Weighted composition operators on spaces and classes of analytic functions

Irina Arévalo Barco

Universidad Autónoma de Madrid
Departamento de Matemáticas

Madrid, Septiembre de 2017

Tesis doctoral dirigida por el profesor Dragan Vukotić Jovšić

A mis padres
y a Marcos

Agradecimientos

Muchas gracias a todos los que me habéis ayudado durante estos años. En primer lugar gracias a mi director, Dragan Vukotić, por acogerme como su alumna y darme la oportunidad de escribir esta tesis, y sobre todo gracias por enseñarme tanto.

También quiero agradecer a mis coautores no sólo el trabajo compartido, sino sobre todo las enseñanzas que recibí de ellos: Rodrigo, Manolo, Luis muchas gracias por todo. Y gracias de todo corazón a María José, por su cariño desde el principio, por ser un espejo donde mirarse, y por abrirme las puertas de su casa.

Quiero agradecer todos estos años a todos los doctorandos del departamento de matemáticas de la UAM, con quienes he convivido en el departamento y fuera de él. Gracias a la familia del despacho 613, Sera, Ana y Félix, por recibirme con los brazos abiertos y por todo lo que les he echado de menos estos últimos meses. Gracias a Iason por las conversaciones matemáticas en el departamento y las discusiones personales en Alemania, y por los cientos de cafés y los miles de tachones en rojo. Gracias a Jose por las infinitas discusiones de política y por saber quién es Chimamanda Ngozi Adichie, a Adrián por dejarme siempre con la boca abierta con su conocimiento enciclopédico, a María por ser el alma de la fiesta, a Leyter por lo mucho que le voy a echar de menos, a Jaime, Bea y Raquel por ser los primeros en apuntarse a cualquier plan, a Nikita por lo que se nota su falta cuando no está en Madrid, a Dani por estar siempre dispuesto a echar una mano a quien lo necesite, a Felipe por su manera de aparecer y desaparecer, y a Marta por su buen humor. Gracias también a Angélica, doctoranda honoraria, por estar tan cerca y ser tan sabia cuando la he necesitado.

Gracias a Mónica y a Matt, por apoyarme en el principio, que es cuando más lo necesitaba. Y a Diana, por lo mismo, por su cariño en la distancia y por los correos kilométricos. Gracias a Jesús, por ser tan como yo en lo bueno y en lo malo. Gracias por ser mi mejor amigo, por aguantarme todo lo que me has aguantado, por estar en algún lugar indeterminado siempre que te he necesitado.

Muchas gracias a Beatriz por ser la mejor persona que conozco. Te mereces todo lo mejor en esta vida. Gracias a Janet, Jose e Ian por hacerme sentir como en casa.

Gracias a mis padres, por apoyarme en todo momento. Por enseñarme a ser quien soy, y a llegar a donde he querido llegar. Gracias por alentar mi curiosidad, por quitarme los mayores obstáculos, por darme todo lo que pudisteis y más. Gracias por vuestro orgullo.

Y, por supuesto, gracias a Marcos. Gracias por entender que no soy persona hasta que me tomo un café, y por no dejarme adoptar diez gatos. Gracias por hacer hueco a mi orgullo, que apenas cabe en nuestra casa. Gracias por compartir estos años, y por crear un futuro juntos. No creo que exista una persona que me entienda mejor que tú para compartir mi vida con ella.

En Madrid, a 6 de junio de 2017.

Contents

Summary

Let F and φ be two analytic functions on the unit disk $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. For an analytic function on the unit disk f, the weighted composition transformation is defined as

$$\mathbf{T}_{F,\varphi} = F(f \circ \varphi).$$

In this thesis we study three different aspects of these maps: as transformations in a non-linear class of analytic functions, as operators between Banach spaces defined axiomatically, and we also consider semigroups of weighted composition operators.

In Chapter 3 we characterize the symbols $\{F, \varphi\}$ such that the transformation $\mathbf{T}_{F,\varphi}$ preserves the class $\mathcal{P}$ of analytic functions on the unit disk with positive real part normalized so that $f(0) = 1$. We give three equivalent conditions for $\mathbf{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$: one in terms of test functions, an analytic one, and a geometrical one. The rest of the chapter is devoted to some discussion on the counterbalance of the behavior of F and φ, and the study of the fixed points of the transformation.

Chapter 4 introduces the family of mixed norm spaces that will be an example for the axiomatic Banach spaces of Chapter 6. We give growth properties of functions in these spaces, and characterize completely the inclusions between the spaces of the family.

In Chapter 5 we study the semigroups of composition operators on the mixed norm spaces defined in the previous chapter. Such semigroup is a family of (weighted, with weight $F \equiv 1$) composition operators $\{\mathbf{C}_{\varphi_t} = \mathbf{C}_t\}$ such that $\mathbf{C}_0$ is the identity and $\mathbf{C}_{t+s} = \mathbf{C}_t \circ \mathbf{C}_s$. We characterize the symbols $\{\varphi_t\}$ such that the semigroup is strongly continuous in the mixed norm spaces, that is, the operators are bounded on the space and for every f in the space

$$\lim_{t\to 0} \|\mathbf{C}_t f - f\| = 0.$$

In the final chapter we study the weighted composition operators acting on general Banach spaces of analytic functions. We will require the spaces to satisfy some natural axioms and characterize the operators that are weighted composition operators and its invertibility in such spaces. We give several examples of spaces that do not satisfy the axioms, in order to check the minimal requirements in the space for the weighted composition operators to have the properties we are interested in.

Resumen

Sean F y φ dos funciones analíticas en el disco unidad $\mathbb{D}$, con $\varphi(\mathbb{D}) \subset \mathbb{D}$. Para una función f analítica en el disco unidad, el operador de composición ponderado se define como

$$\mathbf{T}_{F,\varphi} = F(f \circ \varphi).$$

En esta tesis estudiamos tres aspectos diferentes de estas aplicaciones: como transformaciones en una clase no lineal de funciones analíticas, como operadores entre espacios de Banach definidos axiomáticamente, y también consideramos los semigrupos de operadores de composición.

En el Capítulo 3 caracterizamos los símbolos $\{F, \varphi\}$ tales que la transformación $\mathbf{T}_{F,\varphi}$ preserva la clase $\mathcal{P}$ de funciones analíticas en el disco unidad con parte real positiva y normalizadas de tal manera que $f(0) = 1$. Daremos tres condiciones equivalentes a $\mathbf{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$: una en términos de funciones test, una analítica, y una geométrica. El resto del capítulo está dedicado a discutir el equilibrio entre el comportamiento de F y el de φ, y al estudio de los puntos fijos de la transformación.

En el Capítulo 4 introducimos la familia de espacios de norma mixta, que será un ejemplo para los espacios de Banach definidos axiomáticamente del Capítulo 6. Daremos propiedades de crecimiento de las funciones en estos espacios, y caracterizaremos completamente las inclusiones entre espacios de la familia.

En el Capítulo 5 estudiamos los semigrupos de operadores de composición en los espacios de norma mixta definidos en el capítulo anterior. Un semigrupo es una familia de operadores de composición (ponderados, con peso $F \equiv 1$) $\{\mathbf{C}_{\varphi_t} = \mathbf{C}_t\}$ tales que $\mathbf{C}_0$ es el operador identidad y $\mathbf{C}_{t+s} = \mathbf{C}_t \circ \mathbf{C}_s$. Caracterizamos los símbolos $\{\varphi_t\}$ tales que el semigrupo que induce es fuertemente continuo en los espacios de norma mixta, es decir, tales que los operadores están acotados en el espacio y para cada f en el espacio

$$\lim_{t \to 0} \|\mathbf{C}_t f - f\| = 0.$$

En el capítulo final estudiamos los operadores de composición ponderados que actuan en espacios generales de Banach de funciones analíticas. Pediremos que los espacios cumplan algunos axiomas naturales y caracterizaremos los operadores acotados que son operadores de composición ponderados y su invertibilidad en dichos espacios. Daremos varios ejemplos de espacios que no satisfacen los axiomas, para comprobar los requerimientos mínimos en el espacio para que el operador de composición ponderado tenga las propiedades que nos interesan.

Introduction

Overview of the area

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane and $\mathbb{T} = \{z : |z| = 1\}$ its boundary. Given an analytic function on the unit disk F and an analytic *self-map of the disk* φ, that is, an analytic function on the unit disk such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, the *weighted composition transformation* with symbols F and φ is defined, for any f analytic in the unit disk, as

$$\mathbf{T}_{F,\varphi} f = F(f \circ \varphi).$$

Seen as operators, the weighted composition operators are the most straightforward generalization of two of the most classical of operators between spaces of analytic functions: the composition operators and the pointwise multiplication operators. For both the multiplication or composition operators on some classical spaces such as the Hardy or Bergman spaces of the disk, the properties such as injectivity, surjectivity, or invertibility have already been understood and are relatively simple. However, for a weighted composition operator characterizing such properties does not follow automatically from these earlier results and some hard work is required. It is actually even possible to have a weighted composition operator which is bounded, surjective, and invertible (and even more) and whose individual components (the multiplication and the composition operator) are both unbounded operators, as our Example 4 in Section 6.2.2 shows.

We define the *composition operator* with symbol φ, $\mathbf{C}_\varphi$, as $\mathbf{C}_\varphi f = f \circ \varphi$, with f analytic on the unit disk. The composition operators appear for the first time in an implicit form in 1871, in the work of Schröder ([104]). He was interested in the solutions of the spectral problem: the functions f and the constants λ such that

$$f \circ \varphi = \lambda f.$$

Koenigs ([81]), in 1884, solved this problem in the case when the domain is the unit disk, φ has a fixed point in $\mathbb{D}$ and its derivative at that point has modulus smaller than one. Further work with functions φ with no fixed points in $\mathbb{D}$ was made by Valiron in [122] in the 30's.

The composition operators appear also implicitly in Littlewood's Subordination Theorem [85] in 1925, and in Ryff's work [102] of 1966. The study of the general spectral properties of the composition operators began in 1968, in Nordgren's work

([93]), where he determined the spectra of composition operators when the symbol is an automorphism. The theory was futher developed by Schwartz in his thesis [105] that contains some results on compactness, bounds of the norm, and spectra.

Since then, the composition operators have become a rich field on the borderline between analytic function theory and operator theory. The most frequent goal is to study the properties of the operator $\mathbf{C}_\varphi$ such as boundedness, compactness, norm, spectra... in terms of the function theory and geometric properties of the symbol φ. They have also played a key role in de Branges' proof of the Bieberbach conjecture, since he recognized that the Robertson and Milin conjectures (that imply Bieberbach's) can be interpreted as norm inequalities involving composition operators (see [103, page 20]).

Given two vector (or Banach) spaces X and Y, an analytic function ϕ is a *pointwise multiplier* from X to Y if $\phi X \subset Y$; that is, if $\phi f \in Y$ for any $f \in X$. We denote by $\mathbf{M}(X, Y)$ the space of all multipliers from X to Y, and by $\mathbf{M}(X)$ when $Y = X$. Given $\phi \in \mathbf{M}(X, Y)$, we define the *pointwise multiplication operator with symbol* ϕ, $\mathbf{M}_\phi : X \to Y$, as $\mathbf{M}_\phi f = \phi f$.

The pointwise multiplication operators are among the most classical operators. In the Hardy spaces they are a special case of Toeplitz operators defined, for a function $\phi \in L^\infty(\mathbb{T})$, as

$$T_\phi f = P(\phi f),$$

where P is the Szegö or Riesz projection from $L^p(\mathbb{T})$ onto H^p. It can be defined similarly in other spaces of analytic functions. If the function ϕ is analytic, then the Toeplitz operator is simply the multiplication operator $\mathbf{M}_\phi$. These operators became one of the most popular topics in the theory of operators on spaces of analytic functions in the 20th century, as the number of monographies shows, including [19], [36], [35], [92], [131].

The pointwise multiplication operators have been also studied by many authors and in different contexts, especially in the study of their boundedness, see [17], [18], [62], [87], [125], and [130].

The earliest references to the systematic study of weighted composition operators appear to be [79] and [80], in the 70's, although examples of this kind of operators appeared before. For instance, the weighted composition operators are related to the isometries of the Hardy space H^p, $1 \le p < \infty$, $p \ne 2$, and other spaces. Note that we need to dismiss the Hilbert case H^2, since in Hilbert spaces there are many more isometries. For instance, permutations of the elements of the orthonormal basis are isometries. In 1960, deLeeuw proved that, if T is an onto isometry of H^1, then $T = \mathbf{T}_{\lambda\varphi',\varphi}$, where φ is an automorphism of the disk and $|\lambda| = 1$ (see [75, page 148]). Forelli in [65] extended this result to the Hardy spaces $1 < p < \infty$, $p \ne 2$, proving that if T is an onto isometry of H^p, then $T = \mathbf{T}_{\lambda(\varphi')^{1/p},\varphi}$, with φ an automorphism and $|\lambda| = 1$. The isometries of the weighted Bergman spaces are also weighted composition operators of this type: if T is an onto isometry of A^p_α, then $T = \mathbf{T}_{\lambda(\varphi')^{(2+\alpha)/p},\varphi}$, as Kolaski proved (see [82]).

The weighted composition operators have also proved useful in the study of classical operators, like the Hilbert matrix operator. Let f be an analytic function on the unit disk, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the *Hilbert matrix operator* is defined as

$$Hf(z) = \int_0^1 \frac{f(t)}{1-tz}dt = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)z^n.$$

Diamantopoulos and Siskakis proved in [53] that the Hilbert matrix operator can be written as

$$Hf(z) = \int_0^1 \mathbf{T}_{F_t,\varphi_t} f(z)dt,$$

where

$$F_t(z) = \frac{1}{(t-1)z+1} \text{ and } \varphi_t(z) = \frac{t}{(t-1)z+1},$$

and used the fact that, for $0 < t < 1$, the weighted composition operators $\mathbf{T}_{F_t,\varphi_t}$ are bounded on H^p, $1 \le p \le \infty$, to prove that the Hilbert matrix operator is bounded on H^p, $2 \le p < \infty$. This result was completed by Dostanić, Jevtić and Vukotić in [56], who computed the norm of the operator and extended the results to the context of Bergman spaces, using different techniques.

In 1974 Singh gave another application to the study of weighted composition operators. He proved in [113] that the composition operator $\mathbf{C}_\psi$ is bounded on the Hardy space of the half-plane

$$\Pi = \{w \in \mathbb{C} : \text{Im } w > 0\}$$

if and only if the weighted composition operator $\mathbf{T}_{F,\varphi}$ is bounded on H^p, with $\varphi = \gamma^{-1} \circ \psi \circ \gamma$,

$$F(z) = \left(\frac{1-\varphi(z)}{1-z}\right)^{1/p}, \text{ and } \gamma(z) = i\frac{1+z}{1-z}.$$

The weighted composition operators are also related with Brennan's conjecture which states that, for all simply connected planar domains G and all conformal maps g of G onto the unit disk, the integral

$$\int_G |g'|^p dA$$

is finite for $4/3 < p < 4$. The conjecture can be formulated for univalent maps of $\mathbb{D}$ writing $\tau = g^{-1}$.

Brennan proved in 1978 that the conjecture is true for $4/3 < p < 3 + \delta$ for some number $\delta > 0$, see [39], and several authors afterwards were able to extend the interval, but the conjecture is still open. At the beginning of the XXI century, Shimorin and Smith related the conjecture to the study of a weighted composition operator. Shimorin proved in [112] that the conjecture is equivalent to the property that all weighted composition operators $\mathbf{T}_{(\varphi')^b,\varphi}$ are bounded on A^2 for φ conformal self-maps of the disk and $b \in (-1, 0)$. Smith in [119] considered the weighted composition operator

$$\mathbf{T}_{p,\varphi} f = \left(\frac{\tau'(\varphi(z))}{\tau'(z)}\right)^p f(\varphi(z)),$$

where $\tau = g^{-1}$, and proved that Brennan's conjecture is true if and only if there exists an analytic self-map of the disk φ such that $\mathbf{T}_{p,\varphi}$ is compact on A^2 for $-1/3 < p < 1$.

Related to the weighted composition operators are the semigroups of operators. Given a family of bounded operators $\{T(t)\}$ on a Banach space X of analytic functions depending on a positive parameter t, such family is a semigroup of operators if $T(0)$ is the identity operator and the family has the *semigroup property*, that is,

$$T(t+s) = T(t) \circ T(s).$$

The semigroup is called *strongly continuous* or C_0 if for every $f \in X$ we have $T(t)f \in X$ for all $t \geq 0$ and

$$\lim_{t \to 0^+} \|T(t)f - f\|_X = 0,$$

and *uniformly continuous* if

$$\lim_{t \to 0^+} \|T(t) - I\|_X = 0$$

with I the identity operator. The theory of semigroups of bounded operators began in 1948 with the independent works of Hille [74] and Yosida [128]. They investigated the problem of determining the most general bounded linear operator $T(t)$, $t \geq 0$, such that

$$T(t+s) = T(t) \circ T(s) \quad \text{and} \quad T(0) = I,$$

thus generalizing the exponential functions in infinite dimensional function spaces.

In spaces of analytic functions, the most frequently studied semigroups of operators are the semigroups of composition operators. A family $\{\varphi_t\}$ of analytic self-maps of the disk is a *semigroup of analytic functions* if φ_0 is the identity in $\mathbb{D}$, $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$, and $\varphi_t \to \varphi_0$ as $t \to 0$ uniformly on compact sets of $\mathbb{D}$. If for every $t \geq 0$ the composition operators $\mathbf{C}_{\varphi_t}$ are bounded, then the family $\{\mathbf{C}_{\varphi_t} = \mathbf{C}_t\}$ is a semigroup of operators. This is a particular case of the *semigroups of weighted composition operators*. Given an analytic function $w : \mathbb{D} \to \mathbb{C}$, consider for $t \geq 0$ the weighted composition operator

$$\mathbf{T}_t f(z) = \frac{w(\varphi_t(z))}{w(z)} f(\varphi_t(z)).$$

If every $\mathbf{T}_t$ is bounded, then the family $\{\mathbf{T}_t\}$ is a semigroup. Choosing $w \equiv 1$, we recover the semigroup of composition operators.

The semigroups of analytic functions $\{\varphi_t\}$ were studied by Berkson and Porta in [25]. They found many interesting analytic properties of the semigroup, like the following facts:

1. The function
$$G(z) := \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t}, \; z \in \mathbb{D},$$
called the *infinitesimal generator of the semigroup*, characterizes it uniquely.

2. If the semigroup is not formed by automorphisms of the disk with fixed point in $\mathbb{D}$, there exists a point b in the closed disk, called the *Denjoy-Wolff point* of $\{\varphi_t\}$, such that $\varphi_n \to b$ (choosing the subsequence of $\{\varphi_t\}$ with $t \in \mathbb{N}$) or
$$\lim_{r\to 1} \varphi_t(rb) = b$$
if $b \in \mathbb{T}$.

3. The infinitesimal generator can be written in terms of this point and a positive real-part function as
$$G(z) = (\bar{b}z - 1)(z - b)P(z),\ z \in \mathbb{D}.$$

There are close connections between the semigroup of analytic functions $\{\varphi_t\}$ and the semigroup of operators $\{\mathbf{C}_t\}$ and $\{\mathbf{T}_t\}$. The theory of semigroups studies spectral properties, operator ideal properties or dynamical properties of the semigroup of operators in terms of the theory of functions. It has also been useful in proving properties of operators on spaces of analytic functions. For instance, an application of semigroups of composition operators to finding the full spectrum of some composition operators can be found in [50, Thm. 7.41]. In the case of weighted compostion operators, Siskakis proved in [117] the boundedness and spectral properties of the operator
$$Jf(z) = \frac{1}{1-z}\int_1^z \frac{f(\zeta)}{1+\zeta}d\zeta$$
on the Hardy space. To do this, he compares this operator with the resolvent of the semigroup of weighted composition operators
$$\mathbf{T}_t f(z) = \frac{1-\varphi_t(z)}{1-z} f(\varphi_t(z)),\ \text{where } \varphi_t(z) = \frac{(1+e^t)z + e^t - 1}{(e^t-1)z + 1 + e^t},$$
$z \in \mathbb{D}$, $t \geq 0$. The operator J is related to the classical Cesàro operator that, for an analytic function in the unit disk $f(z) = \sum_{n=0}^{\infty} a_n z^n$, is defined as
$$Cf(z) = \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}d\zeta = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n.$$
Like the Hilbert matrix operator, the Cesàro operator can also be written as an average of weighted composition operators, see [46].

On the other hand, transformations that preserve classes of analytic functions have applications to variational methods for solving non-linear extremal problems in geometric function theory, see [58, Chapter 2] for a partial list of transformations that preserve the class S of normalized univalent functions, and for applications like the proof of the Area and Distortion Theorems. Another application for the transformations that preserve a class of functions is that it could indicate the degree of rigidity when trying to produce new functions in the class from the given ones. Some of the natural examples of transformations are the rotations $e^{-i\theta} f(e^{i\theta} z)$, $0 \leq \theta < 2\pi$, $z \in \mathbb{D}$, and the dilations $f(rz)/r$, $0 < r < 1$, $z \in \mathbb{D}$, which are particular cases of weighted composition transformations, so a natural question would be to characterize all the weighted composition transformations that preserve a specific class. Many classes of analytic functions have a geometric interpretation, which in turn may help us understand better such transformations and the role of the symbols.

Brief description of the thesis

This thesis studies the weighted composition operators and transformations between spaces and classes of analytic functions from different points of view. First, we will introduce in Chapter 1 the classical spaces and classes that will appear afterwards.

The second chapter also has an introductory purpose, and will present the operators that this thesis focuses on: composition, multiplication, and weighted composition operators, keeping our attention to results on boundedness on the classical spaces studied in the first chapter, instead of a complete history of the theory. We will also present some related topics, such as the angular derivative and the iteration theory, two issues of geometric function theory related to the study of composition operators and that we will need later on, and the integral or Volterra operators, linked to the study of semigroups of operators. The last section of Chapter 2 introduces the semigroups of operators, starting with the general theory of semigroups of bounded operators initiated by Hille and Yosida in 1948, and then presenting the theory of semigroups of composition operators.

The third chapter is devoted to the characterization of weighted composition transformations that preserve Carathéodory's Class $\mathcal{P}$. This is the class of analytic functions f on the unit disk with positive real part and normalized so that $f(0) = 1$. It can be seen that every function $f \in \mathcal{P}$ can be written as $\ell \circ \omega$, where ω is some *Schwarz-type function* (that is, $|\omega(z)| \leq 1$ for every $z \in \mathbb{D}$ and $\omega(0) = 0$), and $\ell(z) = \frac{1+z}{1-z}$, $z \in \mathbb{D}$, is the conformal map of the disk onto the right half-plane. We will denote by $\mathcal{L} = \{\ell_\lambda : |\lambda| = 1\}$ the set of rotations of the half-plane function. The main theorem of the chapter is the following result that characterizes the weighted composition transformations that preserve $\mathcal{P}$.

Theorem 3.2. *Let φ be a Schwarz-type function, $F \in \mathcal{P}$, and denote by ω the Schwarz-type function for which $F = \ell \circ \omega$. Then the following conditions are equivalent:*

(a) $\boldsymbol{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$.

(b) $\boldsymbol{T}_{F,\varphi}(\mathcal{L}) \subset \mathcal{P}$.

(c) The inequality

$$4|\varphi(z)| \cdot |\operatorname{Im} \omega(z)| < (1 - |\omega(z)|^2)(1 - |\varphi(z)|^2) \tag{1}$$

holds for all z in $\mathbb{D}$. In other words,

$$2|\varphi(z)| \cdot \left| \frac{\operatorname{Im} F(z)}{\operatorname{Re} F(z)} \right| < 1 - |\varphi(z)|^2\,, \quad \text{for all } z \in \mathbb{D}\,. \tag{2}$$

(d) The inequality

$$|\arg F(z)| < \frac{\pi}{2} - \arcsin \frac{2|\varphi(z)|}{1+|\varphi(z)|^2} \tag{3}$$

holds for all z in $\mathbb{D}$. Note also that

$$\frac{\pi}{2} - \arcsin \frac{2|\varphi(z)|}{1+|\varphi(z)|^2} = \frac{\pi}{2} - \arctan \frac{2|\varphi(z)|}{1-|\varphi(z)|^2} = \arctan \frac{1-|\varphi(z)|^2}{2|\varphi(z)|}\,, \tag{4}$$

where in the case when $\varphi(0) = 0$ the last equality should be understood as the limit $\arctan(+\infty) = \frac{\pi}{2}$.

In the second part of the chapter we will try to understand better this theorem by giving examples of the counterbalance between the range of φ and of F. In Section 3.2.1 we will see that, if one of the symbols is "big", in the sense that the image of the disk under one of the functions φ or ω covers most of the unit disk, then the other has to be small. In Section 3.2.2 we take the other route to show what range must one of the symbols have if the other one has small range. We end the section by proving a result on the boundary behavior of both symbols, showing that, if φ has radial limit of modulus one at a point of the torus, then ω cannot have angular derivative at that point. The last part of the chapter characterizes the fixed points of the weighted composition transformation.

The fourth chapter presents some new results on the study of the mixed norm spaces, a family of spaces related to the Hardy and Bergman spaces. For $p, q, \alpha > 0$, an analytic function on the unit disk f is said to belong to the mixed norm space $H(p, q, \alpha)$ if and only if

$$\alpha q \int_0^1 (1-r)^{\alpha q - 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{q/p} dr < \infty$$

if $q < \infty$, and

$$\sup_{0 \le r < 1} (1-r)^{\alpha} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty$$

if $q = \infty$. Since the spaces $H(p, \infty, \alpha)$ are non-separable, we also define the closure of the polynomials in this space, the space $H_0(p, \infty, \alpha)$, that is, the space of the functions in $H(p, \infty, \alpha)$ such that

$$\lim_{r \to 1} (1-r)^{\alpha} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} = 0.$$

We introduce some results on pointwise growth of functions in these spaces, including the approximation of the norm of the point evaluation functional that will be very useful in the following chapters. The main theorem is the complete characterization of the inclusions between spaces of the family, depending on the three parameters.

Theorem 4.12. *If $p \ge u$, then*

$$H(p, q, \alpha) \subseteq H(u, v, \beta) \Leftrightarrow \begin{cases} \alpha < \beta & \text{or} \\ \alpha = \beta & \text{and} \quad q \le v. \end{cases}$$

Theorem 4.13. *If $p < u$, then*

$$H(p,q,\alpha) \subseteq H(u,v,\beta) \Leftrightarrow \begin{cases} \alpha + \frac{1}{p} < \beta + \frac{1}{u} & \text{or} \\ \alpha + \frac{1}{p} = \beta + \frac{1}{u} & \text{and} \quad q \leq v. \end{cases}$$

This chapter may be of interest by itself and it will also provide an important class of examples of spaces relevant for the last chapter.

The fifth chapter deals with the strong continuity of semigroups of composition operators in several spaces of analytic functions, especially in the mixed norm spaces defined on Chapter 4. We already know that, if $\{\varphi_t\}$ is a semigroup of analytic functions, then the family of operators $\{\mathbf{C}_t = \mathbf{C}_{\varphi_t}\}$ is a semigroup of composition operators on a Banach space of analytic functions X, as long as $\mathbf{C}_t$ are bounded for every $t \geq 0$ on X. Therefore, we first characterize the self-maps of the unit disk that induce bounded composition operators on $H(p,q,\alpha)$.

Proposition 5.1. *Suppose $0 < p, q \leq \infty$ and $0 < \alpha < \infty$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Then $\boldsymbol{C}_\varphi$ is bounded on $H(p,q,\alpha)$ and on $H_0(p,\infty,\alpha)$. Moreover, it holds that*

$$\|\boldsymbol{C}_\varphi\| \lesssim \left(\frac{\|\varphi\|_\infty + |\varphi(0)|}{\|\varphi\|_\infty - |\varphi(0)|} \right)^{\alpha + \frac{1}{p}}.$$

If, in addition, $\varphi(0) = 0$, then $\|\boldsymbol{C}_\varphi\| = 1$.

This result will probably not surprise the experts in the area, given the similarity of the mixed norm spaces to the Hardy and Bergman spaces, but it will also be of help in the last chapter. Once we know that every semigroup of analytic functions induces a semigroup of bounded composition operators, we are ready to characterize those semigroups that are strongly continuous, that is,

$$\lim_{t \to 0^+} \|\mathbf{C}_t f - f\|_{p,q,\alpha} = 0$$

for every $f \in H(p,q,\alpha)$, in terms of $\{\varphi_t\}$. In Section 5.2 we give a general result, in the spirit of the axiomatic results of the final chapter.

Proposition 5.2. *Let $\{\varphi_t\}$ be a semigroup of analytic functions in the unit disk and let X be a Banach space of analytic functions such that*

(i) Polynomials are dense in X;

(ii) There is a constant $C > 0$ such that if f and g belong to X and $|f| \leq |g|$, then $\|f\|_X \leq C\|g\|_X$;

(iii) $M := \limsup_{t \to 0^+} ||\boldsymbol{C}_t|| < +\infty$;

(iv) $\lim_{t \to 0^+} ||\varphi_t - \varphi_0||_X = 0$.

Then the semigroup of operators $\{\boldsymbol{C}_t\}$ is strongly continuous on X.

We will use this proposition to prove that any semigroup of analytic functions induces a strongly continuous semigroup of composition operators in several separable spaces of analytic functions such as the Hardy spaces H^p for $p < \infty$, the Bergman spaces, and the mixed norm spaces $H(p, q, \alpha)$ that are separable, that is, where $q < \infty$. Clearly, we cannot apply this result to the spaces $H(p, \infty, \alpha)$, and actually we know a similar result will not be true, since we have the following example.

Proposition 4.2. *For $0 \leq r < 1$, let $f_r(z) = f(rz)$, $z \in \mathbb{D}$.*

- *If $f \in H(p, q, \alpha)$, $0 < p \leq \infty$, $0 < q, \alpha < \infty$, then $\|f_r - f\|_{p,q,\alpha} \to 0$, as $r \to 1$.*
- *If $f \in H_0(p, \infty, \alpha)$, $0 < p \leq \infty$, $0 < \alpha < \infty$, then $\|f_r - f\|_{p,\infty,\alpha} \to 0$, as $r \to 1$.*

Moreover, if $f \in H(p, \infty, \alpha)$ and $\|f_r - f\|_{p,\infty,\alpha} \to 0$, as $r \to 1$, then $f \in H_0(p, \infty, \alpha)$.

