The shaded region is R_c. Note that $D\check{C}_{r_c/2}(V)$ is a better geometric approximation to R_c than $A_{r_c/2}(V)$.