In other words, let $\varphi_t(z) = e^{-t}z$, for all $t \geq 0$ and $z \in \mathbb{D}$, then $f_r = \mathbf{C}_{\varphi_t} f$ for $t = \log r$, and therefore the semigroup $\{\varphi_t\}$ induces a strongly continuous semigroup of composition operators on $H(p, q, \alpha)$ for $q < \infty$, but not for $H(p, \infty, \alpha)$.

In a similar context, that is, the study of the semigroups of composition operators on the non-separable space $BMOA$, the authors of [29] define the maximal subspace of X such that the semigroup $\{\varphi_t\}$ generates a strongly continuous semigroup of operators on it, denoted by $[\varphi_t, X]$, that is,

$$[\varphi_t, X] = \{f \in X : \|f \circ \varphi_t - f\|_X \to 0 \text{ as } t \to 0\}.$$

In this notation, the second part of Proposition 4.2 becomes

$$[\varphi_t, H(p, \infty, \alpha)] = H_0(p, \infty, \alpha),$$

with $\varphi_t(z) = e^{-t}z$.

In [29] the authors proved that, if X contains the constant functions and the operators $\{T_t\}$ are uniformly bounded, then

$$[\varphi_t, X] = \overline{\{f \in X : Gf' \in X\}},$$

where G is the infinitesimal generator of $\{\varphi_t\}$. With this characterization they were able to relate this subspace with the boundedness of an integral operator on the space. We devote Section 5.3 to study the integral operators on the mixed norm spaces and find, using the characterization of the maximal subspace, that the example with the dilations $\varphi_t(z) = e^{-t}z$ is the model for any semigroup of analytic functions, that is, they are never strongly continuous on the whole space $H(p, \infty, \alpha)$.

Theorem 5.14. *No nontrivial semigroup induces a strongly continuous semigroup of operators on $H(p, \infty, \alpha)$. In other words,*

$$[\varphi_t, H(p, \infty, \alpha)] \subsetneq H(p, \infty, \alpha)$$

for every semigroup of analytic functions $\{\varphi_t\}$.

We are also able to characterize the semigroups $\{\varphi_t\}$ with Denjoy-Wolff point inside the disk (in other words, with fixed point in $\mathbb{D}$), for which the biggest subspace where the semigroup $\{\mathbf{C}_t\}$ is strongly continuous is $H_0(p,\infty,\alpha)$, in terms of the generator of $\{\varphi_t\}$. Recall that the generator G of $\{\varphi_t\}$ can be written as

$$G(z) = (\bar{b}z - 1)(z - b)P(z),\ z \in \mathbb{D},$$

with b the Denjoy-Wolff point and P a function with positive real part.

Theorem 5.16. *Let $\{\varphi_t\}$ be a semigroup with Denjoy-Wolff point $b \in \mathbb{D}$. Then*

$$H_0(p,\infty,\alpha) = [\varphi_t, H(p,\infty,\alpha)] \Leftrightarrow \frac{1}{P} \in H_0(\infty,\infty,1).$$

Moreover, if $\frac{1}{P} \notin H_0(\infty,\infty,1)$ then the space $[\varphi_t, H(p,\infty,\alpha)]$ contains a subspace isomorphic to ℓ_∞.

This theorem leads to characterizations of $H_0(p,\infty,\alpha)$ in terms of the strong continuity of semigroups satisfying the hypothesis above, in the spirit of Proposition 4.2.

The final result proves that, if the Denjoy-Wolff point of the semigroup is on the circle, then the largest subspace where the semigroup $\{\mathbf{C}_t\}$ is strongly continuous is much bigger than $H_0(p,\infty,\alpha)$, since it contains a non-separable subspace.

Theorem 5.17. *For every semigroup of analytic functions with Denjoy-Wolff point $b \in \mathbb{T}$ the maximal subspace $[\varphi_t, H(p,\infty,\alpha)]$ contains a subspace isomorphic to ℓ_∞. Consequently, $[\varphi_t, H(p,\infty,\alpha)] \supsetneq H_0(p,\infty,\alpha)$.*

In the sixth chapter we study the weighted composition operators as transformations acting on spaces of analytic functions defined by a handful of axioms. The spaces we consider will be Banach spaces of analytic functions on $\mathbb{D}$ or on a bounded domain Ω where the pointwise evaluation functionals are bounded, and suppose that they satisfy some natural axioms, such as the boundedness of the shift operator (that is, the multiplication by the identity) or the density of polynomials. Then, we obtain several results on boundedness of the multiplication operator, characterization of the bounded operators that are weighted composition operators, and their invertibility. We also give several examples of spaces that do not satisfy one of the axioms, to emphasize its necessity.

Our first theorem is the following.

Theorem 6.2. *Let $X \subset H(\Omega)$ be a Banach space in which the point evaluations are bounded. Then the following conditions are equivalent:*

(a) $H^\infty(\Omega) = \boldsymbol{M}(X)$.

(b) There is a universal constant $C > 0$ such that if $f \in H(\Omega)$, $g \in X$ and $|f(z)| \le |g(z)|$ holds for all $z \in \mathbb{D}$, then $f \in X$ and $\|f\|_X \le C\|g\|_X$.

Moreover, the least constant possible in the inequality that defines the property (DP) is $C = \|J\|$, where J is the correspondence operator $J:\ H^\infty \to \boldsymbol{M}(X)$, $J(F) = \boldsymbol{M}_F$.

It is easy to see that the Hardy, Bergman and weighted Banach spaces satisfy (*b*), while the Bloch and Dirichlet spaces do not.

In Section 6.2 we consider Banach spaces of analytic functions on $\mathbb{D}$ that satisfy the following set of axioms:

Ax1 All point evaluation functionals Λ_z are bounded on X.

Ax2 The set of all algebraic polynomials of z is contained in X and dense in it (in $\|\cdot\|_X$).

Ax3 The shift operator $S = \mathbf{M}_z$ is bounded on X.

Ax4 $\limsup_{n\to\infty} \|z^n\|^{1/n} = 1$.

Ax5 Every disk automorphism induces a bounded composition operators on X.

These axioms are quite natural and are satisfied by many of the spaces defined in Chapter 1 (see the discussion after the definition of the axioms in Section 6.2.1). With these axioms we are able to generalize two well-known characterizations of the multiplication and composition operators on H^2: a bounded operator T on H^2 is a multiplication operator if and only if it commutes with the shift operator, while T is a composition operator if it is multiplicative in the sense that $T(fg) = Tf \cdot Tg$ for all f, $g \in H^2$ such that also $fg \in H^2$.

Theorem 6.4. *Let $X \subset H(\mathbb{D})$ be a functional Banach space in which the axioms **[Ax1] - [Ax4]** are fulfilled. Let T be a continuous operator on X with the property that $Tz \neq \lambda \cdot T1$ for every unimodular number λ. Then the following conditions are equivalent:*

(a) T is a weighted composition operator;

(b) There exists $\varphi \in H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\mathbf{M}_\varphi T = TS$;

(c) There exists φ, meromorphic in $\mathbb{D}$, such that $\mathbf{M}_\varphi T = TS$;

(d) There exists φ, meromorphic in $\mathbb{D}$, such that $\mathbf{M}_{\varphi^n} T = TS^n$ for all integers $n \geq 0$;

(e) There exists φ, meromorphic in $\mathbb{D}$, such that $\varphi^n \cdot T1 = T(z^n)$ for all integers $n \geq 0$;

(f) There exists $\varphi \in H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi^n \cdot T1 = T(z^n)$ for all integers $n \geq 0$;

(g) $T1 \cdot T(fg) = Tf \cdot Tg$ holds for all functions f, $g \in X$ for which $fg \in X$ as well.

Whenever any of the conditions (a)–(g) is fulfilled, then $\varphi = Tz/(T1)$ is the composition symbol of T.

To show the necessity of Axioms [**Ax2**] and [**Ax4**] we give two examples of spaces of analytic functions not satisfying one of those axioms and where the theorem is false. In the same context of the previous theorem we can study the invertibility of a bounded operator on a space satisfying the Axioms [**Ax1**] - [**Ax5**].

Theorem 6.8. *Let $X \subset H(\mathbb{D})$ be a functional Banach space in which the axioms [**Ax1**] - [**Ax4**] are satisfied and suppose that a weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is bounded in X.*

(a) If $\boldsymbol{T}_{F,\varphi}$ is invertible in X then its composition symbol φ is an automorphism of $\mathbb{D}$, the multiplication symbol F does not vanish in the disk, and the inverse operator $\boldsymbol{T}_{F,\varphi}^{-1}$ is another weighted composition operator $\boldsymbol{T}_{G,\psi}$, whose symbols are:

$$G = \frac{1}{F \circ \varphi^{-1}}, \qquad \psi = \varphi^{-1} .$$

*(b) Assuming that Axiom [**Ax5**] also holds, we have the following characterization.*

The weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is invertible on X if and only if its composition symbol φ is an automorphism of $\mathbb{D}$, the multiplication symbol F does not vanish in the disk, and $1/F \in \boldsymbol{M}(X)$. If this is the case, then F is also a self-multiplier of X and the inverse operator is $\boldsymbol{T}_{G,\psi}$, with G and ψ as above.

We then give an example of a space that fails to satisfy [**Ax2**] and [**Ax5**], and therefore, part (*b*) of Theorem 6.8 does not hold. We are able to find two functions F and φ such that the multiplier and composition operators induced by them are unbounded in the space, but the weighted composition operator $\mathbf{T}_{F,\varphi}$ is not only bounded but also invertible and an involution.

In the last part of the chapter, we consider the more general setting of Banach spaces of analytic functions in a bounded domain Ω. To study these spaces, we define the new set of axioms

A1 All point evaluation functionals Λ_z are bounded on X.

A2 $f_0 \in X$, where $f_0(z) \equiv 1$.

A3 The shift operator is bounded on X.

A4 For every function $f \in X$ we have $\dfrac{|f(z)|}{\|\Lambda_z\|} \to 0$ as $\operatorname{dist}(z, \partial\Omega) \to 0$.

A5 Each automorphism of Ω induces a bounded composition operator in X.

We obtain a similar result on invertibility of the weighted composition operators.

Theorem 6.11. *Let $X \subset H(\Omega)$ be a Banach space which satisfies the set of axioms (**A1**) - (**A4**) and suppose that the weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is bounded in X.*

(a) If $\boldsymbol{T}_{F,\varphi}$ is invertible in X then its composition symbol φ is an automorphism of Ω and the multiplication symbol F does not vanish in Ω.

*(b) If, in addition to the axioms listed, the space X also satisfies Axiom **(A5)** we have the following characterization.*

The weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is invertible on X if and only if its composition symbol φ is an automorphism of Ω, the multiplication symbol F does not vanish in Ω, and $1/F \in \boldsymbol{M}(X)$. If this is the case, then F is also a self-multiplier of X and the inverse operator is $\boldsymbol{T}_{G,\psi}$, with the symbols

$$G = \frac{1}{F \circ \varphi^{-1}}, \qquad \psi = \varphi^{-1}\,.$$

The example after Theorem 6.8 shows the necessity of **(A4)** in this context.

The results of this thesis are contained in the following research papers:

- I. Arévalo, A characterization of the inclusions between mixed norm spaces, *J. Math. Anal. Appl.* **429** (2015), 942–955.
- I. Arévalo, R. Hernández, M. J. Martín and D. Vukotić, On weighted compositions preserving the Carathéodory class, arXiv preprint arXiv:1608.04577 (2016).
- I. Arévalo, M. D. Contreras and L. Rodríguez-Piazza, Semigroups of composition operators and integral operators on mixed norm spaces, arXiv preprint arXiv:1610.08784 (2016).
- I. Arévalo, D. Vukotić, Weighted composition operators in functional Banach spaces: an axiomatic approach, arXiv preprint arXiv:1706.07133 (2017).

This work has been partially supported by the "Ayuda para inicio del Doctorado en Matemáticas del Departamento de Matemáticas de la Universidad Autónoma de Madrid", the MINECO FPI fellowship BES-2013-062668, and the MINECO grants MTM2012-37436- C02-02 and MTM2015-65792-P.

Chapter 1

Spaces and classes of analytic functions

In this chapter we introduce some classical spaces and classes of analytic functions that will appear later in the present work.

In the case of the spaces of analytic functions, one of the properties we will be interested in, given the nature of the operators this work focuses on, is the pointwise growth of the functions in these spaces. Recall that for a Banach space X the *point evaluation functional* is defined as

$$\Lambda_z : X \to \mathbb{C}, \quad \Lambda_z(f) = f(z).$$

In all the spaces listed below the point evaluation functional is bounded, that is, for every $z \in \mathbb{D}$ there exists a constant $C(z)$ such that for every $f \in X$

$$|\Lambda_z f| = |f(z)| \leq \|f\|_X C(z).$$

Therefore,

$$\|\Lambda_z\| := \sup_{\substack{f \in X, \\ f \not\equiv 0}} \frac{|f(z)|}{\|f\|_X} < \infty$$

for every $z \in \mathbb{D}$.

Another interesting property is the conformal invariance. Let

$$\mathrm{Aut}(\mathbb{D}) = \{\lambda \sigma_a : a \in \mathbb{D}, |\lambda| = 1\}.$$

It is well known that the set of all disk automorphisms is $\mathrm{Aut}(\mathbb{D})$, where σ_a are the automorphisms which are involutions, $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. Following [9], a space X of analytic functions in $\mathbb{D}$, defined via a semi-norm ρ, is *conformally invariant* or *Möbius invariant* if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \mathrm{Aut}(\mathbb{D})$, and moreover, $\rho(f \circ \varphi) \leq C\rho(f)$ for some positive constant C and all $f \in X$.

1.1 Hardy Spaces

The Hardy spaces are among the most important spaces of analytic functions. Their study began in 1915 when G. H. Hardy [71] proved, answering a question by Bohr and Landau, that for any analytic function f on $\mathbb{D}$, its integral mean

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \text{ for } 0 < p < \infty,$$

and

$$M_\infty(r,f) = \max_{0\le\theta<2\pi} |f(re^{i\theta})|$$

is an increasing function on r (see [57, Chapter 1]).

The *Hardy space* H^p is the space of analytic functions on the unit disk for which the integral means $M_p(r,f)$ are bounded uniformly in r. For every p, $0 < p < \infty$, the polynomials are dense in H^p, and if $1 \le p \le \infty$, the space H^p is a Banach space with the norm

$$\|f\|_{H^p} = \lim_{r\to 1}\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \text{ when } 1 \le p < \infty,$$

and

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|.$$

Moreover, H^∞ is a Banach algebra and H^2 is a Hilbert space with the inner product

$$\langle f,g\rangle_{H^2} = \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}d\theta.$$

Using polar coordinates one can prove that, if $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$,

$$\langle f,g\rangle_{H^2} = \sum_{n=0}^\infty a_n \bar{b}_n.$$

In particular,

$$\|f\|_{H^2}^2 = \sum_{n=0}^\infty |a_n|^2. \tag{1.1}$$

The theory of Hardy spaces was developed in the 1920s and 1930s by Hardy, Littlewood, Paley, Privalov, the Riesz brothers, the Nevanlinna brothers and many others. Important contributions were later given by Hoffman, Shapiro, Carleson, Flett, Shields, Duren, Khavinson, Rudin and others. There are several monographs on Hardy spaces, to note, for instance, [57], [68], [75] and [83].

One of the differential characteristics that Hardy spaces have is the boundary behavior of their functions.

Theorem 1.1 (Fatou's Theorem)**.** *If $f \in H^p$, $0 < p \le \infty$, the radial limit*

$$\tilde{f}(\theta) = \lim_{r\to 1^-} f(re^{i\theta})$$

exists for almost every $\theta \in [0, 2\pi)$.

Moreover, the existence of the limit can be extended to any region called *Stolz angle*, a non-tangential region associated to any point θ_0 where the radial limit exists. Thus the functions in H^p have non-tangential limits almost everywhere. By Fatou's Theorem, the norm in H^p can be rewritten as

$$\|f\|_{H^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |\tilde{f}(\theta)|^p d\theta \right)^{1/p} = \|\tilde{f}\|_{L^p(\mathbb{T})} \tag{1.2}$$

and the inner product in H^2 as

$$\langle f, g \rangle_{H^2} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(\theta)\overline{\tilde{g}(\theta)} d\theta = \left\langle \tilde{f}, \tilde{g} \right\rangle_{L^2(\mathbb{T})}.$$

Therefore there exists a correspondence between the Hardy space H^p of analytic functions on the unit disk and the space

$$\tilde{\mathcal{H}^p} = \{\tilde{f} \in L^p(\mathbb{T}, d\theta) : \tilde{f}(n) = 0 \text{ for any } n < 0\},$$

where $\tilde{f}(n)$ is the $n-$th Fourier coefficient of $\tilde{f}$.

The Hardy spaces, being L^p- type spaces with respect to a finite measure, the inclusions

$$H^\infty \subset H^p \subset H^q \quad \text{for } 0 < q < p \leq \infty$$

are satisfied. The proof is a direct application of Hölder's inequality.

Another important property of Hardy spaces is the canonical factorization, initiated by Riesz and fully developed by Smirnov. It states that every function $f \in H^p$, $0 < p \leq \infty$, can be written as a product of three unique functions: a Blaschke product, a singular inner function and an outer function. This is a consequence of the behavior of the zeroes of functions in the Hardy space and Fatou's Theorem.

Let k be a non-negative number and $\{a_n\}$ a sequence of complex numbers such that $\sum(1 - |a_n|) < \infty$ (this is called the *Blaschke property*). A *Blaschke product* is an analytic function on the unit disk of the form

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z},$$

$z \in \mathbb{D}$. It satisfies $|B(e^{i\theta})| = 1$ for almost every $e^{i\theta} \in \mathbb{T}$. A *singular inner function* is a function of the form

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right\},$$

with μ a non-decreasing bounded function with $\mu'(t) = 0$ at almost every $t \in [0, 2\pi)$, and satisfies $|S(z)| < 1$ if $|z| < 1$ and $|S(e^{it})| = 1$ at almost every $t \in [0, 2\pi)$ with respect to the measure $\frac{dt}{2\pi}$. An *outer function* in H^p is a function of the form

$$F(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) dt \right\},$$

with $\log \psi \in L^1$ and $\psi \in L^p$.

It can be proved that the zeroes of any Hardy function satisfy the Blaschke property. The canonical factorization of a function $f \in H^p$ will be formed by the Blaschke product constructed with the zeroes $\{a_n\}$ of f, the outer function F with respect to the L^p function $\tilde{f}(e^{it})$, and a singular inner function, see [57].

Moreover, an *inner function* is a function f such that $|f(z)| < 1$ if $|z| < 1$ and $|\tilde{f}(e^{it})| = 1$ in almost every $t \in [0, 2\pi)$ with respect to the measure $\frac{dt}{2\pi}$. They can be factored as $f = e^{i\gamma} BS$. The names of the functions in the canonical factorization appear for the first time in Beurling's work [26] on invariant subspaces for the shift operator on ℓ_2, see also [57, Chapter 2].

The Blaschke products are related not only to the zeroes and canonical characterization of functions in H^p, but also to the unit sphere of H^∞, via Marshall's Theorem, see [83, Chapter VII B].

Theorem 1.2 (Marshall's Theorem)**.** *The unit sphere in H^∞ is the norm-closed convex hull of the set of Blaschke products.*

The functions in H^p satisfy the following sharp growth estimate ([57, Exercise 8.4]):

Proposition 1.3. *If $f \in H^p$ and $z \in \mathbb{D}$, then*

$$|f(z)| \leq \frac{\|f\|_{H^p}}{(1 - |z|^2)^{\frac{1}{p}}}.$$

Given any number $\zeta \in \mathbb{D}$, the equality is attained at ζ only for multiples of the function

$$f_\zeta(z) = \left(\frac{1 - |\zeta|^2}{(1 - \bar{\zeta} z)^2} \right)^{\frac{1}{p}}$$

which satisfies $\|f_\zeta\|_{H^p} = 1$.

This means that the point evaluation functionals Λ_z are bounded for every $z \in \mathbb{D}$ and

$$\|\Lambda_z\| = \frac{1}{(1 - |z|^2)^{\frac{1}{p}}}.$$

Moreover, it can be proved using the subharmonicity of $|f(re^{i\theta})|^p$ that, for every $f \in H^p$,

$$|f(z)| = o(1 - |z|)^{\frac{1}{p}}$$

as $|z| \to 1$.

1.2 Bergman Spaces

The *Bergman space* A^p, $0 < p < \infty$, is the space of analytic functions on the unit disk such that

$$\int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure. That is, A^p is the subspace of $L^p(\mathbb{D}, dA)$ whose elements are analytic functions.

The Bergman spaces were probably studied explicitly (as spaces of analytic functions) for the first time in the works of Djrbashian (see [54], [55]), although the integral expression for analytic functions first appeared in the work of Hardy and Littlewood [72] on properties of the integral means. Djrbashian defined the spaces and gave the integral representation of functions in A^p. However, the development of the theory of reproducing kernels and other important properties of the spaces began with the work of Bergman. See his book [24]. He focused on the Hilbert case A^2 in planar domains Ω with the inner product

$$\langle f, g\rangle_{A^2} = \int_\Omega f(z)\overline{g}(z)dA(z).$$

If $\Omega = \mathbb{D}$, and $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$ belong to A^2, a computation in polar coordinates shows that

$$\langle f, g\rangle_{A^2} = \sum_{n=0}^\infty \frac{a_n \bar{b}_n}{n+1}.$$

In particular, the norm in A^2 can be written as

$$\|f\|_{A^2} = \left(\sum_{n=0}^\infty \frac{|a_n|^2}{n+1}\right)^{1/2}.$$

In general, for any $0 < p < \infty$, the Bergman space A^p is a complete space of analytic functions on the unit disk where polynomials are dense. It becomes a Banach space for $1 \leq p < \infty$ with the norm

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p dz\right)^{1/p}.$$

Being subspaces of L^p with finite measure, the spaces A^p satisfy the inclusions if $p < q$, then $A^q \subset A^p$.

The Bergman spaces are closely related to the Hardy spaces. Rewriting the area integral as

$$\frac{1}{\pi}\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p r d\theta dr = 2\int_0^1 M_p^p(r, f) r dr$$

it is clear that every function in the Hardy space H^p belongs also to the Bergman space A^p. Moreover, Hardy and Littlewood proved $H^p \subseteq A^{2p}$ (see also [89], [126]). Nevertheless, the functions in the Bergman spaces are quite different from those in the Hardy space since they may have a wild boundary behavior. Also, many well-known techniques known for Hardy spaces concerning zero-sets, invariant subspaces, interpolation problems or canonical divisors failed in the Bergman space. In the 90's there were many important advances in the theory of Bergman spaces concerning these topics. The references [59] and [73] discuss this theory.

A natural generalization of theses spaces is the weighted version of this theory. Let w be a *weight*, that is, a non-negative integrable function. The *weighted Bergman space* with weight w, A^p_w, is the space of analytic functions on the unit disk such that

$$\|f\|_{A^p_w} = \left(\int_{\mathbb{D}} |f(z)|^p w(z) dA(z) \right)^{1/p} < \infty.$$

Although A^p_w is a subspace of the space $L^p(w\, dA)$, it is not complete in general. For instance, A^p_w is not complete if $w(z) = 0$ in some annulus $R < |z| < 1$. In fact, it is an open problem to characterize the weights w for which the space A^p_w is complete. Nevertheless, it is known that the *standard radial weights*

$$w_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha$$

with $\alpha > -1$ give rise to spaces A^p_α that are complete. Notice that the case $\alpha = 0$ is the classical Bergman space A^p.

The point evaluation functionals are bounded in these spaces, as the following theorem, proved in the context of the unit ball of $\mathbb{C}^n$ in [123], shows.

Theorem 1.4. *If $f \in A^p_\alpha$, $0 < p < \infty$, and $z \in \mathbb{D}$, then*

$$|f(z)| \leq \frac{\|f\|_{A^p_\alpha}}{(1 - |z|^2)^{\frac{2+\alpha}{p}}}.$$

Given any number $\zeta \in \mathbb{D}$, the equality is attained at ζ only for multiples of the function

$$f_\zeta(z) = \left(\frac{1 - |\zeta|^2}{(1 - \bar{\zeta} z)^2} \right)^{\frac{2+\alpha}{p}}$$

that satisfy $\|f_\zeta\|_{A^p} = 1$.

Therefore, for every $z \in \mathbb{D}$

$$\|\Lambda_z\| = \frac{1}{(1 - |z|^2)^{\frac{2+\alpha}{p}}}.$$

Besides the well-known “big-Oh” growth estimate, we have the estimate

$$|f(z)| = o(1 - |z|)^{\frac{2+\alpha}{p}}$$

as $|z| \to 1$ for every $f \in A^p_\alpha$. This is a consequence of the subharmonicity of $|f|^p$ which yields the inequality

$$\int_{D(a,r)} |f(z)|^p \, dA(z) \leq \int_{\mathbb{D}} |f(z)|^p \, dA(z) = \|f\|^p_{A^p}$$

for $a \in \mathbb{D}$ and $r < 1$, applying the Lebesgue Dominated Convergence Theorem (see [59, Page 7]).

1.3 Dirichlet Space

The Dirichlet space is considered one of the classical spaces of analytic functions on the unit disk, together with the Hardy and Bergman spaces. The definition can be traced back to Beurling's thesis of 1933, and the explicit name of the space to two articles of Beurling and Deny in 1958 and 1959. It is called Dirichlet space in reference to the Dirichlet integral in the method for solving Laplace's equation.

The *Dirichlet space* $\mathcal{D}$ is the space of analytic functions on the unit disk satisfying

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

where dA is the normalized area measure, that is, the analytic functions on the unit disk whose derivative is in the Bergman space A^2. It is a Hilbert space with inner product

$$\langle f, g\rangle_{\mathcal{D}} = f(0)\overline{g}(0) + \int_{\mathbb{D}} f'(z)\overline{g'}(z) dA(z), \tag{1.3}$$

inducing the norm

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

In terms of its Taylor coefficients, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g\rangle_{\mathcal{D}} = a_0\bar{b}_0 + \sum_{n=1}^{\infty} n a_n \bar{b}_n$$

and

$$\|f\|_{\mathcal{D}}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2.$$

From this formula it is easy to see that polynomials are dense in $\mathcal{D}$.

Recalling that the norm in H^2 of a function f in terms of its Taylor coefficients is

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2,$$

it is clear that $\|f\|_{H^2} \leq \|f\|_{\mathcal{D}}$, and therefore $\mathcal{D} \subset H^2$. Nevertheless, there is no inclusion relationship between the Dirichlet space and the H^∞ space. The Dirichlet integral

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z)$$

has a very natural geometric interpretation, since it is exactly the area of the image of the unit disk under f, counted according to multiplicity. Therefore, the univalent (analytic and injective) functions contained in the Dirichlet space are those for which the area of $f(\mathbb{D})$ is finite. Thanks to the Riemman Mapping Theorem it is possible to construct an unbounded function in $\mathcal{D}$, and thus $\mathcal{D} \not\subset H^\infty$. Likewise, taking an infinite

Blaschke product B (that takes each value an infinite number of times), its H^∞ norm is clearly one, while we have, after the change of variables $w = B(z)$, that

$$\int_{\mathbb{D}} |B'(z)|^2 dA(z) = \int_{B(\mathbb{D})} n_B(w) dA(w) = \infty,$$

where

$$n_B(w) = |\{w \in \mathbb{D} : B(w) = z\}|$$

is the Nevanlinna counting function. Therefore $H^\infty \not\subset \mathcal{D}$.

The previous discussion shows that the Dirichlet space has interesting geometric properties. For instance, it was proved in [8] that it is the only (in some sense) Hilbert conformally invariant space. It is also closely related to logarithmic potential theory (see [61], [10]). Moreover, the Dirichlet space is a reproducing kernel Hilbert space. That means that for every $z \in \mathbb{D}$ there exists a kernel $k_z \in \mathcal{D}$ such that for any $f \in \mathcal{D}$,

$$f(z) = \langle f, k_z \rangle$$

(therefore the kernel reproduces the values of the functions of the space). Namely,

$$k_z(w) = 1 + \log \frac{1}{1 - \bar{z}w}$$

is the kernel for the inner product defined by (1.3). Using the Cauchy-Schwarz inequality we obtain the next result.

Proposition 1.5. *If $f \in \mathcal{D}$ and $|z| < 1$, then*

$$|f(z) - f(0)| \leq \|f\|_{\mathcal{D}} \left(\log \frac{1}{1 - |z|^2} \right)^{1/2}.$$

Once again, as in the results stated in the Hardy and Bergman cases, one can show that in fact

$$|f(z)| = o \left(\log \frac{1}{1 - |z|^2} \right)^{1/2}$$

as $|z| \to 1$ for every $f \in \mathcal{D}$ (see [61, Cor. 1.2.2]).

Like in the Bergman spaces, we can define the *weighted Dirichlet spaces* with standard radial weight $\mathcal{D}_\alpha$, $\alpha > -1$, as the space of analytic functions in the unit disk such that

$$\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

It is clear that $\mathcal{D}_0$ is the classical Dirichlet space. Littlewood-Paley's identity (see [50, p. 34]) can be used to prove that $\mathcal{D}_1 = H^2$, and by a well-known norm equivalence, $\mathcal{D}_2 = A^2$.

1.4 Weighted Hilbert spaces

For a sequence $\{\beta(n)\}$ of positive numbers with $\beta(0) = 1$, the *weighted Hilbert spaces* H^2_β consist of all analytic functions in $\mathbb{D}$ with the Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ and such that

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} \beta(n)^2 |a_n|^2.$$

Since the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n\beta(n)}$$

belongs to the space H^2_β, we have that the sequence $\{\beta(n)\}$ must satisfy

$$\limsup_{n\to\infty} \frac{1}{\sqrt[n]{\beta(n)}} \leq 1.$$

The spaces H^2_β are equipped with the following inner product: for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathbb{D}$,

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta(n)^2.$$

In these spaces polynomials are dense, since the Taylor polynomial of each function converges to the same function in the norm.

They are reproducing kernel Hilbert spaces, with the kernel

$$k_z(w) = \sum_{n=0}^{\infty} \frac{\overline{z}^n}{\beta(n)^2} w^n,$$

$w \in \mathbb{D}$, and therefore the point evaluation functionals are bounded, with norm

$$\|\Lambda_z\| = \|k_z\|_\beta = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta(n)^2}.$$

As in the Hardy, Bergman, and Dirichlet case, we also have a "little-oh" growth condition, which in this case depends on the sequence $\{\beta(n)\}$.

Theorem 1.6. *In a weighted Hilbert space H^2_β for which the series*

$$\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2}$$

diverges, we have that, for every $f \in H^2_\beta$,

$$|f(z)| = o\left(\|\Lambda_z\|\right)$$

as $|z| \to 1$.

Theorem 1.6 is obtained by combining the conclusions of Theorem 2.10 and Theorem 2.17 from [50].

This family of spaces was introduced by Shields in 1974 [109], see also [50]. A frequent example of such weighted Hilbert spaces is given by the sequences $\{(n+1)^\gamma\}$, $\gamma < 1$. We recover the Hardy space H^2 with $\gamma = 0$, the Bergman space A^2 with $\gamma = -1/2$, the Dirichlet space $\mathcal{D}$ with $\gamma = 1/2$, and, in general, it can be proved (as was done in [134] and can also be deduced by the work in [41]) that the weighted Dirichlet space D_α is a weighted Hilbert space of this family with $\gamma = (1-\alpha)/2$.

Notice that, as mentioned above, the Dirichlet space $\mathcal{D}$ is the same space as the weighted Hilbert space H^2_β, with $\beta(n) = (n+1)^{1/2}$, but with the norm

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} (n+1)|a_n|^2$$

and the reproducing kernel

$$k_z(w) = \begin{cases} 1, & \bar{z}w = 0, \\ \frac{1}{\bar{z}w} \log \frac{1}{1-\bar{z}w}, & \bar{z}w \neq 0. \end{cases}$$

1.5 Analytic Besov spaces

For $1 < p < \infty$, the *analytic Besov space* B^p is the space of analytic functions in the unit disk whose invariant derivative belongs to the L^p space with respect to the hyperbolic area measure, that is, the space of analytic functions f on the unit disk $\mathbb{D}$ satisfying

$$(p-1) \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \frac{dA(z)}{(1-|z|^2)^2} < \infty.$$

Equivalently, $f \in B^p$ if and only if $f' \in A^p_{p-2}$. The B^p space is a Banach space with the norm

$$\|f\|_{B^p} = \left(|f(0)|^p + (p-1) \int_{\mathbb{D}} (1-|z|^2)^{p-2} |f'(z)|^p dA(z) \right)^{\frac{1}{p}}.$$

The case $p = 2$ is the Dirichlet space.

The space B^1 must be defined in a different fashion, since

$$(1-|z|^2)^{-1} f'(z) \in L^1(\mathbb{D}, dA)$$

if and only if f is constant. It is usually defined in terms of the automorphisms of the disk σ_a ($|a| < 1$) given by $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. Specifically, B^1 is the space of analytic functions f on $\mathbb{D}$ that can be written as

$$f(z) = \sum_{n=1}^{\infty} \lambda_n \sigma_{a_n}(z),$$

for two sequences $\{\lambda_n\}$ in ℓ_1 and $\{a_n\}$ in $\mathbb{D}$. It is a Banach space with the norm

$$\|f\|_{B^1} = \inf\left\{\sum_{n=1}^{\infty} |\lambda_n| : f(z) = \sum_{n=1}^{\infty} \lambda_n \sigma_{a_n}(z)\right\}.$$

It can also be shown that $f \in B^1$ if and only if $\|f''\|_{A^1} < \infty$, see [9, Thm. 8].

The Besov spaces are important examples of conformally invariant spaces, since it was proved in [9] that B^1 is minimal among the "natural" conformally invariant spaces. These spaces are also important in view of their relationship with the Bergman projection, since they arise as the images under the Bergman projection of Lebesgue spaces of the disk with respect to the invariant measure $\frac{dA(z)}{(1-|z|^2)^2}$, and the Hankel operators on Bergman spaces.

This family of spaces does not satisfy the typical chain of inclusions that the L^p spaces with finite measure satisfy since, $\int_{\mathbb{D}} \frac{dA(z)}{(1-|z|^2)^2} = \infty$, but it can be proved that the following inclusions hold

$$B^p \subset B^q \quad \text{if } 1 \leq p < q < \infty.$$

The pointwise growth of the functions in the analytic Besov space is also known (see [132]).

Proposition 1.7. *If $f \in B^p$, $1 < p < \infty$, and $z \in \mathbb{D}$, then*

$$|f(z) - f(0)| \leq C \|f\|_{B^p} \left(\log \frac{2}{1-|z|^2}\right)^{1-\frac{1}{p}}.$$

In [76], the authors prove the following estimate.

Theorem 1.8. *If $f \in B^p$, $1 < p < \infty$, then*

$$|f(z)| = o\left(\log \frac{1}{1-|z|}\right)^{1-\frac{1}{p}},$$

as $|z| \to 1$.

1.6 Bloch Space

The *Bloch space* $\mathcal{B}$ is the space of analytic functions on $\mathbb{D}$ such that $(1-|z|^2)|f'(z)|$ is bounded in $\mathbb{D}$. We can define a seminorm in this space as $\rho_{\mathcal{B}}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$. The norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \rho_{\mathcal{B}}(f)$$

makes the Bloch space into a Banach space.

The quantity $(1-|z|^2)|f'(z)|$ was first explicitly studied in [106] by Seidel and Walsh while studying the radius $d_f(z)$ of the largest disk that lies on a single sheet of the Riemann image of f and is centered at the point $f(z)$, whenever z is not a branch point, that is, if $f'(z) \neq 0$. They showed that

$$d_f(z) \leq (1-|z|^2)|f'(z)|.$$

The name of the Bloch function comes from the Bloch constant, that is

$$\frac{1}{B} = \sup |f'(0)|,$$

where the supremum is taken over all the analytic functions such that $f(0) = 0$ and $d_f(z) \leq 1$. Bloch proved in 1925 ([31]) that $B \geq 1/72$, but the best possible bound is still unknown, in spite of the efforts of authors such as Landau, Ahlfors, and Grunsky. In 1970 ([96]) Pommerenke used the term "Bloch functions" for the first time, and in 1974, in [6], the Bloch space was probably defined for the first time.

In [101] the authors prove that the Bloch space is the biggest "natural" conformally invariant space.

It can be proved that any bounded function belongs to the Bloch space, while the function $f(z) = \log(1-z)$, with the branch of the logarithm appropriately chosen, shows that $\mathcal{B} \not\subset H^\infty$. Moreover, $\mathcal{B} \not\subset H^p$ for any $0 < p \leq \infty$ since we can find Bloch functions with no boundary limits. Nevertheless, $\mathcal{B} \subset A^p$ for any p, in view of the logarithmic growth of Bloch functions. The Bloch space is strongly related to the Bergman spaces since the dual space of A^1 can be identified with the Bloch space (see, for instance, [59, page 48]).

Once again we find that the point evaluation functionals are bounded on $\mathcal{B}$.

Proposition 1.9. *If $f \in \mathcal{B}$, then for any $z \in \mathbb{D}$*

$$|f(z) - f(0)| \leq \frac{1}{2}\,\rho_{\mathcal{B}}(f)\,\log\frac{1+|z|}{1-|z|}.$$

This result cannot be improved to a "little-oh" estimate, since the function $f(z) = \log(1-z)$, satisfies

$$|f(z)| \neq o\left(\log\frac{1+|z|}{1-|z|}\right).$$

This behavior seems linked to the fact that the Bloch seminorm is defined as a "big-Oh" condition on the growth of the derivative of f. Related to the Bloch space is the *little Bloch space* $\mathcal{B}_0$, the closed subspace of the functions f in $\mathcal{B}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

The Bloch and little Bloch spaces are different in several ways. For instance, the little Bloch space does not contain every bounded function. Nevertheless, the functions in the little Bloch space are characterized by the fact that can be approximated by their dilations.

Theorem 1.10. *Let $f \in \mathcal{B}$. For $r \in (0,1)$, let $f_r(z) = f(rz)$, $|z| < 1$. Then $f \in \mathcal{B}_0$ if and only if $\|f_r - f\|_{\mathcal{B}} \to 0$ as $r \to 1^-$.*

The previous result shows also that, unlike the Bloch space, the little Bloch space is separable.

Corollary 1.11. *The space $\mathcal{B}_0$ is the closure of polynomials in $\mathcal{B}$.*

The non-separability of the Bloch space can be proved using the family of functions $f_\lambda(z) = \frac{\lambda}{2}\log\left(\frac{1+\lambda z}{1-\lambda z}\right)$, $z \in \mathbb{D}$, with $|\lambda| = 1$. It is easy to see that the distance between two functions of the family is

$$\|f_\lambda - f_\mu\|_{\mathcal{B}} \geq 1.$$

1.7 Weighted Banach Spaces H_v^∞

In this specific context, a function $v : \mathbb{D} \to \mathbb{R}_+$ will be called a *weight* if it is a bounded continuous positive function. The *weighted Banach spaces* with weight v are the spaces

$$H_v^\infty = \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}$$

and

$$H_v^0 = \{f \in H_v^\infty : \lim_{|z|\to 1} v(z)|f(z)| = 0\}.$$

These spaces appear naturally in the study of the growth of analytic functions, see, for instance, [100], [110], [111], [7]. They are Banach with respect to the norm

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)|.$$

If $\limsup_{|z|\to 1} v(z) > 0$ we have that $H_v^\infty = H^\infty$ and $H_v^0 = \{0\}$. Therefore, we will only be interested in what it is called a *typical weight*, that is, a weight with $\lim_{|z|\to 1} v(z) = 0$. The spaces induced by these typical weights satisfy that $(H_v^0)^{**} = H_v^\infty$, and polynomials are dense in H_v^0.

To each weight (radial or not) there is an associated growth condition. Let

$$B_v = \left\{ f \in H(\mathbb{D}) : |f| \leq \frac{1}{v} \text{ in } \mathbb{D} \right\}$$

be the unit ball of H_v^∞. Then the *associated weight* $\tilde{v} : \mathbb{D} \to \mathbb{R}_+$ is defined by

$$\tilde{v}(z) = \frac{1}{\sup\{|f(z)| : f \in B_v\}},$$

$z \in \mathbb{D}$. In other words, $\tilde{v}$ is the inverse of the norm of the point evaluation functional in H_v^∞. In [27] the authors prove the following result,

Proposition 1.12. *For each $z \in \mathbb{D}$ there exists a $f_z \in B_v$ such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$.*

Unlike the Hardy and Bergman spaces, the estimate

$$\frac{|f(z)|}{\|\Lambda_z\|} \to 0$$

as $|z| \to 1$ does not hold for every $f \in H_v^\infty$, since the function $f_z \in B_v$ such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$ satisfies

$$\frac{|f_z(z)|}{\|\Lambda_z\|} = |f_z(z)|\tilde{v}(z) \geq 1.$$

Nevertheless, it is true for functions in the "little-oh" space H_v^0, just like in the Bloch spaces. Moreover, we also have the following result, similar to the behavior of Bloch functions.

Theorem 1.13. *Let $f \in H_v^\infty$. For $r \in (0,1)$, let $f_r(z) = f(rz)$, $|z| < 1$. Then $f \in H_v^0$ if and only if $\|f_r - f\|_{H_v^\infty} \to 0$ as $r \to 1^-$.*

Therefore, H_v^0 is the closure of polynomials in H_v^∞.

1.8 Caratheodory's Class $\mathcal{P}$

Carathéodory's class $\mathcal{P}$ is the class of analytic functions in the unit disk with positive real part and normalized in such way that $f(0) = 1$. It was studied for the first time in [43] by Carathéodory. Unlike the spaces defined above, $\mathcal{P}$ has no linear structure. Nevertheless, the transformations that preserve this class have been useful in the proofs of several theorems (see [58, Chapter 2]).

Every function f in the class $\mathcal{P}$ can be written as a Poisson-Stieltjes integral

$$f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),$$

where $d\mu \geq 0$ and $\int d\mu(t) = 1$, thanks to the Herglotz Representation Theorem [58, Chapter 1]. An important example of a function in this class is the so-called *half-plane mapping* ℓ given by

$$\ell(z) = \frac{1+z}{1-z}, \qquad z \in \mathbb{D},$$

the conformal map of the disk onto the right half-plane. Actually, via the subordination principle we have that every function f in $\mathcal{P}$ is of the form $\ell \circ \omega$, where ω is some Schwarz-type function (that is, $|\omega(z)| \leq 1$ for every $z \in \mathbb{D}$ and $\omega(0) = 0$). Since every Schwarz-type function has radial limits almost everywhere on the unit circle $\mathbb{T}$ with respect to the normalized arc length measure $dm = d\theta/(2\pi)$, so does every f in $\mathcal{P}$. In the particular case when $\omega(z) = \lambda z$ with $|\lambda| = 1$, we will use the symbol ℓ_λ to denote the functions $\ell \circ \omega$, that is, $\ell_\lambda(z) = (1 + \lambda z)/(1 - \lambda z)$. The Herglotz Representation Theorem can be used to prove that the class $\mathcal{P}$ equals $\overline{\mathrm{co}}\,(\mathcal{L})$, the closed convex hull of the collection $\mathcal{L} = \{\ell_\lambda : |\lambda| = 1\}$ in the topology of uniform convergence on compact subsets of $\mathbb{D}$.

As a consequence of the subordination principle, we have the growth theorem for the functions in the class: for $z \in \mathbb{D}$

$$\ell(-|z|) = \frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|} = \ell(|z|)\,. \tag{1.4}$$

Carathéodory's class is closely related to the class S of normalized univalent (injective and analytic) functions, see for instance [58, Chapter 2] for results linking the subclasses of starlike and convex functions to the class $\mathcal{P}$. Part of the theory related to the celebrated Bieberbach's conjecture is linked to this class.

The following lemma, due to Carathéodory and proved around 1911, allows us to control the behavior of the coefficients of the functions in the class $\mathcal{P}$.

Lemma 1.14. *If* $f \in \mathcal{P}$ *and*

$$f(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

then $|c_n| \leq 2$, $n = 1, 2, \ldots$ *This inequality is sharp for each* n.

Chapter 2

Operators and transformations on spaces and classes of analytic functions

In this second introductory chapter we define some concepts related to operators that we will need later. The chapter begins with a review on boundedness of composition operators and the geometric theory that we need in the study of such operators. The second part is devoted to the pointwise multipliers and their boundedness on classical spaces of analytic functions. The review of characterizations of boundedness of weighted composition operators on several spaces of analytic functions is the third part. In a short fourth subsection we define the integral operators, related with the semigroups of operators, that form the final part of the chapter.

This chapter does not intend to give a complete history of the theory of weighted composition operators, but just to present some results on boundedness of the different operators on the spaces defined in Chapter 1.

2.1 Composition operators and geometric function theory

2.1.1 Composition operators

For a self-map of the disk φ, that is, an analytic function on the unit disk such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, we define the *composition operator* with symbol φ, $\mathbf{C}_\varphi$, as $\mathbf{C}_\varphi f = f \circ \varphi$, with f analytic on the unit disk. Since compositions of analytic functions are still analytic, it is clearly a transformation from the algebra of analytic functions to itself. Moreover, it is a linear operator.

One of the first results on the boundedness of composition operators is Littlewood's Subordination Theorem, of 1925 (see [85] or [57, Section 1.5]).

Theorem 2.1. *Let $f, \omega \in H(\mathbb{D})$ with $\omega(\mathbb{D}) \subset \mathbb{D}$ and $\omega(0) = 0$. Then, for every $p \in (0, \infty]$ and for every $r \in (0, 1)$,*

$$M_p(r, f \circ \omega) \leq M_p(r, f).$$

This theorem is key to prove that every self map of the disk (not only the ones that fix the origin) induces a bounded composition operator on the Hardy spaces.

Corollary 2.2. *If $f \in H^p$, $0 < p < \infty$, and $\varphi \in H(\mathbb{D})$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$, then*

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\frac{1}{p}} \|f\|_{H^p} \leq \|f \circ \varphi\|_{H^p} \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{1}{p}} \|f\|_{H^p}.$$

The proof of both results is based on the subharmonicity of $|f|^p$. Littlewood's Subordination Theorem can also be used to prove an analogous upper bound for the norm of the composition operator on Bergman spaces, while the lower bound was proved in [127].

Theorem 2.3. *If $f \in A^p_\alpha$, $0 < p < \infty$, and $\varphi \in H(\mathbb{D})$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$, then*

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\frac{\alpha+2}{p}} \|f\|_{A^p_\alpha} \leq \|f \circ \varphi\|_{A^p_\alpha} \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{\alpha+2}{p}} \|f\|_{A^p_\alpha}.$$

The boundedness of every composition operator on the Bloch space is a consequence of the Schwarz-Pick Lemma. On the other hand, we have already seen that $H^\infty \not\subset \mathcal{D}$, therefore, since the identity function belongs to the Dirichlet space, there are self-maps of the disk that do not induce bounded composition operators on $\mathcal{D}$. The characterization of the self-maps of the disk φ such that $\mathbf{C}_\varphi$ is bounded on $\mathcal{D}$ was proved in [9], via Carleson measures. Such measures were defined by Lennart Carleson to characterize the interpolating sequences of H^∞, and to prove the Corona Theorem (see [57], [68]). A Borel measure μ in a normed space X of analytic functions in $\mathbb{D}$ is a (X,p)-*Carleson measure* if and only if

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \|f\|^p_X,$$

for every function $f \in X$. In other words, the embedding

$$I : X \to L^p(\mu)$$

is continuous. There are different characterizations of the Carleson measures in terms of test functions or comparison with the Lebesgue measure.

The Carleson measures have become a standard technique in the study of the boundedness of linear operators, in particular of the composition operators. To study the compactness, many authors have used the *vanishing Carleson measures*, that is, the Borel measures that make the embedding above a compact operator.

In the case of the Dirichlet space, the characterization of the boundedness of the composition operator is the following.

Theorem 2.4. *Let $\varphi \in H(\mathbb{D})$. The composition operator $\mathbf{C}_\varphi$ is bounded on $\mathcal{D}$ if and only if the measure $d\mu(w) = n_\varphi(w)dA(w)$ is an $(A^2,2)$-Carleson measure, where $n_\varphi(w)$ is the number of zeroes of the function $\varphi - w$, $w \in \mathbb{D}$.*

On the Besov spaces the characterization is similar to what we have on the Dirichlet space. It can be found in [121].

Theorem 2.5. *Let φ be a holomorphic function on $\mathbb{D}$. Then $\boldsymbol{C}_\varphi$ is a bounded operator on B_p $(1 < p < \infty)$ if and only if $N_p(w,\varphi)dA(w)$ is a (B_p, p)-Carleson measure, where*

$$N_p(w,\varphi) = \sum_{\varphi(z)=w} \left(|\varphi'(z)|(1-|z|^2)\right)^{p-2}$$

is the counting function for the $p-$Besov space.

The weighted Banach spaces are another example of a family of spaces in which not every admissible symbol induces a bounded composition operator. The following characterization can be found, in a more general form, in [32].

Theorem 2.6. *Let v be a weight and φ a self-map of the disk. Then the operator $\boldsymbol{C}_\varphi$ is bounded on H_v^∞ if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\tilde{v}(z)}{\tilde{v}(\varphi(z))} < \infty \tag{2.1}$$

If v satisfies $\lim_{|z|\to 1^-} v(z) = 0$, then 2.1 is equivalent to the operator $\boldsymbol{C}_\varphi$ being bounded on H_v^0.

Recall that $\tilde{v}$ is the inverse of the norm of the point evaluation functional on H_v^∞. An example of an unbounded composition operator is also given in [32]. If $v(z) = e^{-\frac{1}{1-|z|}}$ and $\varphi(z) = \frac{1+z}{2}$, $z \in \mathbb{D}$, then $\tilde{v}(z) = v(z)$ and

$$\frac{\tilde{v}(r)}{\tilde{v}(\varphi(r))} = e^{\frac{1}{1-r}} \to \infty$$

as $r \to 1$, and therefore the composition operator $\mathbf{C}_\varphi$ is not bounded. Actually, in this space the automorphisms of the disk do not induce bounded composition operators in general. For example, taking $\varphi(z) = \sigma_{\frac{1}{2}}$, we get

$$\frac{\tilde{v}(r)}{\tilde{v}\left(\sigma_{\frac{1}{2}}(r)\right)} = \frac{e^{-\frac{1}{1-r}}}{e^{-\frac{1}{1-\frac{1-2r}{2-r}}}} = e^{\frac{1-4r+r^2}{1-r^2}} \to \infty$$

as $r \to 1$, and therefore the composition operator $\mathbf{C}_{\sigma_{\frac{1}{2}}}$ is not bounded on H_v^∞.

Another family of spaces of analytic functions for which not every automorphism of the disk induces a bounded composition operator comes from the weighted Hilbert spaces. For instance, let X be the weighted Hilbert space of analytic functions $f(z) = \sum_{n=0}^\infty a_n z^n$, $z \in \mathbb{D}$, such that

$$\sum_{n=0}^\infty 4^n |a_n|^2 < \infty.$$

Since every function in X is analytic in the disk of radius 2, there are automorphisms of the disk that do not belong to X and therefore do not induce bounded composition operators on X. Nevertheless, for the family of weighted Hilbert spaces given by the sequences $\{(n+1)^\gamma\}$, $n \geq 0$, $\gamma < 2$, which are exactly the weighted Dirichlet spaces $\mathcal{D}_{1-\gamma}$, every composition operator whose symbol is an automorphism of the disk is bounded, as the following theorem proves (see [50, Theorem 3.5]).

Theorem 2.7. *Let ν be a positive function on the unit interval with $\int_{\mathbb{D}} \nu(1-|z|^2)\, dA(z) < \infty$ such that, for each $q > 1$, there is a constant $\kappa = \kappa(q)$ satisfying*

$$\nu(s) \leq \kappa\nu(t) \quad \textit{whenever } s \leq qt.$$

For $1 \leq p < \infty$, suppose X is the Banach space of all analytic functions on the unit disk for which the norm given by

$$\|f\|^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \nu(1-|z|^2)\, \frac{dA(z)}{\pi}$$

is finite. If φ is an automorphism of the unit disk, then $\boldsymbol{C}_\varphi$ is a bounded operator on X.

We just need to apply this theorem with $p = 2$ and $\nu(t) = t^{1-\gamma}$.

The theory of composition operators is extremely rich in content, with many papers produced in the area. More information about boundedness, compactness, spectra, and other topics in theory of composition operators, can be found in the monographs [50] and [108].

2.1.2 Geometric function theory

The theory of composition operators has made use of developments in geometric function theory. For instance, the compactness of the composition operator in several spaces is given in terms of the angular derivative of the symbol, and the dynamics and the spectra of the operator are related to the iteration theory.

Angular derivative

An analytic self-map φ of $\mathbb{D}$ is said to have an *angular derivative* $\varphi'(\zeta)$ (in the restricted sense of Carathéodory [44, § 298-299]) at a point ζ on the unit circle $\mathbb{T}$ if it satisfies the following two conditions:

(a) the non-tangential limit of φ at ζ has modulus one, and

(b) $\varphi'(z)$ has a finite non tangential limit as $z \to \zeta$.

The following theorem, due to Julia and Carathéodory, gives an analytic characterization of existence of the angular derivative (see [50] or [108]).

Theorem 2.8 (Julia-Carathéodory)**.** *Suppose φ is a holomorphic self-map of the disk, and $\zeta \in \mathbb{T}$. Then the following three statements are equivalent:*

(1) $\liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} = \delta < \infty$,

(2) $\angle\lim_{z\to\zeta}\frac{\eta-\varphi(z)}{\zeta-z}$ *exists for some* $\eta\in\mathbb{T}$,

(3) $\angle\lim_{z\to\zeta}\varphi'(z)$ *exists, and* $\angle\lim_{z\to\zeta}\varphi(z)=\eta\in\mathbb{T}$.

Moreover,

1. $\delta>0$,
2. *the boundary points* η *in (2) and (3) are the same,*
3. *the limit of the difference quotient coincides with that of the derivative, with both equal to* $\overline{\zeta}\eta\delta$.

It can also be proved that if φ has angular derivative at a point ζ, then φ is univalent in some Stolz angle with vertex at ζ (see [97, p. 291]). Moreover, φ is conformal and therefore the angles between curves contained in the Stolz angle are preserved.

Examples of functions with angular derivatives are the self-maps of the unit disk that map the disk onto a horodisk, that is, a disk that is tangent to the boundary of $\mathbb{D}$, for instance the function $\varphi(z)=\frac{1+z}{2}$, $z\in\mathbb{D}$, that maps the unit disk to the disk centered at $1/2$ and tangent to $\mathbb{T}$ at the point 1. Examples of functions that have non-tangential limit of modulus one but do not have angular derivatives are the lens maps. For $0<\alpha<1$ we will denote by λ_α the standard *lens map* given by the formula

$$\lambda_\alpha(z)=(\ell^{-1}\circ\ell^\alpha)(z)=\frac{\ell^\alpha(z)-1}{\ell^\alpha(z)+1}, \qquad z\in\mathbb{D},$$

where ℓ is the half-plane mapping.

The lens λ_α is a conformal map of the unit disk onto a lens-shaped region L_α bounded by two circular arcs (symmetric with respect to the real axis) that intersect at the points ±1, forming an angle of opening $\pi\alpha$ at each of these points; see [108, p. 27].

The half-plane map ℓ is bijective between a lens-shaped region L_α and an angle with vertex at the origin and maps in a one-to-one fashion the largest disk contained in the lens-shaped region onto a disk tangent to the legs of the angle.

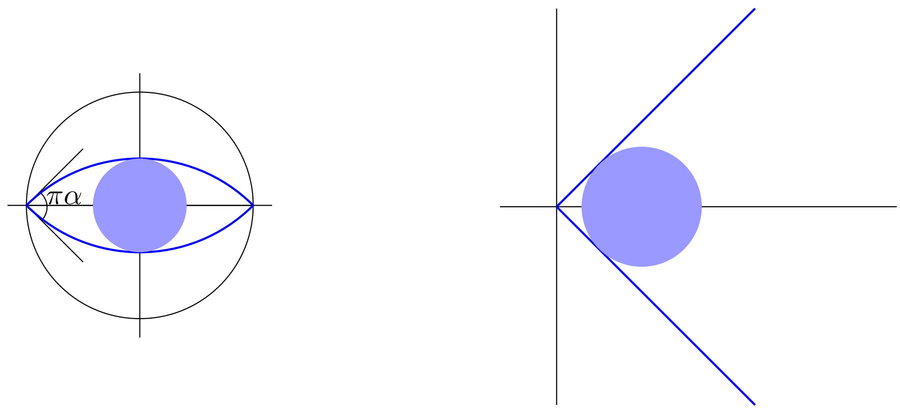

Figure 2.1: Images of L_α, the angle, and the tangent disks

Another characterization of the existence of the angular derivative of a function is given, in terms of the boundary of the image of φ, by the next theorem (see [108, p. 72]).

Theorem 2.9 (Tsuji-Warschawski). *Suppose Ω is a Jordan subdomain of $\mathbb{D}$ whose boundary curve in a neighborhood of 1 has polar equation $1 - r = \gamma(|\theta|)$, where $\gamma : [0, \zeta] \to [0, 1]$ is a continuous, increasing function with $\gamma(0) = 0$. Let φ be a univalent map of $\mathbb{D}$ onto Ω, with $\varphi(1) = 1$. Then φ has an angular derivative at 1 if and only if*

$$\int_0^\zeta \frac{\gamma(\theta)}{\theta^2} d\theta < \infty.$$

Even though the angular derivative of φ need not exist anywhere on $\mathbb{T}$ as a finite number, the function $|\varphi'| : \mathbb{T} \to [0, \infty]$ is well defined in this extended sense; being lower semicontinuous ([38], Lemma 2.5), it attains its minimum on $\mathbb{T}$ (*cf.* also [50, Proposition 2.46]).

One ought to keep in mind that the function $|\varphi'|$ on $\mathbb{T}$ as above, in general, does not coincide at all with the modulus of the boundary values of φ' (if those exist). The most obvious example is the linear map $\varphi(z) = az + b$ onto a disk compactly contained in $\mathbb{D}$, which happens precisely when $|a| + |b| < 1$. Its usual derivative is constant everywhere, while the angular derivative does not exist at any point on the boundary; in this case, we interpret that $|\varphi'(\zeta)| = \infty$ for every point ζ on the unit circle.

The concept of angular derivative is fundamental in the study of compactness of composition operators on Hardy and Bergman spaces, as well as in the iteration of analytic self-maps of the unit disk.

Iteration Theory

If φ is a holomorphic self-map of $\mathbb{D}$ and n is a positive integer, then the n-th iterate of φ is the n-fold composition of φ with itself, denoted by

$$\varphi_n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}},$$

with φ_0 the identity. The iterates of a self-map of the disk are related to the dynamics of the composition operator that it induces, since

$$\mathbf{C}_\varphi^n = \mathbf{C}_{\varphi_n}.$$

The behavior of the family $\{\varphi_n\}$ as $n \to \infty$ is determined by its fixed points, as the following theorem shows, see [108, Chapter 5].

Theorem 2.10 (Denjoy-Wolff). *Suppose φ is a self-map of the disk that is not an automorphism with a fixed point in $\mathbb{D}$. Then there is a point b in the closed unit disk such that $\varphi_n \to b$ uniformly on compact subsets of $\mathbb{D}$.*

Such a point b is called the *Denjoy-Wolff point* of φ. If $b \in \mathbb{D}$, then it is a fixed point of φ, while if $b \in \partial\mathbb{D}$ then it is a boundary fixed point in the sense that

$$\lim_{r\to 1} \varphi(rb) = b.$$

If φ is an elliptic automorphism of the disk, then the iterates never converge to the fixed point (think of $b = 0$ and φ a rotation).

2.2 Pointwise Multipliers

Let X and Y be two Banach spaces of analytic functions on $\mathbb{D}$. An analytic function ϕ is a *pointwise multiplier* from X to Y if $\phi X \subset Y$; that is, if $\phi f \in Y$ for any $f \in X$. We denote by $\mathbf{M}(X, Y)$ the space of all multipliers from X to Y, and by $\mathbf{M}(X)$ when $Y = X$. Given $\phi \in \mathbf{M}(X, Y)$, we define the *pointwise multiplication operator with symbol* ϕ, $\mathbf{M}_\phi : X \to Y$, as $\mathbf{M}_\phi f = \phi f$.

Using the boundedness of the point evaluation functionals we can prove a necessary condition for a function to multiply a Banach space to itself, see [3].

Proposition 2.11. *Let X be a non-trivial Banach space of analytic functions on the unit disk where the point evaluation functionals are bounded. If $\phi \in \boldsymbol{M}(X)$, then $\phi \in H^\infty$ and $\|\phi\|_\infty \leq \|\boldsymbol{M}_\phi\|$.*

Therefore, if a function induces a bounded multiplier from such Banach space into itself, the symbol must be bounded. There are several spaces in which this property is also sufficient, like the Hardy, Bergman, and weighted Banach spaces. Nevertheless, there are bounded functions that do not multiply the Bloch space into itself. The following theorem, from [40], characterizes the space $\mathbf{M}(\mathcal{B})$.

Theorem 2.12. *The function ϕ is a multiplier from $\mathcal{B}$ into itself if and only if $\phi \in H^\infty$ and*

$$|\phi'(z)| = O\left(\frac{1}{(1-|z|)\log(1/1-|z|)}\right), \ |z| \to 1. \tag{2.2}$$

Therefore, ϕ is not only bounded, but also belongs to a smaller space called the *logarithmic Bloch space*. In the Dirichlet space the space of multipliers is also strictly contained in the space of bounded functions, as the following theorem of Stegenga, from [120], shows.

Theorem 2.13. *An analytic function ϕ is a multiplier from $\mathcal{D}$ to itself if and only if ϕ is bounded and there exists a constant A such that*

$$\iint_{\cup S(I_j)} |\phi'|^2 \, dx \, dy \leq A \, \mathrm{Cap}\left(\bigcup_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{I_j}\right),$$

for any finite disjoint collection of arcs $I_1, I_2, \ldots, I_n$ of the unit circle, where Cap *is the logarithmic capacity and $S(I_j)$ is the Carleson square of I_j, $j = 1, 2, \ldots, n$.*

This result is based on the logarithmic capacity (see [69, Chapter III] for more information on Potential Theory) and on Carleson measures of $\mathcal{D}$. The multipliers of Besov spaces can be characterized using Carleson measures too, but in [135] and [67] the authors find a more manageable result. Here $\Delta(w,r)$ denotes the pseudo-hyperbolic disk of center w and radius r,

$$\Delta(w,r) = \left\{ z \in \mathbb{D} \colon \left| \frac{z-w}{1-\overline{w}z} \right| < r \right\}.$$

Theorem 2.14.
(a) If $\phi \in H^\infty \cap \boldsymbol{M}(B^p)$, $1 < p < \infty$ *and* $0 < r < 1$, *then*

$$\sup_{w\in\mathbb{D}} \int\limits_{\Delta(w,r)} (1-|z|^2)^{p-2} |\phi'(z)|^p \left(\log \frac{2}{1-|z|^2} \right)^{p-1} dA(z) < \infty.$$

(b) If $1 < p < \infty$, $\phi \in H^\infty$ *and*

$$\int\limits_{\mathbb{D}} (1-|z|^2)^{p-2} |\phi'(z)|^p \left(\log \frac{2}{1-|z|^2} \right)^{p-1} dA(z) < \infty,$$

then $\phi \in \boldsymbol{M}(B^p)$.

In the last chapter we will see how the space of multipliers depends on the form of the norm of X.

2.3 Weighted Composition Operators

A natural generalization of both the composition and the multiplication operators is the weighted composition operators. For an analytic function F and an analytic self-map φ the *weighted composition operator* with symbols F and φ, $\mathbf{T}_{F,\varphi}$ is the operator

$$\mathbf{T}_{F,\varphi} f = F(f \circ \varphi).$$

Therefore, $\mathbf{T}_{F,\varphi} = \mathbf{M}_F \mathbf{C}_\varphi$. Clearly, if both the multiplication and the composition operator are bounded then the weighted composition operator is bounded too, but the converse is not true. We will later see in the last chapter an example of a bounded weighted composition operator whose multiplication and composition components are both unbounded operators.

We can find bounded weighted composition operators where the multiplier symbol is not a bounded function.

As noted in the introduction, the weighted composition operators are related to many other operators in spaces of analytic functions, such as the isometries of the Hardy and Bergman spaces, the Hilbert matrix, and the composition operators on the Hardy space of the half-plane, and are in the basis of an interesting new technique in the study of Brennan's conjecture. Their applications and the fact that they are a natural generalization to composition operators and pointwise multipliers have led to the beginning of the theory of weighted composition operator and the study of classical concepts in operator theory such as its boundedness, compactness, invertibility, spectra,...

In 2001, Contreras and Hernández–Díaz studied in [49] the boundedness, compactness, weak compactness, and complete continuity of weighted composition operators on Hardy spaces H^p, $1 \leq p < \infty$. The theorem that characterizes the boundedness of the weighted composition operators is the following.

Theorem 2.15. *Let $F \in H^p$ and $\varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and*

$$\mu_{\varphi,F,p}(E) = \int_{\varphi^{-1}(E)\cap\mathbb{T}} |F|^p dm,$$

where E is a measurable subset of the closed unit disk. Then the weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is bounded on H^p, $1 \leq p < \infty$, if and only if $\mu_{\varphi,F,p}$ is a (H^p, p)-Carleson measure on $\overline{\mathbb{D}}$.

In 2004, in [51], Čučkovic and Zhao characterized the boundedness of the weighted composition operators on Bergman spaces in terms of the φ-Berezin transform, that for a function $v \in L^1(\mathbb{D})$ is defined as

$$B_\varphi v(a) = \int_{\mathbb{D}} \frac{(1-|a|^2)^2 v(z)}{|1-\overline{a}\varphi(z)|^4} dA(z).$$

Theorem 2.16. *Let $F, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is bounded on A^2 if and only if $B_\varphi(|F|^2) \in L^\infty(\mathbb{D})$.*

They generalized this result to different Bergman spaces using a generalized Berezin transform in 2007 in [52]. The boundedness of the weighted composition operators on the Dirichlet and Besov spaces was also given in terms of the Berezin transform by Kumar and Singh [84].

Theorem 2.17. *Let $F, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, and suppose that the measure ν, with*

$$\nu(E) = \int_{\varphi^{-1}(E)} |F'(z)|^p (1-|z|^2)^{p-2} dA(z),$$

where E is a measurable subset of the unit disk, is a vanishing (B^p, p)-Carleson measure. Then $\boldsymbol{T}_{F,\varphi}$ is a bounded operator on B^p if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}w|} \right)^p d\mu(w) < \infty,$$

where

$$\mu = \int_{\varphi^{-1}(E)} |F(z)\varphi'(z)|^p (1-|z|^2)^{p-2} dA(z).$$

The boundedness of the weighted composition operator on Bloch spaces was given by Ohno and Zhao in [94].

Theorem 2.18. *Let $F, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is bounded on $\mathcal{B}$ if and only if the following conditions are satisfied:*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|F'(z)| \log \frac{2}{1-|\varphi(z)|^2} < \infty, \text{ and}$$

$$\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2} |F(z)\varphi'(z)| < \infty.$$

In the weighted Banach spaces H_v^∞ the boundedness was characterized also by Contreras and Hernández–Díaz, in [48].

Theorem 2.19. *Let v be a weight and $F, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is bounded on H_v^∞ if and only if*

$$\sup_{z\in\mathbb{D}} |F(z)| \frac{\tilde{v}(z)}{\tilde{v}(\varphi(z))} < \infty.$$

There is a vast literature on weighted composition operators, regarding not only boundedness of the operator between different spaces of analytic functions, but also compactness, weak compactness, essential norm, dynamics... Among the numerous more recent references, we mention [15], [33], [34], [60] and [47].

2.4 Integral operators

Let g be an analytic function on the unit disk and V_g the *integral operator* or *generalized Volterra operator* induced by it, namely,

$$V_g(f)(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta, \quad f \in \mathcal{H}(\mathbb{D})$$

for any $z \in \mathbb{D}$. In the case $g(z) = z$, the integral operator V_g is the classical *Volterra operator*, V, that is,

$$V(f)(z) = \int_0^z f(\zeta)\, d\zeta, \quad f \in \mathcal{H}(\mathbb{D}).$$

This operator was first considered by Pommerenke on the Hardy space H^2 (see [98]), while studying the properties that an analytic function f must satisfy so its exponential e^f belongs to the Hardy space. In the 90's, Aleman and Siskakis developed the theory on Hardy and Bergman spaces. See for example the papers [4], [5]. The following theorem, from [5], gives information on the symbol of bounded and compact integral operators in a general setting.

Theorem 2.20. *Let X a Banach space of analytic functions such that*

1. *the point evaluation functionals Λ_z are bounded in X for every $z \in \mathbb{D}$,*

2. *for every* $\lambda \in \mathbb{T}$ *the rotation operator* $U_\lambda f(z) = f(\lambda z)$, $z \in \mathbb{D}$, *is bounded and* $\sup_{\lambda\in\mathbb{T}} \|U_\lambda\|_X < \infty$, *and*

3. *for some* $s \in (0,1)$ *the composition operator* $\boldsymbol{C}_\psi f = f \circ \psi_s$ *with* $\psi_s(z) = sz+1-s$, $z \in \mathbb{D}$, *is bounded on* X.

Let g *be an analytic function on the unit disk.*

(a) *If* $V_g : X \to X$ *is bounded then* $g \in \mathcal{B}$ *and there is a constant* C *(which does not depend on* g*) such that* $\|g\|_{\mathcal{B}} \le C\|V_g\|$.

(b) *Moreover, if the multiplication operator* $M_z f(z) = zf(z)$ *is bounded on* X *and, for some fixed* $t \in (0,1)$, *it is satisfied that* $\lim_{n\to\infty} \|(tz+1-t)^n f(z)\| = 0$ *for all* $f \in X$, *then the compactness of* V_g *on* X *implies* $g \in \mathcal{B}_0$.

Much more information on integral operators, including a review of the results on Hardy and Bergman spaces and the motivation related to the Cesàro operators, can be found in Aleman's lecture notes [2]. In the next section we will present the relation between integral operators and semigroups of composition operators.

2.5 Semigroups of operators and functions

2.5.1 Semigroups of bounded operators

Let X be a Banach space of analytic functions on the unit disk $\mathbb{D}$. A family $\{T(t)\}_{t\ge 0}$ of bounded operators on X forms a *semigroup* if it satisfies the following

1. $T(0) = I$, where I is the identity in the space of bounded operators on X,

2. $T(t+s) = T(t) \circ T(s)$, for all $t, s \ge 0$.

Such a family of bounded operators $\{T(t)\}$ is called *strongly continuous* or C_0 if for every $f \in X$

$$\lim_{t\to 0^+} \|T(t)f - f\|_X = 0,$$

and *uniformly continuous* if

$$\lim_{t\to 0^+} \|T(t) - I\|_X = 0.$$

Thanks to the strong continuity the semigroup of bounded operators satisfies

$$\|T(t)\| \le Me^{\beta t}$$

for $0 \le t < \infty$ and for some constants $M > 0$ and $\beta < \infty$. Therefore, the semigroup $\{e^{-\beta t}T(t)\}$ is an equibounded strongly continuous semigroup,

$$\|e^{-\beta t}T(t)\| \le M$$

for $0 \leq t < \infty$. The strong continuity is also a natural condition for the applications in physical systems such as the heat equation, to avoid breakdown in time due to small measurement errors in the initial state. Other applications of this theory appear in stochastic processes and integration of the evolution equations, such as diffusion, wave, and Schrödinger equations, see also [129, Chapter IX].

The infinitesimal generator of the semigroup is a linear operator that summarizes the properties of the semigroup and that characterizes it univocally. It is defined as

$$Af = \lim_{t\to 0} \frac{T(t)f - f}{t} = \left.\frac{\partial T(t)f}{\partial t}\right|_{t=0}$$

on the set

$$D(A) = \left\{ f \in X : \lim_{t\to 0} \frac{T(t)f - f}{t} \text{ exists} \right\},$$

called the *domain* of A. Observe that this is a straightforward generalization of the model for semigroups of bounded operators, the case $T(t) = e^{At}$, since

$$T'(t) = Ae^{At} = AT(t).$$

The domain is always dense in X, and it is all of X if and only if the semigroup is uniformly continuous. In that case, the operator A is a bounded operator in X and $T(t) = e^{At}$, recovering the semigroup from the infinitesimal generator. If $T(t)$ is strongly but not uniformly continuous, then it can be proved that A can be approximated by infinitesimal generators A_λ of uniformly continuous semigroups (this is called the *Yosida Approximation*) and therefore

$$T(t) = \lim_{\lambda\to\infty} e^{A_\lambda t},$$

recovering $T(t)$ from A, see [95, Section 1.3].

2.5.2 Semigroups of self-maps of the disk and of composition operators

A family $\{\varphi_t : t \geq 0\}$ of analytic self-maps of the disk $\mathbb{D}$ is a (one-parameter) *semigroup of analytic functions* if it satisfies the following three conditions:

1. φ_0 is the identity in $\mathbb{D}$,

2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,

3. $\varphi_t \to \varphi_0$ as $t \to 0$ uniformly on compact sets of $\mathbb{D}$.

The semigroups of analytic functions can be understood as fractional iterations of φ, thus generalizing the usual discrete iterations φ_n. Some examples of such semigroups are the following:

1. Rotations and dilations: for a constant c with $\operatorname{Re} c \geq 0$, the family

$$\varphi_t(z) = e^{-ct} z$$

is a semigroup. Notice that the images of $\varphi_t(\mathbb{D})$ are disks centered at the origin, and if $\operatorname{Re} c = 0$, then it is a family of rotations.

2. Horodisks: the semigroup

$$\varphi_t(z) = e^{-t} z + 1 - e^{-t}$$

maps the disk to horodisks tangent to the unit circle at 1.

3. Smaller disks: the family

$$\varphi_t(z) = \frac{e^{-t} z}{(e^{-t} - 1)z + 1}$$

is a semigroup in which every self-map of the disk fixes the points 0 and 1. The image of $\mathbb{D}$ is a shrinking tangent disk to $\mathbb{D}$ at 1.

For $t \geq 0$ we can define the composition operators $\mathbf{C}_t$, that is, $\mathbf{C}_t f = f \circ \varphi_t$ for $f \in H(\mathbb{D})$. Let X be a Banach space of analytic functions on the unit disk, then if the operator $\mathbf{C}_t$ is bounded on X, the family $\{\mathbf{C}_t\}$ is an algebraic semigroup of bounded operators on X. The aim of the theory of semigroups of composition operators is to understand operator-theoretic properties, such as strong and uniform continuity, spectrum, ideals or dynamics of the semigroup of composition operators, in terms of geometric function theory.

The study of composition semigroups was started by E. Berkson and H. Porta in 1978. In [25] they found several properties of the semigroups of analytic functions and later applied them to the study of the strong and uniform continuity of semigroups of composition operators in the Hardy spaces. Some of the properties they found are the following.

- If $\{\varphi_t\}$ is a semigroup, then each map φ_t is univalent.
- The *infinitesimal generator* (or simply generator) of $\{\varphi_t\}$ is the function

$$G(z) := \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t}, \; z \in \mathbb{D}.$$

This convergence holds uniformly on compact subsets of $\mathbb{D}$, so $G \in \mathcal{H}(\mathbb{D})$. The generator satisfies

$$G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z) \frac{\partial \varphi_t(z)}{\partial z}$$

and characterizes the semigroup uniquely.

- As a consequence of the Denjoy-Wolff Theorem 2.10, if the semigroup is not formed by automorphisms of the disk with fixed point in $\mathbb{D}$ there exists a point b in the closed disk, called the *Denjoy-Wolff point* of $\{\varphi_t\}$, such that $\varphi_n \to b$ (here we chose the subsequence of $\{\varphi_t\}$ with $t \in \mathbb{N}$). If b is in the interior then it is a fixed point of φ_t, while if b is on the boundary, then it is an attractive fixed point, since
$$\lim_{r\to 1} \varphi_t(rb) = b.$$
- If $\{\varphi_t\}$ is non-trivial, that is, if not every self-map in the semigroup is the identity function, the generator G has a unique representation
$$G(z) = (\bar{b}z - 1)(z - b)P(z),\ z \in \mathbb{D},$$
where $P \in \mathcal{H}(\mathbb{D})$ with $\operatorname{Re} P \geq 0$ in $\mathbb{D}$ and $b \in \overline{\mathbb{D}}$ is the Denjoy-Wolff point of the semigroup. The trivial semigroup $\varphi_t(z) = z$ for every $t \geq 0$ and $z \in \mathbb{D}$ has generator $G \equiv 0$.
- If $\{\varphi_t\}$ is non-trivial, there exists a unique univalent function $h : \mathbb{D} \to \mathbb{C}$, called the *Koenigs function* of $\{\varphi_t\}$ such that:
  - If $b \in \mathbb{D}$ then $h(b) = 0$, $h'(b) = 1$,
  $$h(\varphi_t(z)) = e^{G'(b)t}h(z), \quad z \in \mathbb{D}$$
  for $t \geq 0$, and
  $$h'(z)G(z) = G'(b)h(z), \quad z \in \mathbb{D}.$$
  - If $b \in \mathbb{T}$ then $h(0) = 0$,
  $$h(\varphi_t(z)) = h(z) + t, \quad z \in \mathbb{D}$$
  for $t \geq 0$, and
  $$h'(z)G(z) = 1, \quad z \in \mathbb{D}.$$

In the examples above, we have that:

1. the only fixed point of the semigroup is the origin, and therefore the Denjoy-Wolff point is $b = 0$. The infinitesimal generator is
$$G(z) = -cz,$$
2. the semigroup has a unique fixed point in the boundary, that is also the Denjoy-Wolff point, $b = 1$. The generator is
$$G(z) = 1 - z,$$
3. the semigroup has two fixed points, and the Denjoy-Wolff point is the interior one, $b = 0$. The generator is
$$G(z) = -z(1 - z).$$

With the properties above, Berkson and Porta were able to prove that every semigroup of composition operators is strongly continuous on H^p, $1 \leq p < \infty$, that the infinitesimal generator of $\mathbf{C}_t$ is related with the infinitesimal generator of the semigroup of analytic functions as $Af(z) = G(z)f'(z)$, and that no nontrivial semigroup of analytic functions induces a uniformly continuous semigroup of composition operators on H^p. Similar results were found on Bergman and Dirichlet spaces by Siskakis in [114] and [115]. See also his excelent survey [116]. Nevertheless, on H^∞ the only strongly continuous semigroup of composition operators is the trivial one. This is a consequence of Lotz' Theorems 3.5 and 3.6 in [86] that prove that in a class of spaces that includes H^∞ every strongly continuous semigroup is also uniformly continuous.

Other spaces where not every semigroup of analytic functions induces a strongly continuous semigroup of composition operators are BMOA (see [30] and [29]), the Bloch space (see [29]), and the weighted Banach space (see [22]). In [29] the authors study the strong continuity of semigroups of composition operators on a general Banach space of analytic functions X. They define the *maximal closed linear subspace* of X such that the semigroup $\{\varphi_t\}$ generates a strongly continuous semigroup of operators on it, denoted by $[\varphi_t, X]$, that is,

$$[\varphi_t, X] = \{f \in X : \|f \circ \varphi_t - f\|_X \to 0 \text{ as } t \to 0\}.$$

They are able to characterize this subspace in terms of the generator of the semigroup of analytic functions, as the following theorem shows.

Theorem 2.21. *Let $\{\varphi_t\}$ be a semigroup with generator G and X a Banach space of analytic functions which contains the constant functions and such that* $\sup_{t\in[0,1]} \|\boldsymbol{C}_t\|_X < \infty$. *Then,*

$$[\varphi_t, X] = \overline{\{f \in X : Gf' \in X\}}.$$

Therefore, in order to prove that the semigroup $\{\varphi_t\}$ with generator G induces a strongly continuous semigroup of composition operators on X we need only see that

$$\overline{\{f \in X : Gf' \in X\}} = X.$$

Based on this observation, they give another characterization of $[\varphi_t, X]$ via the integral operator. Given a semigroup $\{\varphi_t\}$ with generator G and Denjoy-Wolff point b, we define the function $\gamma : \mathbb{D} \to \mathbb{C}$, called *the associated g-symbol of* $\{\varphi_t\}$, as

$$\gamma(z) = \int_b^z \frac{\zeta - b}{G(\zeta)}\, d\zeta$$

for $b \in \mathbb{D}$ and

$$\gamma(z) = \int_0^z \frac{1}{G(\zeta)}\, d\zeta$$

for $b \in \partial\mathbb{D}$.

Proposition 2.22. *Let $\{\varphi_t\}$ be a semigroup with associated g-symbol γ. Let X be a Banach space of analytic functions with the properties:*

(i) *X contains the constant functions;*

(ii) *For each* $b \in \mathbb{D}$, $f \in X \Leftrightarrow \frac{f(z)-f(b)}{z-b} \in X$*;*

(iii) *If* $\{\boldsymbol{C}_t\}$ *is the induced semigroup on* X *then* $\sup_{t\in[0,1]} \|\boldsymbol{C}_t\| < \infty$.

Then

$$[\varphi_t, X] = \overline{X \cap (V_\gamma(X) \oplus \mathbb{C})},$$

where V_γ *is the Volterra operator with associated g-symbol defined above.*

Since we can write the generator function G as $G(z) = (\bar{b}z-1)(z-b)P(z)$, $z \in \mathbb{D}$, with $\operatorname{Re} P \geq 0$, and by composition with an automorphism of the disk, if $b \in \mathbb{D}$ we can assume $b = 0$, we may write

$$\gamma(z) = \int_0^z \frac{\zeta}{G(\zeta)} d\zeta = -\int_0^z \frac{1}{P(\zeta)} d\zeta.$$

If $b \in \partial\mathbb{D}$, without loss of generality we can take $b = 1$, so $G(z) = (1-z)^2 P(z)$ and

$$\gamma(z) = \int_0^z \frac{1}{(1-\zeta)^2 P(\zeta)} d\zeta.$$

Therefore, to study the strong continuity of a semigroup of composition operators, we need only study the image of an integral operator (possibly unbounded, as on the Bloch space).

Finally, the authors of [29] are also able to generalize the relation of the infinitesimal generator of the semigroup of composition operators with the infinitesimal generator of the semigroup of analytic functions present in the theory of semigroups on Hardy, Bergman and Dirichlet spaces.

Theorem 2.23. *Let* $\{\varphi_t\}$ *be a semigroup with generator* G *and* X *a Banach space of analytic functions such that* $\{\boldsymbol{C}_t\}$ *is strongly continuous on* X*. Then the infinitesimal generator* A *of* $\{\boldsymbol{C}_t\}$ *is given by* $A(f)(z) = G(z)f'(z)$ *with domain*

$$D(A) = \{f \in X : Gf' \in X\}.$$

Chapter 3

Characterizations of weighted compositions transformations preserving the class $\mathcal{P}$

As indicated in the two previous chapters, the class $\mathcal{P}$ is the class of analytic functions on the unit disk f with positive real part such that $f(0) = 1$, while we denote by $\mathbf{T}_{F,\varphi}$ the weighted composition transformation, that is, for $f \in H(\mathbb{D})$

$$\mathbf{T}_{F,\varphi} f = F(f \circ \varphi).$$

The aim of this chapter is to characterize and understand the weighted composition transformations that preserve the class $\mathcal{P}$. Transformations that preserve classes of analytic functions have proved useful in the proofs of extremal problems for those classes, see [58, Chapter 2]. The class $\mathcal{P}$ has also a clear geometric interpretation. Considering the principal branch of the argument function with values in $(-\pi, \pi]$ we have that for any function f in $\mathcal{P}$, the function $\arg f$ takes on the values only in $(-\pi/2, \pi/2)$ and is a continuous function in the disk. Moreover, the argument of the product of two such functions f and g in $\mathcal{P}$, with values in $(-\pi, \pi)$, is still continuous and the formula

$$\arg(fg) = \arg f + \arg g$$

holds throughout $\mathbb{D}$. This fact will allow us to understand the counterbalance between the images of F and of $f \circ \varphi$, and therefore the behavior of φ.

This chapter is based on the reference [13].

3.1 Multipliers, composition and weighted composition transformations

First we will understand the simplest cases of weighted composition transformations, the multipliers and composition transformations. The following proposition shows that the only multipliers that preserve the class $\mathcal{P}$ are the trivial ones.

Proposition 3.1. *If $F f \in \mathcal{P}$ for all f in $\mathcal{P}$ then $F \equiv 1$.*

Proof. Since $Ff \in \mathcal{P}$, by the growth theorem for the functions in $\mathcal{P}$ we have

$$|F(z)| \cdot |f(z)| \leq \frac{1+|z|}{1-|z|}$$

for all z in $\mathbb{D}$. Also, for any fixed z we may choose f to be a suitable rotation of the half-plane function for which

$$|f(z)| = \frac{1+|z|}{1-|z|}\,.$$

It follows that $|F(z)| \leq 1$ for all $z \in \mathbb{D}$. Since $F(0) = 1$, the maximum modulus principle implies that F is identically constant, hence $F \equiv 1$. □

To characterize the composition transformations that preserve Carathéodory's class, recall that every function $f \in \mathcal{P}$ can be written as $f = \ell \circ \omega$, where ℓ is the half-plane mapping

$$\ell(z) = \frac{1+z}{1-z}$$

and ω is a Schwarz-type function, that is $\omega(\mathbb{D}) \subseteq \mathbb{D}$ and $\omega(0) = 0$. Now, it is easy to see that $f \circ \varphi \in \mathcal{P}$ for every $f \in \mathcal{P}$ if and only if φ is a Schwarz-type function as well.

Before characterizing the weighted composition transformations that preserve the class $\mathcal{P}$, we will see the necessary conditions that the symbols F and φ must satisfy to make the inclusion $\mathbf{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$ possible. First, since the function $f \equiv 1$ belongs to $\mathcal{P}$, we have that $F(f \circ \varphi) = F \in \mathcal{P}$. Moreover, choosing $f(z) = 1 + z$, another function obviously in $\mathcal{P}$, we get

$$1 = F(0)f(\varphi(0)) = 1 + \varphi(0)\,,$$

hence $\varphi(0) = 0$. Thus, from now on we shall always work assuming these hypotheses: $F \in \mathcal{P}$ and φ is a Schwarz-type function.

We now characterize all admissible ordered pairs (F, φ) for which the weighted composition transformation $\mathbf{T}_{F,\varphi}$ preserves the class $\mathcal{P}$. Recall that $\mathcal{L} = \{\ell_\lambda : |\lambda| = 1\}$ is the set of all rotations of the half-plane function $\ell(z) = \frac{1+z}{1-z}$.

Theorem 3.2. *Let φ be a Schwarz-type function, $F \in \mathcal{P}$, and denote by ω the Schwarz-type function for which $F = \ell\circ\omega$. Consider the argument function defined earlier. Then the following conditions are equivalent:*

(a) $\boldsymbol{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$.

(b) $\boldsymbol{T}_{F,\varphi}(\mathcal{L}) \subset \mathcal{P}$.

(c) The inequality

$$4|\varphi(z)| \cdot |\operatorname{Im}\omega(z)| < (1 - |\omega(z)|^2)(1 - |\varphi(z)|^2) \tag{3.1}$$

holds for all z in $\mathbb{D}$. In other words,

$$2|\varphi(z)| \cdot \left|\frac{\operatorname{Im} F(z)}{\operatorname{Re} F(z)}\right| < 1 - |\varphi(z)|^2\,, \quad \text{for all } z \in \mathbb{D}\,. \tag{3.2}$$

(d) The inequality

$$|\arg F(z)| < \frac{\pi}{2} - \arcsin \frac{2|\varphi(z)|}{1+|\varphi(z)|^2} \tag{3.3}$$

holds for all z in $\mathbb{D}$. Note also that

$$\frac{\pi}{2} - \arcsin \frac{2|\varphi(z)|}{1+|\varphi(z)|^2} = \frac{\pi}{2} - \arctan \frac{2|\varphi(z)|}{1-|\varphi(z)|^2} = \arctan \frac{1-|\varphi(z)|^2}{2|\varphi(z)|}\,, \tag{3.4}$$

where in the case when $\varphi(0) = 0$ the last equality should be understood as the limit $\arctan(+\infty) = \frac{\pi}{2}$.

Note that condition (b) simply states that it suffices to test the action of $\mathbf{T}_{F,\varphi}$ on the set $\mathcal{L}$. Condition (c) gives an effective analytic way of testing if a symbol is admissible or not while (d) provides conditions of geometric type.

It should be also noted that in the above result the inequalities in conditions (c) and (d) are both invariant under rotations of φ but not under the rotations in ω (or under the appropriate changes in F).

Proof. We will show that (a) $\Leftrightarrow$ (b), (b) $\Leftrightarrow$ (c), and (c) $\Leftrightarrow$ (d).

$\boxed{\text{(a)} \Leftrightarrow \text{(b)}}$. The implication (a) $\Rightarrow$ (b) is obvious so we only have to see that (b) $\Rightarrow$ (a). Recall that, by the Herglotz Representation Theorem, $\mathcal{P}$ equals $\overline{\mathrm{co}\,(\mathcal{L})}$, the closed convex hull of the collection $\mathcal{L}$ in the topology of uniform convergence on compact subsets of $\mathbb{D}$. Thus, we need only prove that $\mathbf{T}_{F,\varphi}(\mathcal{P}) = \mathbf{T}_{F,\varphi}(\overline{\mathrm{co}\,(\mathcal{L})}) \subset \mathcal{P}$ as long as $\mathbf{T}_{F,\varphi}(\mathcal{L}) \subset \mathcal{P}$.

First, if $\mathbf{T}_{F,\varphi}(\mathcal{L}) \subset \mathcal{P}$ we also have that $\mathbf{T}_{F,\varphi}(\mathrm{co}\,(\mathcal{L})) = \mathrm{co}\,\mathbf{T}_{F,\varphi}(\mathcal{L}) \subset \mathcal{P}$ as the class $\mathcal{P}$ is clearly convex. Moreover, if $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$, then also $F(f_n \circ \varphi) \to F(f \circ \varphi)$ in the same topology. Since $\mathcal{P}$ is a compact family (in the classical terminology, meaning a closed set in the compact-open topology), we get $\mathbf{T}_{F,\varphi}(\mathcal{P}) = \mathbf{T}_{F,\varphi}(\overline{\mathrm{co}\,(\mathcal{L})}) = \overline{\mathbf{T}_{F,\varphi}(\mathrm{co}\,(\mathcal{L}))} \subset \mathcal{P}$.

$\boxed{\text{(b)} \Leftrightarrow \text{(c)}}$. To verify that (3.1) is equivalent to (3.2), one easily checks that if $F = \ell \circ \omega$ then

$$F = \frac{1+\omega}{1-\omega} = \frac{1+2i\mathrm{Im}\,\omega - |\omega|^2}{1-2\mathrm{Re}\,\omega + |\omega|^2}$$

and

$$\left|\frac{\mathrm{Im}\,F(z)}{\mathrm{Re}\,F(z)}\right| = 2\frac{|\mathrm{Im}\,\omega(z)|}{1-|\omega(z)|^2}\,.$$

To see that (b) $\Rightarrow$ (c), suppose that $F(f \circ \varphi) \in \mathcal{P}$ for all f in $\mathcal{L}$. In other words, $F(\ell_\lambda \circ \varphi) \in \mathcal{P}$ for all λ of modulus one and therefore also

$$F(\ell_\lambda \circ \varphi) = \frac{1+\omega_\lambda}{1-\omega_\lambda}$$

for the Schwarz-type functions ω_λ depending on each λ. This leads to the equation

$$\frac{1+\omega}{1-\omega}\frac{1+\lambda\varphi}{1-\lambda\varphi} = \frac{1+\omega_\lambda}{1-\omega_\lambda}$$

which holds in the entire unit disk. Solving for ω_λ, we get

$$\omega_\lambda = \frac{\lambda\varphi + \omega}{1 + \lambda\varphi\omega}.$$

The condition $|\omega_\lambda| < 1$ in $\mathbb{D}$ is equivalent to

$$|\lambda\varphi + \omega|^2 < |1 + \lambda\varphi\omega|^2$$

which amounts to the inequality

$$|\varphi|^2 + |\omega|^2 + 2\mathrm{Re}\,\{\lambda\varphi\overline{\omega}\} < 1 + |\varphi\omega|^2 + 2\mathrm{Re}\,\{\lambda\varphi\omega\}. \tag{3.5}$$

Grouping the terms in (3.5) we obtain

$$2\mathrm{Re}\,\{\lambda\varphi(z)(\overline{\omega(z)} - \omega(z))\} < (1 - |\omega(z)|^2)(1 - |\varphi(z)|^2)$$

for each z in $\mathbb{D}$ and for arbitrary λ with $|\lambda| = 1$. For each point z we can choose the argument of λ appropriately so as to get

$$2\mathrm{Re}\,\{\lambda\varphi(z)(\overline{\omega(z)} - \omega(z))\} = 4|\varphi(z)| \cdot |\mathrm{Im}\,\omega(z)|.$$

Since this is valid at every point z in the disk, the statement (3.1) follows.

To see that (c) $\Rightarrow$ (b), it suffices to observe that

$$2\mathrm{Re}\,\{\lambda\varphi(z)(\overline{\omega(z)} - \omega(z))\} \le 4|\varphi(z)| \cdot |\mathrm{Im}\,\omega(z)|$$

and it is now easy to reverse the steps in the above proof.

(c) $\Leftrightarrow$ (d). Since $F \in \mathcal{P}$, we know that $|\arg F| < \pi/2$. Thus, inequality (3.2) is clearly equivalent to

$$2|\varphi(z)| \cdot |\tan(\arg F(z))| < 1 - |\varphi(z)|^2, \qquad z \in \mathbb{D},$$

which is the same as

$$|\tan(\arg F(z))| < \frac{1 - |\varphi(z)|^2}{2|\varphi(z)|}, \qquad z \in \mathbb{D},$$

understanding the right-hand side as $+\infty$ when $\varphi(z) = 0$. The inverse tangent function is odd so this is the same as

$$|\arg F(z)| < \arctan\frac{1 - |\varphi(z)|^2}{2|\varphi(z)|}, \qquad z \in \mathbb{D}.$$

Equalities (3.4) follow by elementary trigonometry, so the proof is complete. □

3.2 Some consequences and discussions

In this section we will apply Theorem 3.2 to specific examples of symbols to understand the relation between the range of F and of φ. In the first subsection we will see what happens when one of the symbols is "big", in the sense that the image of either φ or ω covers most of the unit disk. In the second subsection we will have one of the symbols bounded, and we will check what conditions must the other symbol satisfy in order to have the transformation preserve $\mathcal{P}$. In the last subsection we will compare the boundary behavior of the symbols.

3.2.1 Some rigidity principles

Recall that, as we saw in Section 1.1, functions in H^∞ have radial limits $\varphi(\zeta) = \lim_{r\to 1}\varphi(r\zeta)$ for almost every point ζ on the unit circle $\mathbb{T}$ with respect to the normalized Lebesgue arc length measure, and that a function is inner if $|\varphi(z)| \leq 1$ for all z in $\mathbb{D}$ (equivalently, $\|\varphi\|_\infty \leq 1$) and also $|\varphi(\zeta)| = 1$ almost everywhere on $\mathbb{T}$. The following result generalizes our Proposition 3.1.

Proposition 3.3. *Let $F \in \mathcal{P}$ and let φ be inner. Then $\boldsymbol{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$ if and only if $F \equiv 1$.*

Proof. The bounded functions φ and ω have radial limits almost everywhere on the circle. Thus, for almost every $\zeta \in \mathbb{T}$ we may pass to the limit as $z \to \zeta$ in inequality (3.1) to conclude that $\operatorname{Im}\omega(\zeta) = 0$ almost everywhere on $\mathbb{T}$. To see that this implies that ω is zero, let g be the analytic function $g = \exp\{i\omega\}$, then it has boundary values on the circle have modulus one almost everywhere, and thus $\|g\|_\infty = 1$. Moreover, since $g(0) = 1$ it follows that $g \equiv 1$, and from here $\omega \equiv 0$ (that is, $F \equiv 1$). ☐

Here is the counterpart of this statement with assumptions on ω.

Proposition 3.4. *Let $F = \ell \circ \omega$, where ω is an inner function. Then $\boldsymbol{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$ if and only if $\varphi \equiv 0$.*

In the proof we will need the following classical fact that follows from the theorems of Fatou and Nevanlinna [57, Theorem 2.2]: each function in H^∞ has radial limits almost everywhere, and $\log|f(e^{i\theta})|$ is integrable unless $f \equiv 0$. In particular, if $f \in H^p$ satisfies $f = 0$ in a set of positive measure of the boundary, then $f \equiv 0$ because $\log|f(e^{i\theta})|$ is not integrable.

Proof. After passing to the radial limits in (3.1) we get that $\varphi \operatorname{Im}\omega = 0$ almost everywhere on the unit circle.

If $\varphi = 0$ only on a set of measure zero on the circle, then $\operatorname{Im}\omega = 0$ almost everywhere on the circle. From the proof of the previous theorem we know that $\omega \equiv 0$, which contradicts our initial assumption. Hence $\varphi = 0$ on a set of positive measure. It follows that $\varphi \equiv 0$. ☐

It is easily seen from Theorem 3.2 that any admissible multiplication symbol F can only carry a very small portion of the boundary of the unit disk to the imaginary axis.

Proposition 3.5. *Let $F \in \mathcal{P}$, let φ and ω be two Schwarz-type functions, $\varphi \not\equiv 0$, and suppose that $F = \ell \circ \omega$ and $\boldsymbol{T}_{F,\varphi}$ preserves $\mathcal{P}$ as before. Denote the radial limits of F again by F and let*

$$A = \{\zeta \in \mathbb{T} : \operatorname{Re} F(\zeta) = 0\} = \{\zeta \in \mathbb{T} : |\omega(\zeta)| = 1, \omega(\zeta) \neq 1\}.$$

Then $m(A) = 0$.

Proof. Suppose $m(A) > 0$. After passing on to the radial limits in (3.1), we obtain

$$4|\varphi(\zeta)| \cdot |\operatorname{Im} \omega(\zeta)| \leq (1 - |\omega(\zeta)|^2)(1 - |\varphi(\zeta)|^2)$$

for almost all ζ with $|\zeta| = 1$. If $\zeta \in A$ then $|\omega(\zeta)| = 1$ and therefore $\varphi \operatorname{Im} \omega = 0$ holds at almost every point of A (note that φ may not have radial limits at some subset of A of total measure zero). Since the measure of A is positive and $\varphi \not\equiv 0$, we must have $\operatorname{Im} \omega(\zeta) = 0$, that is, either $\omega = 1$ or $\omega = -1$ on a set of positive measure in A. The first case is excluded by the definition of A and the second case implies that $\omega \equiv -1$ in $\mathbb{D}$, which is impossible in view of the assumption that $\omega(0) = 0$. This shows that $m(A) = 0$. □

In the context of (linear) weighted composition transformations the case in which $F = \phi'$ is often important. However, in our context it should be noted that in this case we only obtain another rigidity situation. Namely, assuming that $\mathbf{T}_{\phi',\phi}(\mathcal{P}) \subset \mathcal{P}$ and choosing $f \equiv 1$ we get $\phi' \in \mathcal{P}$ hence $\phi'(0) = 1$. The case of equality in the Schwarz lemma forces $\phi(z) = z$, hence $F = \phi' \equiv 1$, so our transformation $\mathbf{T}_{\phi',\phi}$ reduces to the identity map.

3.2.2 Cases where one of the symbols has small range

Many non-trivial examples of weighted composition transformations that preserve class $\mathcal{P}$ are possible when $\varphi(\mathbb{D})$ is compactly contained in $\mathbb{D}$ or $F(\mathbb{D})$ is contained in a sector, as the following results show.

Proposition 3.6. *Let $F \in \mathcal{P}$ and let φ be a Schwarz-type function such that $\|\varphi\|_\infty = R < 1$. Then whenever the function F satisfies*

$$|\arg F(z)| < \frac{\pi}{2} - \arcsin \frac{2R}{1 + R^2}$$

for all z in $\mathbb{D}$, we have that $\boldsymbol{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$.

Proof. Follows from criterion (d) of Theorem 3.2 and the fact that the function $2x/(1+x^2)$ is increasing in the interval $(0, 1)$. □

Example 1. An explicit example is $\varphi(z) = Rz$, $0 < R < 1$, and

$$F(z) = \left(\frac{1+z}{1-z}\right)^{\varepsilon}, \qquad 0 < \varepsilon < 1 - \frac{2}{\pi} \arcsin \frac{2R}{1 + R^2},$$

a conformal map of the unit disk onto an angular sector with vertex at the origin. Condition (3.3) is clearly satisfied.

Proposition 3.7. *Let $F \in \mathcal{P}$ and $K = \sup_{z\in\mathbb{D}} |\arg F(z)| < \frac{\pi}{2}$. Write $K = \arcsin \frac{2R}{1+R^2}$, $0 \le R < 1$. If φ is a Schwarz-type function such that*

$$\|\varphi\|_\infty \le \frac{1-R}{1+R}$$

then $\boldsymbol{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$.

Proof. By assumption,

$$|\arg F(z)| \le K = \arcsin \frac{2R}{1+R^2}\,, \qquad z \in \mathbb{D}\,.$$

In view of condition (3.3) from Theorem 3.2 it suffices to check that

$$\arcsin \frac{2R}{1+R^2} < \frac{\pi}{2} - \arcsin \frac{2|\varphi(z)|}{1+|\varphi(z)|^2}$$

holds for all z in $\mathbb{D}$. Equivalently,

$$\frac{2|\varphi(z)|}{1+|\varphi(z)|^2} < \sin\left(\frac{\pi}{2} - \arcsin \frac{2R}{1+R^2}\right) = \cos\left(\arcsin \frac{2R}{1+R^2}\right)$$

must hold throughout $\mathbb{D}$. This will certainly be satisfied if

$$\frac{2\|\varphi\|_\infty}{1+\|\varphi\|_\infty^2} \le \cos\left(\arcsin \frac{2R}{1+R^2}\right) \tag{3.6}$$

in view of monotonicity of the function $u(x) = \frac{2x}{1+x^2}$ in $[0, 1)$. But (3.6) is clearly equivalent to

$$\frac{2\|\varphi\|_\infty}{1+\|\varphi\|_\infty^2} \le \cos\left(\arcsin \frac{2R}{1+R^2}\right) = \sqrt{1-\left(\frac{2R}{1+R^2}\right)^2} = \frac{1-R^2}{1+R^2}\,.$$

This yields an elementary quadratic inequality in $\|\varphi\|_\infty$ which is easily seen to be satisfied whenever

$$0 \le \|\varphi\|_\infty \le \frac{1-R}{1+R}\,.$$

This proves the statement. □

We now formulate a counterpart of Proposition 3.6 with similar hypotheses on ω instead of φ which follows from our previous result.

Corollary 3.8. *Let $F = \ell \circ \omega \in \mathcal{P}$, where ω is a Schwarz-type function. If $\|\omega\|_\infty < 1$ and φ is a Schwarz-type function such that*

$$\|\varphi\|_\infty < \frac{1-\|\omega\|_\infty}{1+\|\omega\|_\infty}$$

then $\boldsymbol{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$.

Proof. Let $R = \|\omega\|_\infty < 1$. Then the function F is clearly subordinated to the function

$$\ell_R(z) = \frac{1+Rz}{1-Rz}$$

in the usual sense that $F = \ell_R \circ \left(\frac{\omega}{R}\right)$. Thus, $F(\mathbb{D}) \subset \ell_R(\mathbb{D})$. To prove the result, we will use Proposition 3.7, and therefore we need to bound the argument of F. By the subordination, it is enough to bound the argument of ℓ_R. It is plain that $\ell_R(\mathbb{D})$ is the disk whose diameter has endpoints

$$\ell_R(-1) = \frac{1-R}{1+R}, \qquad \ell_R(1) = \frac{1+R}{1-R},$$

hence its center and radius are respectively

$$C = \frac{1+R^2}{1-R^2}, \qquad \rho = \frac{2R}{1-R^2}.$$

Let us denote by $C_R = \{z \in \mathbb{C} : |z - C| = \rho\}$ the boundary of this disk. Let a be the point of intersection of the circle C_R with its tangent from the origin in the upper half-plane. By looking at the right triangle determined by the origin and the points a and C, we infer that

$$\arg a = \arcsin \frac{\rho}{C} = \arcsin \frac{2R}{1+R^2}.$$

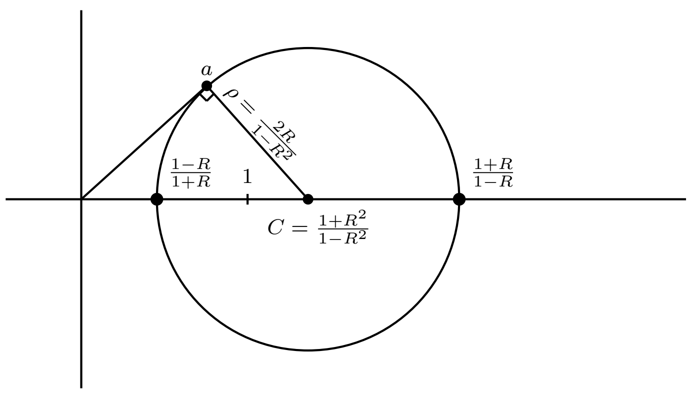

Figure 3.1: The circle C_R

One argues similarly for the point of tangent in the lower half-plane and obtains that, for every z in $\mathbb{D}$,

$$|\arg F(z)| < \arcsin \frac{2R}{1+R^2}.$$

The conclusion now follows from Proposition 3.7. □

Notice that the fact that $F(\mathbb{D})$ is contained in a sector means that, for every $z \in \mathbb{D}$,

$$|\tan(\arg F(z))| = 2\frac{|\operatorname{Im} \omega(z)|}{1-|\omega(z)|^2} \leq \tan \frac{\alpha\pi}{2}$$

with $\alpha \in (0,1)$. In terms of the Schwarz-type function ω, this means that

$$2|\operatorname{Im}\omega(z)| \leq \tan\frac{\alpha\pi}{2}(1-|\omega(z)|^2).$$

Since the image of

$$\frac{2}{C}|y| \leq 1-x^2-y^2,$$

with $C<\infty$ and $x+iy\in\mathbb{D}$ is the intersection of two disks of radius $1+\frac{1}{C^2}$ and centers $\frac{i}{C}$ and $\frac{-i}{C}$, $F(\mathbb{D})$ contained in a sector means that $\omega(\mathbb{D})$ is contained in a lens-shaped region. Our next result essentially shows that when ω fills the region, that is, when the multiplication symbol is obtained by composing the half-plane map with a lens map, the statements of Proposition 3.6 and Proposition 3.7 can be unified into a single “if and only if” statement.

Proposition 3.9. *Let $F=\ell\circ\lambda_\alpha$, where λ_α is a lens map, and let φ be a Schwarz-type function. Then $\boldsymbol{T}_{F,\varphi}(\mathcal{P})\subset\mathcal{P}$ if and only if*

$$\|\varphi\|_\infty \leq \frac{1-R}{1+R},$$

where $\frac{2R}{1+R^2}=\sin\frac{\alpha\pi}{2}$.

Proof. Suppose first that $\|\varphi\|_\infty \leq \frac{1-R}{1+R}$. Then, since

$$\sup_{z\in\mathbb{D}}|\arg\, F(z)| = \frac{\alpha\pi}{2} = K = \arcsin\frac{2R}{1+R^2},$$

by Proposition 3.7 it follows that $\mathbf{T}_{F,\varphi}(\mathcal{P})\subset\mathcal{P}$. Now, if $\mathbf{T}_{F,\varphi}(\mathcal{P})\subset\mathcal{P}$, by condition (3.3) of Theorem 3.2 we have

$$|\arg\, F(z)| + \arcsin\frac{2|\varphi(z)|}{1+|\varphi(z)|^2} < \frac{\pi}{2}.$$

Therefore, for almost every $\zeta\in\mathbb{T}$,

$$\arcsin\frac{2|\varphi(\zeta)|}{1+|\varphi(\zeta)|^2} \leq \frac{\pi}{2}-\frac{\alpha\pi}{2} = \frac{\pi}{2}-\arcsin\frac{2R}{1+R^2}.$$

Then, as in the proof of Proposition 3.7,

$$\frac{2\|\varphi\|_\infty}{1+\|\varphi\|_\infty^2} \leq \sin\left(\frac{\pi}{2}-\arcsin\frac{2R}{1+R^2}\right) = \frac{1-R^2}{1+R^2},$$

and from here,

$$\|\varphi\|_\infty \leq \frac{1-R}{1+R}.$$

□

3.2.3 Composition symbols with radial limits of modulus one and/or angular derivatives

Our next result shows that if φ possesses even a mildly reasonable boundary behavior at a point on the unit circle then ω automatically cannot be "too good" at the same point.

Theorem 3.10. *Let $F \in \mathcal{P}$, let φ and ω be two Schwarz-type functions, $\varphi \not\equiv 0$, $F = \ell \circ \omega$ and suppose that at some point ζ on the unit circle the function φ has radial limit of modulus one. Then if the transformation $\boldsymbol{T}_{F,\varphi}$ preserves $\mathcal{P}$, the function ω cannot have angular derivative at ζ.*

Proof. Note that $T_{F,\phi}$ preserves $\mathcal{P}$ if and only if the transformation $T_{F_\lambda,\phi_\lambda}$ preserves $\mathcal{P}$, where $F_\lambda(z) = F(\lambda z)$ and $\phi_\lambda(z) = \phi(\lambda z)$, whenever $|\lambda| = 1$. Hence, we may assume without loss of generality that $\zeta = 1$.

Suppose that ω has angular derivative at $\zeta = 1$. Then the radial limit $\omega(1)$ exists and $|\omega(1)| = 1$. Taking the angular limit as $z \to 1$ in (3.1), we conclude that $\text{Im}\,\{\omega(1)\} = 0$. Thus, either $\omega(1) = -1$ or $\omega(1) = 1$.

Let us first consider the case $\omega(1) = -1$. Since at $\zeta = 1$ the angular derivative of ω is neither 0 nor ∞, we know (see Section 2.1.2) that it is actually univalent in some Stolz domain with vertex at $z = 1$:

$$\Delta = \{z \colon |\arg(1-z)| < \theta,\ r < |z| < 1\}$$

for suitable $r \in (0,1)$ and $\theta > 0$. Also, as is also noted in Section 2.1.2, the function ω preserves angles between curves contained in $\Delta \cup \{1\}$ that meet at $z = 1$. This shows that there exists a curve $\gamma : [0,1] \to \Delta \cup \{1\}$ with $\gamma(1) = 1$ and which is mapped by ω onto some non-horizontal segment

$$S = \{-1 + se^{i\alpha_0} \colon 0 \le s \le s_0\}\,, \quad 0 < |\alpha_0| < \frac{\pi}{2}\,,$$

for an appropriate value of s_0. (To see this, it suffices to look at the image under ω of the suitable Stolz domain mentioned earlier with vertex at 1, which will contain another Stolz domain with vertex at -1, and to select α_0 and s_0 so that the segment S is contained in this new Stolz angle and is not contained in the real axis). Keeping in mind that $F = \ell \circ \omega$ and ℓ is a Möbius transformation which maps the diameter $(-1,1)$ to the positive semi-axis, we see that

$$\arg\, F(\gamma(t)) = \arg\, \ell(\omega(\gamma(t))) \to \alpha_0\,, \quad \text{as} \quad t \to 1^-\,.$$

Therefore, taking the limit as $z \to 1$ along γ in (3.3), we obtain $|\alpha_0| \le 0$, which is contrary to our construction of the segment S. This completes the proof in the case when $\omega(1) = -1$.

By (3.1), $T_{F,\phi}$ preserves $\mathcal{P}$ if and only if $T_{G,\phi}$ with $G = (1-\omega)/(1+\omega)$ does, so we can argue as above in the case when $\omega(1) = 1$ to get a contradiction again. $\square$

There are two ways in which the function ω can fail to have angular derivative: either it does not have a radial limit of modulus one or it does but the differential quotient fails to have a limit at the point in question. Here is an example of the first kind. It deals with the map φ such that $\varphi(\mathbb{D})$ has a tangential contact with the unit circle. The price we pay for this is that ω is a dilated self-map of the disk (hence, in this example $\omega(\mathbb{D})$ is compactly contained in $\mathbb{D}$).

Example 2. For $K \geq 3/2$, let

$$\varphi(z) = \frac{z(1+z)}{2}, \qquad \omega(z) = \frac{z(2-z)}{2K}.$$

Both are clearly Schwarz-type functions. Obviously, $\varphi(1) = 1$ and φ is conformal at $z = 1$ since $\varphi'(1) \neq 0$. For a sufficiently large value of K (which will be determined below) one can also check that our condition (3.1) is satisfied, hence $\mathbf{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$. Indeed, it is immediate that

$$\operatorname{Im} \omega(z) = \frac{y(1-x)}{K}, \qquad z = x + iy.$$

Checking our condition (3.1) in this case reduces to verifying that

$$\frac{2\,|z(1+z)y|\,(1-x)}{K} < \left(1 - \left|\frac{z(1+z)}{2}\right|^2\right)\left(1 - \left|\frac{z(2-z)}{2K}\right|^2\right)$$

holds for all z in $\mathbb{D}$. (Note that as $z \to 1$, both sides tend to zero but the strict inequality is maintained). Since $x^2 + y^2 = |z|^2 < 1$, it is clear that

$$\frac{2\,|z(1+z)y|\,(1-x)}{K} < \frac{4(1-x)}{K}$$

while the right-hand side can be estimated from below as follows:

$$\begin{aligned}
\left(1 - \left|\frac{z(1+z)}{2}\right|^2\right)\left(1 - \left|\frac{z(2-z)}{2K}\right|^2\right) &> \left(1 - \frac{|1+z|^2}{4}\right)\left(1 - \frac{9}{4K^2}\right) \\
&= \left(1 - \frac{(1+x)^2 + y^2}{4}\right)\left(1 - \frac{9}{4K^2}\right) \\
&\geq \left(1 - \frac{2+2x}{4}\right)\left(1 - \frac{9}{4K^2}\right) \\
&= \frac{1-x}{2}\left(1 - \frac{9}{4K^2}\right)
\end{aligned}$$

so it is only left to check that

$$\frac{4(1-x)}{K} < \frac{1-x}{2}\left(1 - \frac{9}{4K^2}\right)$$

for K large enough and $|x| < 1$, which is clear. The inequality holds for all $K > 4 + \frac{\sqrt{73}}{2}$.

The natural question arises as to whether it is possible to have an example where both φ and ω can have radial limits of modulus one at the same point (obviously, without having an angular derivative at the point in question) but the weighted composition $\mathbf{T}_{F,\varphi}$ still preserves $\mathcal{P}$. The following example, illustrated by the figure below, gives an affirmative answer.

Example 3. Consider the planar domain

$$\Omega = \{x + iy : \ 4|y|\sqrt{x^2 + y^2} < (1 - x^2 - y^2)^2\} \,,$$

clearly symmetric with respect to both the real and imaginary axes. Let ω be a conformal map of $\mathbb{D}$ onto Ω which fixes the origin. Starting with the subdomain of Ω in the upper half-plane and using the Schwarz reflection principle, one can also choose ω in such a fashion that it fixes the diameter $(-1, 1)$ and $\omega(1) = 1$ in the sense of a radial limit. Let $\varphi = \omega$. It can now easily be checked that our condition (3.1) is satisfied, hence $\mathbf{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$.

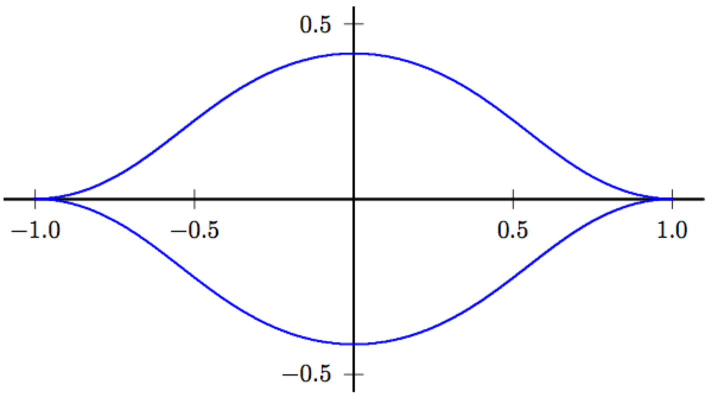

Figure 3.2: The boundary of the leaf-shaped region Ω.

Note, however, in relation to this “leaf-shaped” region that our mapping $\omega = \varphi$ has boundary contact with the unit circle but does not have angular derivative at $z = 1$. The intuitive reason for this is that the corners at -1 and 1 are contained in lens-shaped regions and lens maps do not have angular derivatives. A rigorous proof of this fact can be given by using subordination, or by noting that, if ω had angular derivative at the point 1, then it would preserve angles in a Stolz domain with vertex at 1 (see Section 2.1.2), but the shape of the image of $\mathbb{D}$ by ω prevents this behaviour.

Alternatively, one can write the equation of the boundary of Ω in polar coordinates:

$$4r^2|\sin\theta| = (1 - r^2)^2 \,.$$

Solving for r, one obtains

$$1 - r^2(\theta) = 2\left(\sqrt{\sin|\theta| + \sin^2|\theta|} - \sin|\theta|\right).$$

Let $1 - r(\theta) = \gamma(|\theta|)$, with $\gamma : [0,\pi] \to [0,1]$. Then for a fixed $1 > \varepsilon > 0$ we have

$$\begin{aligned}
\int_0^{\pi} \frac{\gamma(|\theta|)}{\theta^2}\,d\theta &\geq \int_0^{\varepsilon} \frac{\gamma(|\theta|)}{\theta^2}\,d\theta = \int_0^{\varepsilon} \frac{1-r^2(\theta)}{\theta^2(1+r(\theta))}\,d\theta \\
&\geq \int_0^{\varepsilon} \frac{2\left(\sqrt{\sin\theta + \sin^2\theta} - \sin\theta\right)}{2\theta^2}\,d\theta \\
&\geq \int_0^{\varepsilon} \frac{\sqrt{\sin\theta + \sin^2\theta} - \theta}{\theta^2}\,d\theta \geq \int_0^{\varepsilon} \frac{\sqrt{\sin\theta} - \theta}{\theta^2}\,d\theta \\
&\approx \int_0^{\varepsilon} \frac{\sqrt{\theta}(1-\sqrt{\theta})}{\theta^2}\,d\theta \geq \int_0^{\varepsilon} \frac{1-\sqrt{\varepsilon}}{\theta^{3/2}}\,d\theta = \infty
\end{aligned}$$

Thus, by the Tsuji-Warschawski Theorem 2.9 the function φ has no angular derivative at 1.

3.3 Fixed points of weighted composition transformations that preserve $\mathcal{P}$

In the last section of this chapter, we find the fixed points of the weighted composition transformations in $\mathcal{P}$. Even though we are working in a non-linear context, it is possible to adapt the arguments typical for such situations, see [108, Sect. 6.1].

Theorem 3.11. *Let $\boldsymbol{T}_{F,\varphi}$ be a weighted composition transformation such that $\boldsymbol{T}_{F,\varphi}(\mathcal{P}) \subset \mathcal{P}$, where $F = \ell \circ \omega$, φ and ω are Schwarz-type functions, and φ is not a rotation. Then $\boldsymbol{T}_{F,\varphi}$ has a unique fixed point which is obtained by iterating $\boldsymbol{T}_{F,\varphi}$ applied to arbitrary f in $\mathcal{P}$.*

In the case when φ is inner but not a rotation, the unique fixed point is the constant function one.

Proof. We first show that the limit of iterates of $\mathbf{T}_{F,\varphi}$ applied to an arbitrary function f in $\mathcal{P}$ is a fixed point of the transformation. Recall that the n-th iterations of φ were defined in Section 2.1.2 as $\varphi_{n+1} = \varphi_n \circ \varphi$, $n \geq 0$, with φ_0 the identity function. Let $f \in \mathcal{P}$. It is easy to see by induction that

$$F(F\circ\varphi)\ldots(F\circ\varphi_{n-1})(f\circ\varphi_n) = \mathbf{T}^n_{F,\varphi} f \in \mathcal{P}$$

for any integer $n \geq 1$. By our assumptions on φ, the origin is its only fixed point in $\mathbb{D}$. Since φ is not a disk automorphism, it follows that $\varphi_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ by the Denjoy-Wolff Theorem 2.10, and therefore, $f \circ \varphi_n \to 1$ uniformly on compact subsets as $n \to \infty$. On the other hand,

$$\prod_{k=0}^{n-1} (F \circ \varphi_k) = \prod_{k=0}^{n-1} \frac{1 + \omega \circ \varphi_k}{1 - \omega \circ \varphi_k},$$

so proving the uniform convergence on compact subsets of the infinite product

$$\prod_{k=0}^{\infty} (F \circ \varphi_k)$$

is equivalent to proving the convergence on compact subsets of $\mathbb{D}$ of the sums

$$\sum_{k=0}^{n-1} \left| 1 - \frac{1 + \omega \circ \varphi_k}{1 - \omega \circ \varphi_k} \right| = 2 \sum_{k=0}^{n-1} \left| \frac{\omega \circ \varphi_k}{1 - \omega \circ \varphi_k} \right|.$$

For $r \in (0,1)$ fixed, let $m(r) = \max_{|z| \le r} |\varphi(z)|$. Let $\delta = m(r)/r$. Clearly, $\delta < 1$ since φ is not a rotation. Applying the Schwarz lemma to $\varphi(rw)/m(r)$, we get $|\varphi(rw)|/m(r) \le |w|$ for any $w \in \mathbb{D}$, and from here

$$|\varphi(z)| \le \frac{m(r)}{r} |z| = \delta |z|$$

whenever $|z| \le r$. Iterating this inequality, we get

$$|\varphi_k(z)| \le \delta |\varphi_{k-1}(z)| \le \ldots \le \delta^k |z|$$

for $|z| \le r$. Using the fact that ω is a Schwarz-type function, we obtain

$$\left| 1 - \frac{1 + \omega \circ \varphi_k}{1 - \omega \circ \varphi_k} \right| = 2 \left| \frac{\omega \circ \varphi_k}{1 - \omega \circ \varphi_k} \right| \le 2 \frac{|\omega \circ \varphi_k|}{1 - |\omega \circ \varphi_k|} \le 2 \frac{|\varphi_k|}{1 - |\varphi_k|} \le 2 \frac{\delta^k r}{1 - \delta^k r} \le \frac{2r}{1 - r} \delta^k$$

in the disk $\{z : |z| \le r\}$. Thus, the series

$$\sum_{k=0}^{\infty} \left| 1 - \frac{1 + \omega \circ \varphi_k}{1 - \omega \circ \varphi_k} \right|$$

converges uniformly on compact subsets of the disk and the infinite product $\prod_{k=0}^{\infty} (F \circ \varphi_k)$ is uniformly convergent on compact subsets to some function G analytic in $\mathbb{D}$. Moreover, since $\mathcal{P}$ is a compact class, $G \in \mathcal{P}$. Combining both limits, we obtain

$$\mathbf{T}_{F,\varphi}^n f = F(F \circ \varphi) \ldots (F \circ \varphi_{n-1})(f \circ \varphi_n) \to G$$

uniformly on compact subsets as $n \to \infty$ for any $f \in \mathcal{P}$. Now we can see that G is a fixed point of the transformation. Applying $\mathbf{T}_{F,\varphi}$ to G we have

$$\begin{aligned}
\mathbf{T}_{F,\varphi} G = F(G \circ \varphi) &= F \left(\lim_{n \to \infty} \prod_{k=0}^{n-1} (F \circ \varphi_k) \right) \circ \varphi = F \left(\lim_{n \to \infty} \prod_{k=0}^{n-1} (F \circ \varphi_{k+1}) \right) \\
&= F \left(\lim_{n \to \infty} \prod_{k=1}^{n} (F \circ \varphi_k) \right) = \lim_{n \to \infty} \prod_{k=0}^{n} (F \circ \varphi_k) = G.
\end{aligned}$$

Note that G as constructed above does not depend on the initial choice of the function f in $\mathcal{P}$. Now it is clear that this G is the only fixed point of $\mathbf{T}_{F,\varphi}$, because if $g \in \mathcal{P}$ satisfies $\mathbf{T}_{F,\varphi} g = g$, iterating the transformation we get

$$g = \mathbf{T}^n_{F,\varphi} g \to G$$

uniformly on compact subsets of $\mathbb{D}$.

It is only left to check our final comment in the statement of the theorem. Since φ is an inner function, by Proposition 3.3 we have $F \equiv 1$, so the equation for the fixed point, $f \circ \varphi = f$, is the classical Schröder's equation for the composition operator $\mathbf{C}_\varphi$ corresponding to the eigenvalue $\lambda = 1$. The only solution f for this equation is the constant, since, if f satisfies $f \circ \varphi = f$, then iterating we have

$$f = f \circ \varphi = f \circ \varphi \circ \varphi = \ldots = f \circ \varphi_n.$$

Again, as $\varphi_n \to 0$ uniformly on compact subsets of the unit disk as $n \to \infty$, this means that $f = f \circ \varphi_n \to f(0) = 1$ uniformly on compact subsets of the unit disk as $n \to \infty$, and therefore $f \equiv 1$. □

The case when φ is a rotation leads to the well-known case of fixed points from the theory of composition operators, describing a trichotomy: the identity map, a rational rotation or an irrational rotation.

Proposition 3.12. *Let $\boldsymbol{T}_{F,\varphi}$ be a weighted composition transformation that preserves $\mathcal{P}$, where $F = \ell \circ \omega$, ω is a Schwarz-type functions, and φ is a rotation. Then the set of all fixed points of $\boldsymbol{T}_{F,\varphi}$ is as follows:*

(a) all of $\mathcal{P}$, if $\varphi(z) \equiv z$;

(b) the functions with n-fold symmetry: $f(z) = g(z^n)$, $g \in \mathcal{P}$, whenever $\varphi(z) = \lambda z$, where $\lambda^n = 1$ for some $n > 1$;

(c) only the constant function one, if $\varphi(z) = \lambda z$, where $|\lambda| = 1$ and $\lambda^n \neq 1$ for all $n \in \mathbb{N}$.

Proof. Since φ is an inner function, Proposition 3.3 forces $F \equiv 1$, hence $\mathbf{T}_{F,\varphi} f = f \circ \varphi$. Part (a) now follows trivially.

Parts (b) and (c) follow readily by comparing the Taylor series of both sides of the equality $f(\lambda z) = f(z)$ in the disk. □

Chapter 4

Mixed norm spaces

In this chapter we introduce the mixed norm spaces, a family of spaces related to the Hardy, Bergman and weighted Banach spaces. They have an interesting dual behavior depending on one of the parameters. If the parameter is finite, the norm is given by an integral, and the behavior is similar to the Bergman spaces, while if the parameter is infinite, the norm is given by a supremum, and therefore they are closer to the weighted Banach spaces.

After an introductory section to define these spaces, we will give pointwise and mean estimates in Section 2, and then we will characterize the inclusions between the different spaces.

This chapter is based on the paper [11].

4.1 Definition

The *mixed norm space* $H(p, q, \alpha)$, $p, q, \alpha > 0$, is the space of analytic functions on the unit disk such that

$$\int_0^1 (1-r)^{\alpha q - 1} M_p^q(r, f)\, dr < \infty,$$

for $q < \infty$, and

$$\sup_{0 \le r < 1} (1-r)^{\alpha} M_p(r, f) < \infty$$

for $q = \infty$.

Like in the Bergman spaces case, this expression first appears in Hardy and Littlewood's paper on properties of the integral mean [72], in the following theorem relating the mixed norm spaces and the Hardy spaces.

Theorem 4.1 (Hardy, Littlewood, 1932)**.** *If $f \in H^t$, $0 < t < p$ and $q \ge t$, then*

$$\int_0^1 (1-r)^{q\left(\frac{1}{t} - \frac{1}{p}\right) - 1} M_p^q(r, f)\, dr \le C \|f\|_{H^t}.$$

They were explicitly defined in Flett's works [63], [64]. Since then, these spaces have been studied by many authors (see [1], [28], [41], [66], [118]). Recently, the mixed norm spaces are mentioned in the works [20], [21], [11], and the monograph [78].

For any $0 < p, q \leq \infty$, $0 < \alpha < \infty$ the space $H(p, q, \alpha)$ is a complete subspace of the space $L(p, q, \alpha)$ of measurable functions in $\mathbb{D}$ (see [23]), and for $q \geq 1$ they are Banach spaces with the norm

$$\|f\|_{p,q,\alpha} = \left(\alpha q \int_0^1 (1-r)^{\alpha q - 1} M_p^q(r, f)\, dr \right)^{1/q},$$

for $q < \infty$, and

$$\|f\|_{p,\infty,\alpha} = \sup_{0 \leq r < 1} (1-r)^{\alpha} M_p(r, f).$$

As in the case of the Bloch and Weighted Banach spaces, we have a Banach space given by a supremum. We define the “little-oh” space $H_0(p, \infty, \alpha)$ as the space of the functions f in $H(p, \infty, \alpha)$ such that

$$\lim_{r \to 1} (1-r)^{\alpha} M_p(r, f) = 0.$$

We have the following results for these spaces [78, Proposition 7.1.3.]:

Proposition 4.2. *For* $0 \leq r < 1$, *let* $f_r(z) = f(rz)$, $z \in \mathbb{D}$.

- *If* $f \in H(p, q, \alpha)$, $0 < p \leq \infty$, $0 < q, \alpha < \infty$, *then* $\|f_r - f\|_{p,q,\alpha} \to 0$, *as* $r \to 1$.
- *If* $f \in H_0(p, \infty, \alpha)$, $0 < p \leq \infty$, $0 < \alpha < \infty$, *then* $\|f_r - f\|_{p,\infty,\alpha} \to 0$, *as* $r \to 1$.

Moreover, if $f \in H(p, \infty, \alpha)$ *and* $\|f_r - f\|_{p,\infty,\alpha} \to 0$, *as* $r \to 1$, *then* $f \in H_0(p, \infty, \alpha)$.

A first consequence of Proposition 4.2 is that polynomials are dense in $H(p, q, \alpha)$, $0 < p \leq \infty$, $0 < q, \alpha < \infty$ and $H_0(p, \infty, \alpha)$, $0 < p \leq \infty$, $0 < \alpha < \infty$. The closure in $H(p, \infty, \alpha)$ of the set of all analytic polynomials is $H_0(p, \infty, \alpha)$.

This result was also proved by Lusky in [88] in a more general setting. He also proved the following theorem.

Theorem 4.3. *The space* $H(p, q, \alpha)$ *is reflexive for* $1 < q < \infty$ *and*

$$H_0(p, \infty, \alpha)^{**} = H(p, \infty, \alpha).$$

These spaces are closely related to the spaces already studied in the first chapter, since we can identify the weighted Bergman space A^p_α, $0 < p < \infty$, $\alpha > -1$, with the space $H\left(p, p, \frac{\alpha+1}{p}\right)$ and the Hardy space H^p with the limit case $H(p, \infty, 0)$. They are also related to other spaces of analytic functions, such as Besov and Lipschitz spaces, via fractional derivatives (see [78, Chapter 7]), and, for $q = \infty$, with the Weighted Banach spaces H_v^∞ with weight $v(r) = (1-r)^\alpha$.

Familiar examples of analytic functions on the unit disk are the functions of type $(1-z)^{-\gamma}$, with γ a real constant. It is well known that such function is in the Hardy space H^p if and only if $\gamma < 1/p$ and in the Bergman space A^p if and only if $\gamma < 2/p$. The following lemma determines when these functions belong to $H(p, q, \alpha)$ (see [20]).

Lemma 4.4. *Let* $0 < p \leq \infty$, $0 < \alpha < \infty$. *The functions* $f(z) = \frac{1}{(1-z)^\gamma}$ *belong to* $H(p,q,\alpha)$, $0 < q < \infty$, *if and only if* $\gamma < \alpha + 1/p$, *and to* $H(p,\infty,\alpha)$ *if and only if* $\gamma \leq \alpha + 1/p$.

Starting with these examples we can search for functions with faster growth for $z \in \mathbb{R}$, $0 < z < 1$. The following lemma gives us examples of functions which attain the critical exponent shown in the last lemma, but still belong to the space (see [20]).

Lemma 4.5. *Let* $0 < p \leq \infty$, $0 < \alpha < \infty$. *The functions*

$$f(z) = \frac{1}{(1-z)^{\alpha+1/p}} \left(\log \frac{e}{1-z} \right)^{-c}$$

belong to $H(p,q,\alpha)$ *if and only if* $c > 1/q$ *for* $q < \infty$, *and* $c \geq 0$ *for* $q = \infty$.

Another well-known class of analytic functions is the class of lacunary series. By a classical theorem of Paley [78, Theorem 6.2.2], such series belongs to the Hardy space H^p if and only if the sequence formed by its coefficients belongs to the ℓ_2 space. In that case (and only then) the function has radial limits almost everywhere, and otherwise, has radial limits almost nowhere. The following result appears in [78, Thm. 8.1.1], based on [90].

Lemma 4.6. *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^{2^{n-1}}$ *and* $0 < p,q \leq \infty$, $0 < \alpha < \infty$. *Then* $f \in H(p,q,\alpha)$ *if and only if* $\{2^{-n\alpha} a_n\} \in \ell_q$.

In particular, there are functions with radial limits almost nowhere in each $H(p,q,\alpha)$ with $\alpha > 0$ (for instance, the lacunary series with coefficients equal to 1 satisfies $\sum_{n=0}^{\infty} 2^{-n\alpha q} |a_n|^q < \infty$ for every $0 < p,q \leq \infty$, $0 < \alpha < \infty$, but $\sum_{n=0}^{\infty} |a_n|^2 = \infty$). Therefore, the Hardy space does not contain any $H(p,q,\alpha)$ with $\alpha > 0$.

Other examples of functions in the mixed norm spaces can be found via derivatives. The following lemma, based on a theorem by Hardy and Littlewood (see [57, Thm. 5.5]), characterizes the inclusion of derivatives of functions in a mixed norm space to a space of the same family.

Lemma 4.7. *For* $0 < p \leq \infty$ *and* $\alpha > 0$,

(1) $M_p(r,f) = O(1-r)^{-\alpha} \Leftrightarrow M_p(r,f') = O(1-r)^{-(\alpha+1)}$ *(that is,* $f \in H(p,\infty,\alpha)$ *if and only if* $f' \in H(p,\infty,\alpha+1)$*).*

(2) $M_p(r,f) = o(1-r)^{-\alpha} \Leftrightarrow M_p(r,f') = o(1-r)^{-(\alpha+1)}$ *(that is,* $f \in H_0(p,\infty,\alpha)$ *if and only if* $f' \in H_0(p,\infty,\alpha+1)$*).*

In particular, the differentiation operator D, given by $Df(z) = f'(z)$, $z \in \mathbb{D}$, is bounded from $H(p,\infty,\alpha)$ (resp. $H_0(p,\infty,\alpha)$) to $H(p,\infty,\alpha+1)$ (resp. $H_0(p,\infty,\alpha+1)$).

4.2 Pointwise and mean estimates

If f is a function in $H(p,q,\alpha)$, we have the following estimate for its integral means.

Lemma 4.8. *If $f \in H(p,q,\alpha)$, $0 < p \leq \infty$, $0 < q, \alpha < \infty$, then*

$$M_p(r,f) = o\left((1-r)^{-\alpha}\right)$$

as $r \to 1$.

Proof. Since the integral

$$\alpha q \int_0^r (1-\rho)^{\alpha q - 1} M_p^q(\rho, f)\, d\rho$$

converges to $\|f\|_{p,q,\alpha}^q$ as $r \to 1$, then for every $\varepsilon > 0$ there exists r_0 such that

$$\alpha q \int_r^1 (1-\rho)^{\alpha q - 1} M_p^q(\rho, f)\, d\rho < \varepsilon \tag{4.1}$$

for every $r > r_0$. Therefore, since the integral means are increasing as functions of r, we get

$$\begin{aligned}(1-r)^{\alpha q} M_p^q(r,f) &= \alpha q \int_r^1 (1-\rho)^{\alpha q - 1} M_p^q(r, f)\, d\rho \\ &\leq \alpha q \int_r^1 (1-\rho)^{\alpha q - 1} M_p^q(\rho, f)\, d\rho < \varepsilon.\end{aligned}$$

□

Moreover, it follows from the proof that if $f \in H(p,q,\alpha)$, then

$$M_p(r,f) \leq \frac{\|f\|_{p,q,\alpha}}{(1-r)^\alpha} \tag{4.2}$$

since we can bound the integral in (4.1) by the norm of f instead of ε. Notice that, taking supremum over r, we get

$$\|f\|_{p,\infty,\alpha} \leq \|f\|_{p,q,\alpha}, \tag{4.3}$$

and therefore $H(p,q,\alpha) \subseteq H(p,\infty,\alpha)$ for every $0 < p, q \leq \infty$, $0 < \alpha < \infty$.

Although the result in Lemma 4.8 fails for $q = \infty$ as the function $f(z) = (1-z)^{-\alpha-1/p}$ shows, the above bound for the integral mean still holds since

$$\|f\|_{p,\infty,\alpha} = \sup_{0 \leq \rho < 1} (1-\rho)^\alpha M_p(\rho, f) \geq (1-r)^\alpha M_p(r,f)$$

for any r, $0 < r < 1$, and therefore

$$M_p(r,f) \leq \frac{\|f\|_{p,\infty,\alpha}}{(1-r)^\alpha} \tag{4.4}$$

for $f \in H(p,\infty,\alpha)$.

We can obtain an analogous result to the little-oh estimation in Hardy and Bergman spaces if $q < \infty$.

Proposition 4.9. *If $f \in H(p,q,\alpha)$, $0 < p \le \infty$, $0 < q, \alpha < \infty$, then*

$$|f(z)| = o\left((1-|z|)^{\alpha+1/p}\right)$$

as $|z| \to 1$.

In the proof we will use the following identity.

Lemma 4.10. *For $0 < p, q, \alpha < \infty$ and $z \in \mathbb{D}$,*

$$\int_{|z|}^{1} (1-\rho)^{\alpha q-1}(\rho-|z|)^{q/p}\, d\rho = B(\alpha q, q/p+1)\,(1-|z|)^{\alpha q+q/p},$$

where $B(a,b) = \int_0^1 (1-x)^{a-1}x^{b-1}\,dx$, $a, b > 0$, is the Beta function.

Proof. With the change of variables $x = \frac{\rho-|z|}{1-|z|}$,

$$\begin{aligned}\int_{|z|}^{1} (1-\rho)^{\alpha q-1}(\rho-|z|)^{q/p}\, d\rho &= \int_0^1 (1-x)^{\alpha q-1}(1-|z|)^{\alpha q-1}x^{q/p}(1-|z|)^{q/p}(1-|z|)\,dx \\ &= (1-|z|)^{\alpha q+q/p}\int_0^1 (1-x)^{\alpha q-1}x^{q/p}\,dx.\end{aligned}$$

□

Next, we prove Proposition 4.9.

Proof of Proposition 4.9. If $p = \infty$, it is easy to see that, for r close enough to 1 (as in Lemma 4.8),

$$\begin{aligned}|f(re^{i\theta})|^q(1-r)^{\alpha q} &= \alpha q\,|f(re^{i\theta})|^q \int_r^1 (1-\rho)^{\alpha q-1}\,d\rho \\ &\le \alpha q \int_r^1 (1-\rho)^{\alpha q-1} M_\infty^q(\rho,f)\,d\rho < \varepsilon.\end{aligned} \tag{4.5}$$

If $0 < p < \infty$, we estimate the integral mean $M_p(r,f)$ using the Poisson integral: Let $\rho \in (0,1)$ and define $f_\rho(z) = f(\rho z)$, for $f \in \mathcal{H}(\mathbb{D})$ and $z \in \mathbb{D}$. Since $f_\rho \in H^\infty$ for any $f \in H(p,q,\alpha)$ and $r < \rho$ we have, as in [66],

$$\begin{aligned}|f(re^{i\theta})|^p &\le \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{it})|^p \frac{\rho^2-r^2}{|\rho - re^{i(\theta-t)}|^2}\,dt \le \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{it})|^p \frac{\rho^2-r^2}{(\rho-r)^2}\,dt \\ &\le \frac{2}{\rho-r}\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{it})|^p\,dt = \frac{2}{\rho-r}\,M_p^p(\rho,f).\end{aligned}$$

Hence,

$$|f(re^{i\theta})|(\rho-r)^{1/p} \le 2^{1/p} M_p(\rho,f). \tag{4.6}$$

Let $\varepsilon > 0$, then for r close to 1 we have, using the above estimate and Lemma 4.10, as in (4.1),

$$\begin{aligned} &\frac{\alpha q}{2^{q/p}} B(\alpha q, q/p+1)\,|f(re^{i\theta})|^q(1-r)^{\alpha q+q/p} \qquad (4.7)\\ &= \frac{\alpha q}{2^{q/p}}|f(re^{i\theta})|^q \int_r^1 (1-\rho)^{\alpha q-1}(\rho-r)^{q/p}\,d\rho \\ &\le \alpha q \int_r^1 (1-\rho)^{\alpha q-1} M_p^q(\rho,f)\,d\rho < \varepsilon. \end{aligned}$$

□

From the above proof, the following known pointwise estimate also follows: if we bound the integral in (4.7) by $\|f\|_{p,q,\alpha}$, then

$$|f(z)| \le m \frac{\|f\|_{p,q,\alpha}}{(1-|z|)^{\alpha+1/p}}, \qquad (4.8)$$

with

$$m = \begin{cases} \dfrac{2^{1/p}}{(\alpha q\, B(\alpha q, q/p+1))^{1/q}} & \text{if } p<\infty \\ 1 & \text{if } p=\infty. \end{cases} \qquad (4.9)$$

One should notice that, once again, this proposition does not hold for $q = \infty$, as the function $f(z) = (1-z)^{-\alpha-1/p}$ shows. However, the pointwise estimation is still true (see [78, Prop. 7.1.1]).

Note also that the norm of the point evaluation functional Λ_z can be estimated as follows:

$$\|\Lambda_z\| \le \frac{m}{(1-|z|)^{\alpha+\frac{1}{p}}}, \qquad (4.10)$$

with m as in (4.9).

Now, for a given z in $\mathbb{D}$, we will find a function f_z in $H(p,q,\alpha)$ with pointwise growth of maximal order, that is, $\|f_z\| \approx 1$ and

$$|f_z(z)| \approx \|\Lambda_z\| \approx \frac{1}{(1-|z|)^{\alpha+1/p}}.$$

Here we give a general version of a well-known fact for Bergman spaces.

Proposition 4.11. *For $z \in \mathbb{D}$, $0 < p, q \le \infty$, $0 < \alpha < \infty$ and $s > 0$, the functions*

$$f_z(w) = \frac{(1-|z|^2)^s}{(1-\bar{z}w)^{\alpha+\frac{1}{p}+s}}$$

satisfy $|f_z(z)| \approx \|\Lambda_z\|$ and $\|f_z\|_{p,q,\alpha} \approx 1$.

Proof. First we check that f_z belongs to $H(p,q,\alpha)$ and estimate its norm:

If $w = re^{i\theta}$, then, as in [57, page 65],

$$M_p^p(r,f_z) = \int_0^{2\pi} \frac{(1-|z|^2)^{ps}}{|1-\bar{z}re^{i\theta}|^{p(\alpha+\frac{1}{p}+s)}}\frac{d\theta}{2\pi} \approx \frac{(1-|z|^2)^{ps}}{(1-r|z|)^{p(\alpha+\frac{1}{p}+s)-1}} = \frac{(1-|z|^2)^{ps}}{(1-r|z|)^{(\alpha+s)p}},$$

for $p<\infty$ and

$$M_\infty(r,f_z) \approx \frac{(1-|z|^2)^s}{(1-r|z|)^{\alpha+s}}.$$

Therefore, if $q<\infty$ and $0<p\le\infty$,

$$\begin{aligned}\|f_z\|_{p,q,\alpha}^q &= \alpha q \int_0^1 (1-r)^{\alpha q-1} M_p^q(r,f_z)\,dr \\ &\approx \alpha q\,(1-|z|)^{sq}\int_0^1 (1-r)^{\alpha q-1}\frac{1}{(1-r|z|)^{(\alpha+s)q}}\,dr.\end{aligned}$$

Now, on the one hand,

$$\begin{aligned}\|f_z\|_{p,q,\alpha}^q &\approx \alpha q\,(1-|z|)^{sq}\int_0^1 (1-r)^{\alpha q-1}\frac{1}{(1-r|z|)^{(\alpha+s)q}}\,dr \\ &\ge \alpha q\,(1-|z|)^{sq}\int_{|z|}^1 (1-r)^{\alpha q-1}\frac{1}{(1-r|z|)^{(\alpha+s)q}}\,dr \\ &\ge \alpha q\,\frac{(1-|z|)^{sq}}{(1-|z|^2)^{(\alpha+s)q}}\int_{|z|}^1 (1-r)^{\alpha q-1}\,dr \\ &\approx \frac{1}{(1-|z|)^{\alpha q}}(1-|z|)^{\alpha q} = 1\end{aligned}$$

and, on the other hand, integrating by parts and using $(1-r)^{\alpha q} \le (1-r|z|)^{\alpha q}$,

$$\begin{aligned}\|f_z\|_{p,q,\alpha}^q &\approx (1-|z|)^{sq}\int_0^1 \alpha q\,(1-r)^{\alpha q-1}\frac{1}{(1-r|z|)^{(\alpha+s)q}}\,dr \\ &= (1-|z|)^{sq}\left(1-(\alpha+s)q\,|z|\int_0^1 (1-r)^{\alpha q}\frac{1}{(1-r|z|)^{(\alpha+s)q+1}}\,dr\right) \\ &\le (1-|z|)^{sq}\left(1-(\alpha+s)q\,|z|\int_0^1 (1-r|z|)^{-(sq+1)}\,dr\right) \\ &= (1-|z|)^{sq}\left(1-\frac{\alpha+s}{s}\left((1-|z|)^{-sq}-1\right)\right) \\ &= \left(1+\frac{\alpha+s}{s}\right)(1-|z|)^{sq} - \frac{\alpha+s}{s} \approx 1.\end{aligned}$$

If $q=\infty$,

$$\|f_z\|_{p,\infty,\alpha} = \sup_{0\le r<1}(1-r)^\alpha M_p(r,f_z) \approx \sup_{0\le r<1}(1-r)^\alpha\frac{(1-|z|)^s}{(1-r|z|)^{\alpha+s}}.$$

Since $1-r \leq 1-r|z|$ and $1-|z| \leq 1-r|z|$, we have

$$\|f_z\|_{p,\infty,\alpha} \approx \sup_{0\leq r<1}(1-r)^{\alpha}\frac{(1-|z|)^s}{(1-r|z|)^{\alpha+s}} \leq \sup_{0\leq r<1}(1-r|z|)^{\alpha}\frac{(1-r|z|)^s}{(1-r|z|)^{\alpha+s}} = 1$$

and

$$\begin{aligned}\|f_z\|_{p,\infty,\alpha} &\geq \sup_{|z|<r<1}(1-r)^{\alpha}\frac{(1-|z|)^s}{(1-r|z|)^{\alpha+s}} \geq \frac{(1-|z|)^s}{(1-|z|^2)^{\alpha+s}}\sup_{|z|<r<1}(1-r)^{\alpha}\\ &= \frac{(1-|z|)^s(1-|z|)^{\alpha}}{(1-|z|^2)^{\alpha+s}} \approx 1.\end{aligned}$$

Now that we know that $f_z \in H(p,q,\alpha)$, we see easily that

$$|f_z(z)| = \frac{(1-|z|^2)^s}{(1-|z|^2)^{\alpha+\frac{1}{p}+s}} = \frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}} \approx \frac{1}{(1-|z|)^{\alpha+\frac{1}{p}}}$$

and from here

$$\|\Lambda_z\| \approx \|\Lambda_z\|\|f_z\|_{p,q,\alpha} \geq |f_z(z)| \approx \frac{1}{(1-|z|)^{\alpha+\frac{1}{p}}}.$$

With (4.10), we get

$$|f_z(z)| \approx \|\Lambda_z\| \approx \frac{1}{(1-|z|)^{\alpha+\frac{1}{p}}}.$$

□

4.3 Inclusions between mixed norm spaces

In this section, we determine the inclusions between different mixed norm spaces depending on the three parameters. The main theorems are the following. To avoid repetitions, we recall here that we are assuming our parameters to be $0 < \alpha, \beta < \infty$ and $0 < p, q, u, v \leq \infty$.

Theorem 4.12. *If $p \geq u$, then*

$$H(p,q,\alpha) \subseteq H(u,v,\beta) \Leftrightarrow \begin{cases}\alpha < \beta & \textit{or}\\ \alpha = \beta & \textit{and} \quad q \leq v.\end{cases}$$

Theorem 4.13. *If $p < u$, then*

$$H(p,q,\alpha) \subseteq H(u,v,\beta) \Leftrightarrow \begin{cases}\alpha + \frac{1}{p} < \beta + \frac{1}{u} & \textit{or}\\ \alpha + \frac{1}{p} = \beta + \frac{1}{u} & \textit{and} \quad q \leq v.\end{cases}$$

It is worth noticing that we need α to be greater than zero as we stated when these spaces were defined. In the limit case $\alpha = 0$, by a theorem by Hardy and Littlewood (related to the Isoperimetric Inequality, see [89], [126]), we have $H^p \subseteq A^{2p}$. That is, $H(p,\infty,0) \subseteq H(2p,2p,1/2p)$, although these parameters do not satisfy Theorem 4.13.

Notice also that it is only to be expected that the relation between the spaces would depend on the relation between the parameters p and u, since, ultimately, in order to compare the different spaces we need to compare the sizes of the integral means. In turn, the integral means relate in a different fashion according to the parameters p and u.

Therefore, in order to prove these theorems we will need the following estimates of the integral means, which can be found in the literature (see [57, Thm. 5.9], [72]).

Lemma 4.14. *If $f \in H(p,q,\alpha)$ and $q \leq v < \infty$, then*

$$M_p^v(r,f) \leq \|f\|_{p,q,\alpha}^{v-q} (1-r)^{-\alpha(v-q)} M_p^q(r,f).$$

Proof. If $f \in H(p,q,\alpha)$, by the bound on the integral mean (4.2)

$$M_p(r,f) \leq \|f\|_{p,q,\alpha} (1-r)^{-\alpha},$$

and since $q \leq v < \infty$,

$$M_p^v(r,f) = M_p^{v-q}(r,f) M_p^q(r,f) \leq \|f\|_{p,q,\alpha}^{v-q} (1-r)^{-\alpha(v-q)} M_p^q(r,f).$$

□

If f belongs to $H(p,q,\alpha)$ and $u > p$ we have the following bound for $M_u(r,f)$.

Lemma 4.15. *If $f \in H(p,q,\alpha)$ and $p < u$, then*

$$M_u(r,f) \leq m^{1-\frac{p}{u}} \|f\|_{p,q,\alpha} (1-r)^{-\alpha+\frac{1}{u}-\frac{1}{p}},$$

where

$$m = \frac{2^{1/p}}{(\alpha q\, B(\alpha q, q/p+1))^{1/q}}.$$

Proof. The pointwise inequality (4.8)

$$M_\infty(r,f) \leq m \|f\|_{p,q,\alpha} (1-r)^{-\alpha-\frac{1}{p}}$$

is the case $u = \infty$. Now if $u < \infty$,

$$\begin{aligned} M_u(r,f) &= \left(\int_0^{2\pi} |f(re^{i\theta})|^{u-p} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/u} \leq M_\infty^{1-\frac{p}{u}}(r,f)\, M_p^{\frac{p}{u}}(r,f) \qquad (4.11) \\ &\leq m^{1-\frac{p}{u}} \|f\|_{p,q,\alpha}^{1-\frac{p}{u}} (1-r)^{(1-\frac{p}{u})(-\alpha-\frac{1}{p})} \|f\|_{p,q,\alpha}^{\frac{p}{u}} (1-r)^{-\alpha\frac{p}{u}} \\ &= m^{1-\frac{p}{u}} \|f\|_{p,q,\alpha} (1-r)^{-\alpha+\frac{1}{u}-\frac{1}{p}}. \end{aligned}$$

□

The growth property for the integral means is well known: for an analytic function f on the disk, $M_p(r,f) \leq M_u(r,f)$ when $p \leq u$. Furthermore, if f belongs to $H(p,q,\alpha)$, we also have the following property that quantifies how the integral means decrease.

Lemma 4.16 (Lemma 5, [16])**.** *If $0 < p \le u \le \infty$, then there exists a constant $C > 0$ such that, for every $f \in \mathcal{H}(\mathbb{D})$ and $0 < r < 1$,*

$$M_u(r, f) \le C(1-r)^{\frac{1}{u}-\frac{1}{p}} M_p(r, f).$$

Now we can prove the theorems.

Proof of Theorem 4.12. Throughout this proof, we will assume that $p \ge u$. The key to proving the sufficiency is the inequality of the integral means: if $p \ge u$, then $M_u(r, f) \le M_p(r, f)$.

We suppose first that $\alpha < \beta$. Then, since $M_p(r, f) \le \|f\|_{p,q,\alpha}(1-r)^{-\alpha}$ by (4.2), we have that, if v is finite,

$$\begin{aligned}
\|f\|_{u,v,\beta}^v &= \beta v \int_0^1 (1-r)^{\beta v-1} M_u^v(r, f)\, dr \le \beta v \int_0^1 (1-r)^{\beta v-1} M_p^v(r, f)\, dr \\
&\le \beta v\, \|f\|_{p,q,\alpha}^v \int_0^1 (1-r)^{\beta v-1}(1-r)^{-\alpha v}\, dr \\
&= \beta v\, \|f\|_{p,q,\alpha}^v \int_0^1 (1-r)^{v(\beta-\alpha)-1}\, dr = \frac{\beta}{\beta-\alpha}\|f\|_{p,q,\alpha}^v,
\end{aligned}$$

and, by (4.3),

$$\|f\|_{u,\infty,\beta} = \sup_{0\le r<1} (1-r)^\beta M_u(r, f) \le \sup_{0\le r<1} (1-r)^\alpha M_p(r, f) = \|f\|_{p,\infty,\alpha} \le \|f\|_{p,q,\alpha}.$$

Therefore, $f \in H(u, v, \beta)$ for every $f \in H(p, q, \alpha)$.

Now, if $\alpha = \beta$ and $q \le v$, by Lemma 4.14,

$$\begin{aligned}
\|f\|_{u,v,\beta}^v &= \beta v \int_0^1 (1-r)^{\beta v-1} M_u^v(r, f)\, dr \le \beta v \int_0^1 (1-r)^{\beta v-1} M_p^v(r, f)\, dr \\
&\le \beta v\, \|f\|_{p,q,\alpha}^{v-q} \int_0^1 (1-r)^{\beta v-1}(1-r)^{-\alpha(v-q)} M_p^q(r, f)\, dr \\
&= \beta v\, \|f\|_{p,q,\alpha}^{v-q} \int_0^1 (1-r)^{\alpha q-1} M_p^q(r, f)\, dr \\
&= \frac{\beta v}{\alpha q}\, \|f\|_{p,q,\alpha}^{v-q}\, \|f\|_{p,q,\alpha}^q = \frac{v}{q}\, \|f\|_{p,q,\alpha}^v
\end{aligned}$$

if $v < \infty$, and, again by (4.3),

$$\|f\|_{u,\infty,\beta} = \sup_{0\le r<1} (1-r)^\beta M_u(r, f) \le \sup_{0\le r<1} (1-r)^\alpha M_p(r, f) = \|f\|_{p,\infty,\alpha} \le \|f\|_{p,q,\alpha}.$$

Hence, in both cases $H(p, q, \alpha) \subseteq H(u, v, \beta)$, and the sufficiency is proven.

For the necessity, we need to see that $H(p, q, \alpha) \not\subseteq H(u, v, \beta)$ when the parameters do not relate as in the statement of the theorem. For this, consider a function of type $f(z) = \sum_{n=1}^\infty a_n\, z^{2^{n-1}}$ as in Lemma 4.6. Recall that f belongs to $H(p, q, \alpha)$ if and only if $\{2^{-\alpha n} a_n\} \in \ell_q$.

If $\alpha > \beta$, let $f(z) = \sum_{n=1}^{\infty} 2^{n\beta} z^{2^{n-1}}$. Since

$$\left\{\frac{a_n}{2^{\alpha n}}\right\} = \left\{\frac{1}{2^{n(\alpha-\beta)}}\right\} \in \ell_q,$$

the function f belongs to $H(p,q,\alpha)$, but

$$\left\{\frac{a_n}{2^{\beta n}}\right\} = \{1\} \notin \ell_v,$$

so this function does not belong to $H(u,v,\beta)$, and therefore $H(p,q,\alpha) \not\subseteq H(u,v,\beta)$ if $\alpha > \beta$.

If $\alpha = \beta$ and $q > v$, we take $f(z) = \sum_{n=1}^{\infty} 2^{n\alpha} n^{-1/v} z^{2^{n-1}}$. Similarly,

$$\left\{\frac{a_n}{2^{\alpha n}}\right\} = \left\{\frac{2^{n\alpha} n^{-1/v}}{2^{n\alpha}}\right\} = \left\{\frac{1}{n^{1/v}}\right\} \in \ell_q,$$

and $f \in H(p,q,\alpha)$, but

$$\left\{\frac{a_n}{2^{\beta n}}\right\} = \left\{\frac{2^{n\alpha} n^{-1/v}}{2^{\beta n}}\right\} = \left\{\frac{1}{n^{1/v}}\right\} \notin \ell_v,$$

so it does not belong to $H(u,v,\beta)$, and hence $H(p,q,\alpha) \not\subseteq H(u,v,\beta)$ if $\alpha = \beta$ and $q > v$.

□

Proof of Theorem 4.13. As in the last proof, from now on we will assume $p < u$. Firstly we shall see that if $f \in H(p,q,\alpha)$ and the parameters are ordered as in the statement, then $f \in H(u,v,\beta)$.

If $\alpha < \beta + \frac{1}{u} - \frac{1}{p}$, by Lemma 4.15,

$$\begin{aligned}
\|f\|_{u,v,\beta}^v &= \beta v \int_0^1 (1-r)^{\beta v - 1} M_u^v(r,f)\, dr \\
&\le \beta v\, m^{v(1-\frac{p}{u})} \|f\|_{p,q,\alpha}^v \int_0^1 (1-r)^{\beta v-1} (1-r)^{v(-\alpha+\frac{1}{u}-\frac{1}{p})}\, dr \\
&= \beta v\, m^{v(1-\frac{p}{u})} \|f\|_{p,q,\alpha}^v \int_0^1 (1-r)^{v(\beta-\alpha+\frac{1}{u}-\frac{1}{p})-1}\, dr = \frac{\beta\, m^{v(1-\frac{p}{u})}}{\beta-\alpha+\frac{1}{u}-\frac{1}{p}} \|f\|_{p,q,\alpha}^v
\end{aligned}$$

for $v < \infty$, and

$$\begin{aligned}
\|f\|_{u,\infty,\beta} &= \sup_{0\le r<1} (1-r)^{\beta} M_u(r,f) \\
&\le m^{1-\frac{p}{u}} \|f\|_{p,q,\alpha} \sup_{0\le r<1} (1-r)^{\beta}(1-r)^{-\alpha+\frac{1}{u}-\frac{1}{p}} = m^{1-\frac{p}{u}} \|f\|_{p,q,\alpha}.
\end{aligned}$$

If $\alpha = \beta + \frac{1}{u} - \frac{1}{p}$ and $q \le v$, by Lemmas 4.16 and 4.14,

$$\begin{aligned}
\|f\|_{u,v,\beta}^v &= \beta v \int_0^1 (1-r)^{\beta v - 1} M_u^v(r,f)\,dr \\
&\le \beta v\, C^v \int_0^1 (1-r)^{\beta v-1}(1-r)^{v(\frac{1}{u}-\frac{1}{p})} M_p^v(r,f)\,dr \\
&= \beta v\, C^v \int_r^1 (1-r)^{\alpha v-1} M_p^v(r,f)\,dr \\
&\le \beta v\, C^v \, \|f\|_{p,q,\alpha}^{v-q} \int_0^1 (1-r)^{\alpha v-1}(1-r)^{-\alpha(v-q)} M_p^q(r,f)\,dr \\
&= \frac{\beta v}{\alpha q}\, C^v \, \|f\|_{p,q,\alpha}^v .
\end{aligned}$$

If $v = \infty$, in a similar way,

$$\begin{aligned}
\|f\|_{u,\infty,\beta} = \sup_{0\le r<1} (1-r)^\beta M_u(r,f) &\le C \sup_{0\le r<1} (1-r)^\beta (1-r)^{\frac{1}{u}-\frac{1}{p}} M_p(r,f) \\
= C \sup_{0\le r<1} (1-r)^\alpha M_p(r,f) &= C\,\|f\|_{p,\infty,\alpha} \le C\,\|f\|_{p,q,\alpha}.
\end{aligned}$$

Finally, we need to see that $H(p,q,\alpha) \not\subseteq H(u,v,\beta)$ when the parameters are not as in the assumptions of the statement. If $\alpha + \frac{1}{p} > \beta + \frac{1}{u}$, Lemma 4.4 tells us that

$$f(z) = \frac{1}{(1-z)^{\beta+1/u}}$$

belongs to $H(p,q,\alpha)$ but not $H(u,v,\beta)$, and this proves that $H(p,q,\alpha) \not\subseteq H(u,v,\beta)$ when $\alpha + \frac{1}{p} > \beta + \frac{1}{u}$.

If $\alpha + \frac{1}{p} = \beta + \frac{1}{u}$ and $q > v$, by Lemma 4.5 the function

$$f(z) = \frac{1}{(1-z)^{\alpha+1/p}} \left(\log \frac{e}{1-z} \right)^{-1/v}$$

is an example of a function in $H(p,q,\alpha)$ which is not in $H(u,v,\beta)$, and hence $H(p,q,\alpha) \not\subseteq H(u,v,\beta)$ for $\alpha + \frac{1}{p} = \beta + \frac{1}{u}$ and $q > v$.

□

Chapter 5

Semigroups of composition operators on mixed norm spaces

This chapter deals with the semigroups of composition operators on mixed norm spaces. As we saw in Section 2.5.2, given a family of analytic self-maps of the disk $\mathbb{D}$ $\{\varphi_t\}$, $t \geq 0$, such that φ_0 is the identity, $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$, and $\varphi_t \to \varphi_0$ as $t \to 0$ uniformly on compact sets of $\mathbb{D}$ (that is, a semigroup of analytic functions) then, if for every $t \geq 0$ the composition operator $\mathbf{C}_{\varphi_t} = \mathbf{C}_t$ is bounded on $H(p,q,\alpha)$, then the family $\{\mathbf{C}_t\}$ is a semigroup of composition operators. We are interested in the strong and uniform continuity of the semigroup, that is, in knowing whether

$$\lim_{t\to 0^+} \|\mathbf{C}_t f - f\|_{p,q,\alpha} = 0$$

for every $f \in H(p,q,\alpha)$, or

$$\lim_{t\to 0^+} \|\mathbf{C}_t - I\|_{p,q,\alpha} = 0$$

where I is the identity operator.

We already saw an example of semigroup of composition operators on mixed norm spaces.

Proposition 4.2. *For $0 \leq r < 1$, let $f_r(z) = f(rz)$, $z \in \mathbb{D}$.*

- *If $f \in H(p,q,\alpha)$, $0 < p \leq \infty$, $0 < q, \alpha < \infty$, then $\|f_r - f\|_{p,q,\alpha} \to 0$, as $r \to 1$.*
- *If $f \in H_0(p,\infty,\alpha)$, $0 < p \leq \infty$, $0 < \alpha < \infty$, then $\|f_r - f\|_{p,\infty,\alpha} \to 0$, as $r \to 1$.*

Moreover, if $f \in H(p,\infty,\alpha)$ and $\|f_r - f\|_{p,\infty,\alpha} \to 0$, as $r \to 1$, then $f \in H_0(p,\infty,\alpha)$.

This result can be also written in terms of semigroups of composition operators on mixed norm spaces. If we define $\varphi_t(z) = e^{-t}z$, for all $t \geq 0$ and $z \in \mathbb{D}$, then $f_r = \mathbf{C}_{\varphi_t} f$ for $t = \log r$, and therefore $\{\varphi_t\}$ induces a strongly continuous semigroup of composition operators on $H(p,q,\alpha)$ for $q < \infty$ and

$$[\varphi_t, H(p,\infty,\alpha)] = H_0(p,\infty,\alpha).$$

Recall that the maximal closed linear subspace $[\varphi_t, X]$ of a Banach space X with respect to the semigroup $\{\varphi_t\}$, defined in Section 2.5.2, is the subspace such that $\{\varphi_t\}$ generates a strongly continuous semigroup of operators on it, that is,

$$[\varphi_t, X] = \{f \in X : \|f \circ \varphi_t - f\|_X \to 0 \text{ as } t \to 0\}.$$

In this chapter we will see that this is the expected behavior for every semigroup of analytic functions: they induce strongly continuous semigroups of composition operators on $H(p, q, \alpha)$ for $q < \infty$ and on $H_0(p, \infty, \alpha)$ (separable spaces), but not in $H(p, \infty, \alpha)$, a non-separable space. To do this, in the first section we will check that every self-map of the disk induces a bounded composition operator, and therefore every semigroup of analytic functions induces a semigroup of bounded composition operators. In the second section we will deal with separable spaces and will give a general theorem on strong continuity of semigroups of composition operators for separable spaces. To study the non-separable cases, we will use the characterizations given in Section 2.5.2 in terms of an integral operator, so the third section is devoted to the study of the integral operators on $H(p, \infty, \alpha)$. Finally, in the last section we will use the results from the previous section to characterize the semigroups of analytic functions that induce strongly continuous semigroups of composition operators.

The results in this chapter can be found in [12].

5.1 Composition operators on mixed norm spaces

For our study of semigroups of composition operators, the first thing we need to check is that these operators are bounded on the mixed norm spaces and to find a bound on the norm.

Proposition 5.1. *Suppose* $0 < p, q \leq \infty$ *and* $0 < \alpha < \infty$*, and let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an analytic function. Then* $\boldsymbol{C}_\varphi$ *is bounded on* $H(p, q, \alpha)$ *and on* $H_0(p, \infty, \alpha)$*. Moreover, it holds that*

$$\|\boldsymbol{C}_\varphi\| \lesssim \left(\frac{\|\varphi\|_\infty + |\varphi(0)|}{\|\varphi\|_\infty - |\varphi(0)|}\right)^{\alpha + \frac{1}{p}}.$$

If, in addition, $\varphi(0) = 0$*, then* $\|\boldsymbol{C}_\varphi\| = 1$*.*

Proof. If $\varphi(0) = 0$, then by Littlewood's Subordination Theorem 2.1, $M_p(r, f \circ \varphi) \leq M_p(r, f)$, and $\|\mathbf{C}_\varphi\| \leq 1$. The constant function $f(z) \equiv 1$ shows the equality can hold.

If $\varphi(0) \neq 0$, we get the bound arguing as in [114, Lemma 1]. Assume firstly that $||\varphi||_\infty = 1$. Fix $0 < r < 1$. Applying the Schwarz-Pick Lemma, we have

$$|\varphi(z)| \leq \frac{|\varphi(0)| + r}{1 + |\varphi(0)|r}.$$

Now, since

$$\frac{|\varphi(0)| + r}{1 + |\varphi(0)|r} \leq \frac{(1 - |\varphi(0)|)r + 2|\varphi(0)|}{1 + |\varphi(0)|} = R$$

we have $|\varphi(z)| \leq R$ for $|z| \leq r$. From here clearly $M_\infty(r, f\circ\varphi) \leq M_\infty(R, f)$. For $0 < p < \infty$, let u be the harmonic majorant of $|f|^p$ on $|z| \leq R$ such that $u = |f|^p$ on $|z| = R$. Then $|f(z)|^p \leq u(z)$ on $|z| \leq R$, and hence $|f(\varphi(z))|^p \leq u(\varphi(z))$ for $|z| \leq r$. Therefore,

$$M_p^p(r, f\circ\varphi) = \int_0^{2\pi} |f(\varphi(re^{i\theta}))|^p \frac{d\theta}{2\pi} \leq \int_0^{2\pi} u(\varphi(re^{i\theta})) \frac{d\theta}{2\pi} = u(\varphi(0)).$$

To simplify the notation, let us write

$$\frac{R + |\varphi(0)|}{R - |\varphi(0)|} = \psi(R).$$

Then, by Harnack's inequality,

$$u(\varphi(0)) \leq \psi(R)\, u(0) = \psi(R) \int_0^{2\pi} |f(Re^{i\theta})|^p \frac{d\theta}{2\pi}.$$

Therefore,

$$M_p(r, f\circ\varphi) \leq \psi(R)^{\frac{1}{p}} M_p(R, f)$$

for $p < \infty$ and

$$M_\infty(r, f\circ\varphi) \leq M_\infty(R, f).$$

Using that ψ is a decreasing function, we obtain

$$\psi(R) = \frac{R + |\varphi(0)|}{R - |\varphi(0)|} \leq \frac{3 + |\varphi(0)|}{1 - |\varphi(0)|}$$

for $R \in \left[\frac{2|\varphi(0)|}{1+|\varphi(0)|}, 1\right]$.

Now we can use these inequalities (assuming $1/\infty = 0$ so $\psi(R)^{\frac{1}{\infty}} = 1$) to bound the norm of $f\circ\varphi$ by the norm of f. For $q < \infty$,

$$\|f\circ\varphi\|_{p,q,\alpha}^q = \alpha q \int_0^1 (1-r)^{\alpha q - 1} M_p^q(r, f\circ\varphi)\, dr \leq \alpha q \int_0^1 (1-r)^{\alpha q - 1} \psi(R)^{\frac{q}{p}} M_p^q(R, f)\, dr.$$

Next we use the change of variables

$$t = R = \frac{(1 - |\varphi(0)|)r + 2|\varphi(0)|}{1 + |\varphi(0)|}$$

getting

$$1 - r = \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}(1-t) \quad \text{and} \quad dr = \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} dt.$$

Hence, the norm of the composition operator can be bounded as follows

$$\begin{aligned}
\|f\circ\varphi\|_{p,q,\alpha}^q &\leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{\alpha q} \alpha q \int_{\frac{2|\varphi(0)|}{1+|\varphi(0)|}}^1 (1-t)^{\alpha q - 1} \psi(t)^{\frac{q}{p}} M_p^q(t, f)\, dt \\
&\leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{\alpha q} \left(\frac{3 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{\frac{q}{p}} \alpha q \int_0^1 (1-t)^{\alpha q - 1} M_p^q(t, f)\, dt \\
&\leq C \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{\alpha q + \frac{q}{p}} \|f\|_{p,q,\alpha}^q.
\end{aligned}$$

And for $H(p,\infty,\alpha)$, just like above,

$$\begin{aligned}\|f\circ\varphi\|_{p,\infty,\alpha} &= \sup_{0<r<1}(1-r)^{\alpha}M_p(r,f) \le \sup_{0<r<1}(1-r)^{\alpha}\psi(R)^{\frac{1}{p}}M_p(R,f)\\ &= \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha}\sup_{\frac{2|\varphi(0)|}{1+|\varphi(0)|}<t<1}(1-t)^{\alpha}\left(\frac{t+|\varphi(0)|}{t-|\varphi(0)|}\right)^{\frac{1}{p}}M_p(t,f)\\ &\le C\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha+\frac{1}{p}}\|f\|_{p,\infty,\alpha}.\end{aligned}$$

If $||\varphi||_\infty < 1$, writing $U(z) = ||\varphi||_\infty z$, then $\mathbf{C}_\varphi = \mathbf{C}_U \circ \mathbf{C}_{\varphi/||\varphi||}$. Applying the previous two cases, we conclude the result.

The proof for $H_0(p,\infty,\alpha)$ is analogous. □

5.2 Semigroups of composition operators on separable mixed norm spaces

Our first goal is to study the semigroups of composition operators on mixed norm spaces for $q < \infty$. We will see that these spaces behave as the Hardy and Bergman spaces studied in [25] and [114].

The next result will help us prove the strong continuity of semigroups of composition operators on Banach spaces of analytic functions where polynomials are dense. The ideas behind the next proposition are taken from [116].

Proposition 5.2. *Let $\{\varphi_t\}$ be a semigroup of analytic functions in the unit disk and let X be a Banach space of analytic functions such that*

(i) Polynomials are dense in X;

(ii) There is a constant $C > 0$ such that if f and g belong to X and $|f| \le |g|$, then $\|f\|_X \le C\|g\|_X$;

(iii) $M := \limsup_{t\to 0^+} ||\boldsymbol{C}_t|| < +\infty$;

(iv) $\lim_{t\to 0^+} ||\varphi_t - \varphi_0||_X = 0$.

Then the semigroup of operators $\{\boldsymbol{C}_t\}$ is strongly continuous on X.

Proof. We have to prove that, given $f \in X$, it is satisfied that

$$\|f\circ\varphi_t - f\|_X \to 0 \quad \text{as } t \to 0.$$

Let us fix $n \in \mathbb{N}$. We have that

$$\varphi_t^n(z) - z^n = \left(\sum_{k=0}^{n-1}\varphi_t^k(z)z^{n-1-k}\right)(\varphi_t(z) - z).$$

Hence,

$$|\varphi_t^n(z) - z^n| \leq n|\varphi_t(z) - z|.$$

By (ii) and (iv), we deduce that

$$\|\varphi_t^n - \varphi_0^n\|_X \leq nC\|\varphi_t - \varphi_0\|_X \to 0 \quad \text{as } t \to 0.$$

Then, for every polynomial p, we have that

$$\|p \circ \varphi_t - p\|_X \to 0. \tag{5.1}$$

Fix $\varepsilon > 0$. Since polynomials are dense in X (by (i)), for every $f \in X$ there exists a polynomial p such that $||p - f||_X < \varepsilon$ and

$$\begin{aligned}\|f \circ \varphi_t - f\|_X &\leq \|f \circ \varphi_t - p \circ \varphi_t\|_X + \|p \circ \varphi_t - p\|_X + \|p - f\|_X \\ &\leq (\|\mathbf{C}_t\| + 1)\|p - f\|_X + \|p \circ \varphi_t - p\|_X \leq (\|\mathbf{C}_t\| + 1)\varepsilon + \|p \circ \varphi_t - p\|_X.\end{aligned}$$

By (iii) and (5.1), we get $\limsup_{t\to 0^+} \|f \circ \varphi_t - f\|_X \leq M\varepsilon$. The arbitrariness of ε implies that $\lim_{t\to 0^+} \|f \circ \varphi_t - f\|_X = 0$. □

It is easy to see that, for $q < \infty$, the mixed norm spaces $H(p, q, \alpha)$ satisfy the axioms of Proposition 5.2.

Theorem 5.3. (a) *Every semigroup of analytic functions generates a strongly continuous semigroup of operators on $H(p, q, \alpha)$ for $q < \infty$.*

(b) *No non-trivial semigroup of analytic functions induces a uniformly continuous semigroup of composition operators on $H(p, q, \alpha)$ for $q < \infty$.*

Proof. a) Let $\{\varphi_t\}$ be a semigroup of analytic functions in the unit disk. By Proposition 5.1, we have that $\limsup_{t\to 0^+} ||\mathbf{C}_t|| < +\infty$. Moreover, since φ_t tends to φ_0 uniformly on compact subsets of the unit disk, $M_p(r, \varphi_t - \varphi_0) \to 0$, and by Lebesgue's Dominated Convergence Theorem, $\|\varphi_t - \varphi_0\|_{p,q,\alpha} \to 0$. Thus, by Proposition 5.2, the semigroup $\{\mathbf{C}_t\}$ is strongly continuous on $H(p, q, \alpha)$.

b) Let us recall that a semigroup of operators is uniformly continuous on X if its infinitesimal generator, defined in Section 2.5.1, is a bounded operator on it. On the other hand, given a semigroup $\{\varphi_t\}$ with infinitesimal generator G, the infinitesimal generator of $\{\mathbf{C}_t\}$ is given by $A(f) = Gf'$. So, suppose that A is bounded on $H(p, q, \alpha)$. Then

$$||Af||_{p,q,\alpha} = ||Gf'||_{p,q,\alpha} \leq ||A||||f||_{p,q,\alpha}$$

for every $f \in H(p, q, \alpha)$. In particular, for $f_n(z) = z^n$ we have $\|A\|^q\|f_n\|_{p,q,\alpha}^q \geq n^q\|Gf_{n-1}\|_{p,q,\alpha}^q$. Now let $\delta \in (0, 1)$ be such that, for every $n \in \mathbb{N}$,

$$\int_\delta^1 (1 - r)^{\alpha q - 1} r^{(n-1)q} dr \geq \frac{1}{2}\int_0^1 (1 - r)^{\alpha q - 1} r^{(n-1)q} dr.$$

Hence, since the integral means are increasing functions of r,

$$\begin{aligned}\|A\|^q\|f_n\|_{p,q,\alpha}^q \geq n^q\|Gf_{n-1}\|_{p,q,\alpha}^q &= \alpha q n^q \int_0^1 (1-r)^{\alpha q-1} r^{(n-1)q} M_p^q(r,G)\,dr\\ &\geq \alpha q n^q \int_\delta^1 (1-r)^{\alpha q-1} r^{(n-1)q} M_p^q(r,G)\,dr\\ &\geq \alpha q n^q M_p^q(\delta,G) \int_\delta^1 (1-r)^{\alpha q-1} r^{(n-1)q} dr\\ &\geq \frac{\alpha q n^q}{2} M_p^q(\delta,G) \int_0^1 (1-r)^{\alpha q-1} r^{(n-1)q} dr\\ &\geq \frac{\alpha q n^q}{2} M_p^q(\delta,G) \int_0^1 (1-r)^{\alpha q-1} r^{nq} dr = \frac{1}{2} n^q \|f_n\|_{p,q,\alpha}^q M_p^q(\delta,G).\end{aligned}$$

From here,

$$nM_p(\delta,G) \leq 2^{\frac{1}{q}}\|A\|$$

for $n \in \mathbb{N}$, therefore $M_p(\delta,G) = 0$ and that means $G \equiv 0$. □

The same result can be used to see that the separable spaces $H_0(p,\infty,\alpha)$ behave in a similar way.

Proposition 5.4. *Every semigroup of analytic functions generates a strongly continuous semigroup of operators on $H_0(p,\infty,\alpha)$, but no non-trivial semigroup of analytic functions induces a uniformly continuous semigroup of composition operators on it.*

Proof. Let $\{\varphi_t\}$ be a semigroup of analytic functions in the unit disk. By Proposition 5.1, we have that $\limsup_{t\to 0^+} ||\mathbf{C}_t|| < +\infty$. Let $\varepsilon > 0$ and $r_0 < 1$ such that $(1- r_0)^\alpha < \varepsilon/4$. Since $\varphi_t \to \varphi_0$ uniformly on compact sets, in particular in $\overline{D(0,r_0)}$, if t is small enough then $M_p(r,\varphi_t-\varphi_0) < \varepsilon$ for $r \leq r_0$. Hence, using that $M_p(r,\varphi_t-\varphi_0) \leq 2(M_p(r,\varphi_t)+M_p(r,\varphi_0)) \leq 4$, we have that

$$\begin{aligned}\|\varphi_t-\varphi_0\|_{p,\infty,\alpha} &= \sup_{0\leq r<1}(1-r)^\alpha M_p(r,\varphi_t-\varphi_0)\\ &= \max\left\{\sup_{0\leq r\leq r_0}(1-r)^\alpha M_p(r,\varphi_t-\varphi_0), \sup_{r_0<r<1}(1-r)^\alpha M_p(r,\varphi_t-\varphi_0)\right\}\\ &\leq \max\left\{\varepsilon \sup_{0\leq r\leq r_0}(1-r)^\alpha, 4\sup_{r_0<r<1}(1-r)^\alpha\right\} = \varepsilon.\end{aligned}$$

Thus, by Proposition 5.2, the semigroup $\{\mathbf{C}_t\}$ is strongly continuous on $H_0(p,\infty,\alpha)$.

As in the proof of Theorem 5.3 we will show that the infinitesimal generator of $\{\mathbf{C}_t\}$, $A(f)(z) = G(z)f'(z)$ with G the generator of the semigroup $\{\varphi_t\}$, is not a bounded operator on $H_0(p,\infty,\alpha)$. For this, suppose $\|Af\|_{p,\infty,\alpha} = \|Gf'\|_{p,\infty,\alpha} \leq \|A\|\|f\|_{p,\infty,\alpha}$ for every $f \in H_0(p,\infty,\alpha)$. In particular, if $f_n(z) = z^n$, $n \geq 1$, then $\|A\|\|f_n\|_{p,\infty,\alpha} \geq n\|Gf_{n-1}\|_{p,\infty,\alpha}$. Let $\delta \in (0,1)$ be such that, for all n,

$$\sup_{\delta<r<1}(1-r)^\alpha r^{n-1} \geq \frac{1}{2}\sup_{0<r<1}(1-r)^\alpha r^n.$$

Hence, since the integral means are increasing functions of r,

$$\begin{aligned}\|A\|\|f_n\|_{p,\infty,\alpha} &\geq n\|Gf_{n-1}\|_{p,\infty,\alpha} = n \sup_{0<r<1}(1-r)^\alpha M_p(r, f_{n-1}G) \\ &\geq n \sup_{\delta<r<1}(1-r)^\alpha r^{n-1} M_p(r,G) \geq n\, M_p(\delta, G) \sup_{\delta<r<1}(1-r)^\alpha r^{n-1} \\ &\geq \frac{n}{2} M_p(\delta, G) \sup_{0<r<1}(1-r)^\alpha r^n = \frac{n}{2} M_p(\delta, G)\|f_n\|_{p,\infty,\alpha}.\end{aligned}$$

That is,

$$nM_p(\delta, G) \leq 2\|A\|$$

for $n \in \mathbb{N}$, thus $M_p(\delta, G) = 0$ and $G \equiv 0$. $\square$

Nevertheless, we cannot apply the last theorem to $H(p, \infty, \alpha)$ since polynomials are not dense. In the following sections we will deal with this space.

5.3 Integral operators on mixed norm spaces

In Section 2.5.2 we defined the maximal closed linear subspace of X such that the semigroup $\{\varphi_t\}$ generates a strongly continuous semigroup of operators on it,

$$[\varphi_t, X] = \{f \in X : \|f \circ \varphi_t - f\|_X \to 0 \text{ as } t \to 0\}$$

and gave several characterizations. The first one is in terms of the infinitesimal generator of the semigroup $\{\varphi_t\}$,

$$[\varphi_t, X] = \overline{\{f \in X : Gf' \in X\}},$$

(see Theorem 2.21), as long as X is a Banach space of analytic functions that contains the constants and such that $M = \sup_{t\in[0,1]} \|\mathbf{C}_t\|_X < \infty$. Clearly the mixed norm spaces satisfy the conditions, by virtue of Theorem 5.1.

Another characterization is in terms of the integral operator V_g, which, for a holomorphic function g, is defined as

$$V_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\, d\zeta, \quad f \in \mathcal{H}(\mathbb{D})$$

for any $z \in \mathbb{D}$. According to Proposition 2.22, if the space X satisfies the conditions above and also that for each $b \in \mathbb{D}$, $f \in X \Leftrightarrow \frac{f(z)-f(b)}{z-b} \in X$ (a property that, again, the mixed norm spaces satisfy), then

$$[\varphi_t, X] = \overline{X \cap (V_\gamma(X) \oplus \mathbb{C})},$$

where

$$\gamma(z) = \int_0^z \frac{\zeta}{G(\zeta)} d\zeta = -\int_0^z \frac{1}{P(\zeta)} d\zeta$$

if the Denjoy-Wolff point of the semigroup is inside the unit disk, and

$$\gamma(z) = \int_0^z \frac{1}{(1-\zeta)^2 P(\zeta)} d\zeta$$

if it belongs to the boundary.

We will use the last characterization to find the maximal closed linear subspace $[\varphi_t, H(p, \infty, \alpha)]$, so first we will study the boundedness of the integral operator on mixed norm spaces. For the sake of completeness, we start with the spaces $H(p, q, \alpha)$ with $q < \infty$.

Recall that we denote by V the Volterra operator

$$V(f)(z) = \int_0^z f(\zeta)\, d\zeta.$$

Proposition 5.5. *Let g be an analytic function on the unit disk, V_g the integral operator induced by g, and $0 < p \leq \infty$, $0 < \alpha, q < \infty$.*

- *$V_g : H(p, q, \alpha) \to H(p, q, \alpha)$ is bounded if and only if $g \in \mathcal{B}$.*
- *$V_g : H(p, q, \alpha) \to H(p, q, \alpha)$ is compact if and only if $g \in \mathcal{B}_0$.*

Proof. The sufficiency in both statements is given by Theorem 2.20.

If $g \in \mathcal{B}$ then the multiplication operator $\mathbf{M}_{g'}$ is bounded from $H(p, q, \alpha)$ to $H(p, q, \alpha+1)$ and (1) in Lemma 4.7 shows that V is a bounded operator from $H(p, q, \alpha+1)$ to $H(p, q, \alpha)$, so $V_g = V \circ \mathbf{M}_{g'}$ is bounded.

If $g \in \mathcal{B}_0$, using again that $V_g = V \circ \mathbf{M}_{g'}$, and that V is bounded, the result follows from the compactness of the multiplication operator $\mathbf{M}_{g'} : H(p, q, \alpha) \to H(p, q, \alpha + 1)$. Indeed, let $\{f_n\}$ be a sequence in the unit ball of $H(p, q, \alpha)$ that converges to zero uniformly on compact subsets of the unit disk. We have to prove that $||\mathbf{M}_{g'} f_n||_{p,q,\alpha} \to 0$ as $n \to \infty$. Since $g \in \mathcal{B}_0$, for $\varepsilon > 0$ there exists a $R < 1$ such that $|g'(z)|(1 - |z|) < \varepsilon$ for $|z| \geq R$. Now, let $N_0 \in \mathbb{N}$ be such that $|f_n(z)| \leq \varepsilon/||g||_{\mathcal{B}}$ for $n \geq N_0$ and $|z| \leq R$. Then

$$\begin{aligned}
\|g' f_n\|^q_{p,q,\alpha+1} &= (\alpha+1)q \int_0^1 (1-r)^{(\alpha+1)q-1} M_p^q(r, g' f_n)\, dr \\
&\leq (\alpha+1)q \int_0^1 (1-r)^{(\alpha+1)q-1} \left(\sup_{\theta \in [0,2\pi]} |g'(re^{i\theta})| \right)^q M_p^q(r, f_n)\, dr \\
&\leq \|g\|^q_{\mathcal{B}} (\alpha+1)q \int_0^R (1-r)^{\alpha q-1} M_p^q(r, f_n)\, dr \\
&+ \varepsilon^q (\alpha+1)q \int_R^1 (1-r)^{\alpha q-1} M_p^q(r, f_n)\, dr \leq 2\frac{\alpha+1}{\alpha} \varepsilon^q.
\end{aligned}$$

Then $\|g' f_n\|_{p,q,\alpha} \to 0$ and therefore $\mathbf{M}_{g'} : H(p, q, \alpha) \to H(p, q, \alpha+1)$ is compact. □

Now we prove directly the boundedness on $H(p, \infty, \alpha)$ and $H_0(p, \infty, \alpha)$.

Proposition 5.6. *Let g be an analytic function in the unit disk. The following are equivalent:*

(a) $V_g : H(p,\infty,\alpha) \to H(p,\infty,\alpha)$ *is bounded;*

(b) $V_g : H_0(p,\infty,\alpha) \to H_0(p,\infty,\alpha)$ *is bounded;*

(c) $V_g : H_0(p,\infty,\alpha) \to H(p,\infty,\alpha)$ *is bounded;*

(d) $g \in \mathcal{B}$.

Proof. If $g \in \mathcal{B}$ and $f \in H(p,\infty,\alpha)$, then

$$\begin{aligned}\|g'f\|_{p,\infty,\alpha+1} &= \sup_{0\le r<1}(1-r)^{\alpha+1}M_p(r,g'f)\\ &\le ||g||_{\mathcal{B}} \sup_{0\le r<1}\frac{(1-r)^{\alpha+1}}{1-r}M_p(r,f) = ||g||_{\mathcal{B}}\,\|f\|_{p,\infty,\alpha}.\end{aligned}$$

So, by Lemma 4.7, (d) implies (a) and (c). The same inequalities show that if $f \in H_0(p,\infty,\alpha)$, then $g'f \in H_0(p,\infty,\alpha+1)$. Again, by Lemma 4.7, (d) implies (b).

On the other hand, suppose $g'f \in H(p,\infty,\alpha+1)$ for every $f \in H_0(p,\infty,\alpha)$. This means that the operator $\mathbf{M}_{g'}$ is bounded from $H_0(p,\infty,\alpha)$ into $H(p,\infty,\alpha+1)$. Let us denote by M the norm of this operator. Then, by Proposition 4.9, there exists a constant $C>0$ such that, for every $z \in \mathbb{D}$,

$$|g'(z)f(z)| \le \frac{C\|g'f\|_{p,\infty,\alpha+1}}{(1-|z|)^{\alpha+1+\frac{1}{p}}} \le \frac{CM\|f\|_{p,\infty,\alpha}}{(1-|z|)^{\alpha+1+\frac{1}{p}}}.$$

Choosing $f_z \in H_0(p,\infty,\alpha)$ as

$$f_z(w) = \frac{(1-|z|^2)^{\alpha+\frac{1}{p}}}{(1-\overline{w}z)^{2(\alpha+\frac{1}{p})}},$$

a function that satisfies $|f_z(z)| = (1-|z|^2)^{-(\alpha+\frac{1}{p})}$ and $\|f_z\|_{p,\infty,\alpha} \approx 1$, we get

$$|g'(z)f_z(z)| = \frac{|g'(z)|}{(1-|z|^2)^{\alpha+\frac{1}{p}}} \lesssim \frac{CM}{(1-|z|)^{\alpha+1+\frac{1}{p}}}.$$

From here it is clear that $g \in \mathcal{B}$. This argument shows that (b) or (c) implies (d). □

The remaining case, that is, the boundedness of the integral operator from the bigger space $H(p,\infty,\alpha)$ to the smaller space $H_0(p,\infty,\alpha)$ will be of interest for us in the study of semigroups of composition operators on $H(p,\infty,\alpha)$. First, we will need two lemmas. As is usual, given an analytic function $f \in \mathcal{H}(\mathbb{D})$, we denote by $\widehat{f}(k)$ its k'th Taylor coefficient.

Lemma 5.7. *For every $N \in \mathbb{N}$ there exists a polynomial G_N satisfying:*

(a) $\widehat{G_N}(k) \geq 0$, *for every* $k \in [N, 3N]$,

(b) $\widehat{G_N}(k) = 0$, *for every* $k \notin (N, 3N)$,

(c) $\|G_N\|_{H^1} = 1$, *and*

(d) $\|G_N\|_{H^\infty} = \sum_{k=N}^{3N} \widehat{G_N}(k) = N$.

(e) *For* $1 < p < +\infty$, $\|G_N\|_{H^p} \leq N^{\frac{1}{p'}}$, *where* $\frac{1}{p'} + \frac{1}{p} = 1$.

Proof. Let F_{N-1} be the $(N-1)$'th Fejér Kernel, namely

$$F_{N-1}(e^{it}) = \sum_{k=-(N-1)}^{N-1} \Big(1 - \frac{|k|}{N}\Big) e^{ikt}, \qquad e^{it} \in \mathbb{T}.$$

It is known that $\|F_{N-1}\|_{L^1(\mathbb{T})} = 1$. Define $\tilde{G}_N(z) = \zeta^{2N} F_{N-1}(\zeta)$ for $\zeta \in \mathbb{T}$, then $\|\tilde{G}_N\|_{L^1(\mathbb{T})} = 1$ and

$$\|\tilde{G}_N\|_{L^\infty(\mathbb{T})} = \tilde{G}_N(1) = \sum_{k=N}^{3N} \widehat{\tilde{G}_N}(k) = N.$$

Moreover, $\widehat{\tilde{G}_N}(k) \geq 0$, for every $k \in [N, 3N]$, and $\widehat{\tilde{G}_N}(k) = 0$, for every $k \notin (N, 3N)$. The polynomial G_N defined in the unit disk with boundary values $\tilde{G}_N$ satisfies the same properties, by the equality of norms (1.2), that proves properties $(a) - (d)$. The last property comes from the interpolation inequality $\|f\|_p \leq \|f\|_\infty^{\frac{1}{p'}} \|f\|_1^{\frac{1}{p}}$. □

Lemma 5.8. *Suppose* $g \in \mathcal{B} \backslash \mathcal{B}_0$. *Then there exist* $\delta \in (0, \pi/8)$, *an increasing sequence* $\{r_n\}_n$ *in* $(0,1)$, *and a sequence* $\{t_n\}_n$ *in* $[0, 2\pi)$ *satisfying:*

(a) *For every* $n \in \mathbb{N}$ *and every* $t \in [-\delta(1-r_n), \delta(1-r_n)]$ *we have*

$$(1 - r_n) \left| g'\left(r_n e^{i(t_n + t)}\right)\right| \geq \delta.$$

(b) $\lim_{n\to\infty} r_n = 1$.

Proof. Write $M = ||g||_{\mathcal{B}}$. Since $g \notin \mathcal{B}_0$ there exists an $\eta > 0$, an increasing sequence $\{r_n\}_n$ in $(0,1)$ with $\lim_{n\to\infty} r_n = 1$, and a sequence $\{t_n\}_n$ in $[0, 2\pi)$, such that

$$(1 - r_n)|g'(r_n e^{it_n})| \geq \eta, \qquad \text{for all } n. \tag{5.2}$$

Since $g \in \mathcal{B}$,

$$\sup_{z\in\mathbb{D}} (1 - |z|)|g'(z)| = M,$$

and, from the Maximum Modulus Principle we have, for $|z| \leq \frac{1+r_n}{2}$,

$$|g'(z)| \leq \frac{2M}{1 - r_n}.$$

Now, let $|z| \le r_n$. Then, $D\left(z, \frac{1-r_n}{2}\right) \subseteq D\left(0, \frac{1+r_n}{2}\right)$, and by Cauchy's inequality

$$|g''(z)| \le \frac{\max\limits_{w\in D\left(z,\frac{1-r_n}{2}\right)} |g'(w)|}{\frac{1-r_n}{2}} \le 2\frac{\max\limits_{w\in D\left(0,\frac{1+r_n}{2}\right)} |g'(w)|}{1-r_n} \le \frac{4M}{(1-r_n)^2}.$$

Take $\delta > 0$ and $|s-t_n| < \delta(1-r_n)$. Noticing that

$$|r_n e^{is} - r_n e^{it_n}| \le |e^{is} - e^{it_n}| = \left|e^{\frac{i(t_n-s)}{2}}\right| \left|e^{\frac{i(s-t_n)}{2}} - e^{-\frac{i(s-t_n)}{2}}\right| = 2\left|\sin\frac{s-t_n}{2}\right| \le 2\left|\frac{s-t_n}{2}\right|,$$

we obtain

$$|g'(r_ne^{is}) - g'(r_ne^{it_n})| \le |r_ne^{is} - r_ne^{it_n}|\frac{4M}{(1-r_n)^2} \le \frac{4|s-t_n|M}{(1-r_n)^2} \le \frac{4\delta M}{1-r_n} \le \frac{\eta/2}{1-r_n},$$

if δ is small enough. From here and from (5.2),

$$|g'(r_ne^{is})| \ge |g'(r_ne^{it_n})| - |g'(r_ne^{is}) - g'(r_ne^{it_n})| \ge \frac{\eta}{1-r_n} - \frac{\eta}{2(1-r_n)} = \frac{\eta}{2(1-r_n)},$$

and letting $s = t + t_n$ we have, for $|t| \le \delta(1-r_n)$,

$$(1-r_n)|g'(r_ne^{i(t+t_n)})| \ge \frac{\eta}{2} \ge \delta,$$

if δ is small enough. □

Before stating the main result of this section, recall that, given two Banach spaces X and Y, an operator $T : X \to Y$ *fixes a copy of a Banach space* Z if there exists a subspace X_0 of X such that X_0 is isomorphic to Z and $T : X_0 \to T(X_0)$ is an isomorphism. Studying spaces and operators for which this happens is a question of interest in the theory of Banach spaces.

Theorem 5.9. *If $g \in \mathcal{B} \setminus \mathcal{B}_0$, then the operator $V_g : H_0(p,\infty,\alpha) \to H_0(p,\infty,\alpha)$ fixes a copy of c_0 and the operator $V_g : H(p,\infty,\alpha) \to H(p,\infty,\alpha)$ fixes a copy of ℓ_∞. Consequently this last operator has a non-separable image.*

Proof. Since $g \in \mathcal{B}$, we have that $V_g : H(p,\infty,\alpha) \to H(p,\infty,\alpha)$ is bounded. Once again we write V_g as $V \circ \mathbf{M}_{g'}$, with $\mathbf{M}_{g'} : H(p,\infty,\alpha) \to H(p,\infty,\alpha+1)$ the operator of multiplication by g' and $V : H(p,\infty,\alpha+1) \to H(p,\infty,\alpha)$ the Volterra operator. We know that, if we avoid the constant functions, V is an isomorphism. More specifically, if we call X the one-codimensional subspace of $H(p,\infty,\alpha)$ defined by

$$X = \{f \in H(p,\infty,\alpha) : f(0) = 0\},$$

then $V : H(p,\infty,\alpha+1) \to X$ is an onto isomorphism whose inverse is the differentiation. Therefore we only need to prove that $\mathbf{M}_{g'} : H(p,\infty,\alpha) \to H(p,\infty,\alpha+1)$ fixes a copy of ℓ_∞ if $g \in \mathcal{B} \setminus \mathcal{B}_0$.

To do this we need to construct a bounded linear operator $\Phi\colon \ell_\infty \to H(p,\infty,\alpha)$ such that, for certain $C > 0$,

$$C\|\mathbf{M}_{g'}(\Phi\mathbf{a})\|_{p,\infty,\alpha+1} \geq \|\mathbf{a}\|_{\ell_\infty}, \qquad \text{for all } \mathbf{a} \in \ell_\infty. \tag{5.3}$$

Notice that, if $\Phi\colon \ell_\infty \to H(p,\infty,\alpha)$ is a bounded operator and 5.3 is satisfied, then $\|\Phi\mathbf{a}\| \leq \|\Phi\|\|\mathbf{a}\|_{\ell_\infty}$ and

$$C\|\mathbf{M}_{g'}(\Phi\mathbf{a})\|_{p,\infty,\alpha+1} \geq \|\mathbf{a}\|_{\ell_\infty} \geq \frac{\|\Phi\mathbf{a}\|}{\|\Phi\|}, \qquad \text{for all } \mathbf{a} \in \ell_\infty, \tag{5.4}$$

and therefore $\mathbf{M}_{g'}$ is bounded away from zero on the set $X_0 = \Phi\mathbf{a}$, $\mathbf{a} \in \ell_\infty$. This means that it is an isomorphism between X_0 and $\mathbf{M}_{g'}(X_0)$. Moreover, since $\mathbf{M}_{g'}$ is bounded, then $C\|\mathbf{M}_{g'}(\Phi\mathbf{a})\|_{p,\infty,\alpha+1} \leq C\|\mathbf{M}_{g'}\|\|\Phi\mathbf{a}\|_{p,\infty,\alpha}$ and, from 5.3,

$$C\|\mathbf{M}_{g'}\|\|\Phi\mathbf{a}\|_{p,\infty,\alpha} \geq C\|\mathbf{M}_{g'}(\Phi\mathbf{a})\|_{p,\infty,\alpha+1} \geq \|\mathbf{a}\|_{\ell_\infty}, \qquad \text{for all } \mathbf{a} \in \ell_\infty, \tag{5.5}$$

and thus Φ is bounded away from zero, so it is an isomorphism between ℓ_∞ and $\Phi\mathbf{a} = X_0$.

Since $g \in \mathcal{B} \setminus \mathcal{B}_0$ we can apply Lemma 5.8 to find a $\delta \in (0, \pi/8)$, an increasing sequence $\{r_n\}_n$ in $(0,1)$ with $r_n \to 1$ and a sequence $\{t_n\}_n$ in $[0, 2\pi)$ such that for every $n \in \mathbb{N}$ and every $t \in [-\delta(1-r_n), \delta(1-r_n)]$ we have

$$(1-r_n)\left|g'\left(r_n e^{i(t_n+t)}\right)\right| \geq \delta.$$

Fix $\beta > 0$ large enough (this β will depend on $||g||_{\mathcal{B}}$, δ, α and p). Passing to a subsequence if necessary, we can assume that $r_n \geq 1/2$, for all n and

$$\frac{1-r_n}{1-r_{n+1}} \geq \beta, \qquad \text{for all } n. \tag{5.6}$$

Now consider a sequence $\{N_n\}_n$ of positive integers such that

$$N_n(1-r_n) \in [1,2], \qquad \text{for all } n \in \mathbb{N}. \tag{5.7}$$

By (5.6) and (5.7) we have, if β is big enough,

$$\frac{N_{n+1}}{N_n} \geq \frac{\beta}{2} \geq 3, \qquad \text{for all } n \in \mathbb{N}.$$

Let $\nu = \alpha + \frac{1}{p} - 1 = \alpha - \frac{1}{p'}$. For every $n \in \mathbb{N}$ define the function g_n by

$$g_n(z) = N_n^\nu G_{N_n}(e^{-it_n}z), \qquad z \in \mathbb{D}.$$

The G_N's are given in Lemma 5.7 and the t_n's in Lemma 5.8.

Observe that, for every $r \in (0,1)$ and every $t \in [0, 2\pi)$, we have

$$|g_n(re^{it})| \leq N_n^\nu \sum_{k=N_n}^{3N_n} \widehat{G_{N_n}}(k) r^{N_n} = r^{N_n} N_n^{1+\nu} = e^{N_n \log r + (\alpha + \frac{1}{p})\log N_n}. \tag{5.8}$$

This and the fact that $N_n \geq 3^{n-1}$ yield that, for every $\mathbf{a} = \{a_n\}_n \in \ell_\infty$, the series

$$\sum_{n=1}^{\infty} a_n g_n(z)$$

converges uniformly on compact subsets of $\mathbb{D}$ and its sum defines a function $\Phi\mathbf{a}$ holomorphic on $\mathbb{D}$.

Let us see that $\Phi\mathbf{a}$ belongs to $H(p,\infty,\alpha)$ for all $\mathbf{a} \in \ell_\infty$. Since $\widehat{g_n}(k) = 0$, for $k \leq N_n$, we have the estimate, for $r \in (0,1)$ and $n \in \mathbb{N}$,

$$M_p(r, g_n) \leq N_n^{\nu} r^{N_n} \|G_{N_n}\|_{H^p} \leq r^{N_n} N_n^{\nu+\frac{1}{p'}} = r^{N_n} N_n^{\alpha}.$$

Consequently, for $\mathbf{a} \in \ell_\infty$, we have

$$M_p(r, \Phi\mathbf{a}) \leq \|\mathbf{a}\|_{\ell_\infty} \sum_{n=1}^{\infty} M_p(r, g_n) \leq \|\mathbf{a}\|_{\ell_\infty} \sum_{n=1}^{\infty} r^{N_n} N_n^{\alpha}.$$

Take $r \in (0,1)$ and, putting $r_0 = 0$, define $l \in \mathbb{N}$ by the condition $r_{l-1} < r \leq r_l$. Thus, in the case $l \geq 2$, we have

$$\begin{aligned}(1-r)^{\alpha} \sum_{n=1}^{l-1} r^{N_n} N_n^{\alpha} &\leq (1-r_{l-1})^{\alpha} \sum_{n=1}^{l-1} N_n^{\alpha} = [(1-r_{l-1})N_{l-1}]^{\alpha} \sum_{n=1}^{l-1} \left(\frac{N_n}{N_{l-1}}\right)^{\alpha} \\ &\leq 2^{\alpha} \sum_{n=1}^{l-1} \left(\frac{\beta}{2}\right)^{-\alpha(l-1-n)} \leq 2^{\alpha} \sum_{k=0}^{\infty} \left(\frac{\beta}{2}\right)^{-k\alpha} := A < +\infty.\end{aligned}$$

For $n \geq l$ we have, using $\log r \leq r-1$ and $N_n(1-r) \geq N_n(1-r_n) \geq 1$,

$$\begin{aligned}\log\big[(1-r)^{\alpha} r^{N_n} N_n^{\alpha}\big] &\leq \alpha \log\big[N_n(1-r)\big] + N_n(r-1) \\ &\leq C_\alpha - \frac{N_n(1-r)}{2} \leq C_\alpha - \frac{N_n(1-r_l)}{2},\end{aligned}$$

for certain $C_\alpha > 0$ satisfying $\alpha \log x \leq C_\alpha + \frac{x}{2}$, for all $x \geq 1$. Therefore

$$\sum_{n=l}^{\infty} (1-r)^{\alpha} r^{N_n} N_n^{\alpha} \leq \sum_{n=l}^{\infty} e^{C_\alpha} e^{-\frac{\beta^{n-l}}{2}} = e^{C_\alpha} \sum_{j=0}^{\infty} e^{-\frac{\beta^j}{2}} := B < +\infty.$$

Putting all together we have, for all $r \in (0,1)$,

$$(1-r)^{\alpha} M_p(r, \Phi\mathbf{a}) \leq (A+B)\|\mathbf{a}\|_{\ell_\infty}.$$

Taking the supremum over r we see that $\Phi\mathbf{a} \in H(p,\infty,\alpha)$ and $\Phi\colon \ell_\infty \to H(p,\infty,\alpha)$ is a bounded linear operator.

It remains to prove (5.3). We can assume $\mathbf{a} = \{a_n\}_n \in \ell_\infty$ and $\|\mathbf{a}\|_{\ell_\infty} = 1$. Pick $l \in \mathbb{N}$ such that $|a_l| \geq 1/2$. We are going to prove that, for certain $\eta > 0$ we have

$$|\Phi\mathbf{a}(r_l e^{i(t+t_l)})| \geq \eta(1-r_l)^{-\frac{1}{p}-\alpha}, \qquad \text{for all } t \in [-\delta(1-r_l), \delta(1-r_l)]. \tag{5.9}$$

This and Lemma 5.8(a) yield

$$|\mathbf{M}_{g'}(\Phi\mathbf{a})(r_l e^{i(t+t_l)})| \geq \eta\delta(1-r_l)^{-\frac{1}{p}-1-\alpha}, \qquad \text{for all } t \in [-\delta(1-r_l), \delta(1-r_l)],$$

and consequently we get (5.3), since

$$\begin{aligned} M_p(r_l, \mathbf{M}_{g'}(\Phi\mathbf{a})) &= \left(\int_0^{2\pi} |\mathbf{M}_{g'}(\Phi\mathbf{a})(r_l e^{i(\theta+t_l)})|^p \frac{d\theta}{2\pi}\right)^{1/p} \\ &\geq \left(\int_{-\delta(1-r_l)}^{\delta(1-r_l)} \eta^p\delta^p(1-r_l)^{(-\frac{1}{p}-1-\alpha)p}\frac{d\theta}{2\pi}\right)^{1/p} \\ &= \eta\delta(1-r_l)^{-\frac{1}{p}-1-\alpha}\left(\frac{\delta(1-r_l)}{\pi}\right)^{1/p} \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{M}_{g'}(\Phi\mathbf{a})\|_{p,\infty,\alpha+1} &= \sup_{r\leq 1}(1-r)^{1+\alpha}M_p(r, \mathbf{M}_{g'}(\Phi\mathbf{a})) \geq (1-r_l)^{1+\alpha}M_p(r_l, \mathbf{M}_{g'}(\Phi\mathbf{a})) \\ &\geq \eta\delta\left(\frac{\delta}{\pi}\right)^{\frac{1}{p}} := \frac{1}{C}. \end{aligned}$$

Let us prove (5.9). By the definition of $\Phi\mathbf{a}$, for every $t \in \mathbb{R}$, we have

$$\begin{aligned} |\Phi\mathbf{a}(r_l e^{i(t+t_l)})| &\geq |a_l||g_l(r_l e^{i(t+t_l)})| - \sum_{n\neq l}|a_n||g_n(r_l e^{i(t+t_l)})| \\ &\geq \frac{1}{2}\left|g_l(r_l e^{i(t+t_l)})\right| - \sum_{n\neq l} M_\infty(r_l, g_n). \end{aligned} \tag{5.10}$$

By (5.7), for $t \in [-\delta(1-r_l), \delta(1-r_l)]$ and $k \leq 3N_l$, we have $|kt| \leq 6\delta \leq \pi/3$ and $\cos(kt) \geq 1/2$. Therefore

$$\begin{aligned} \left|g_l(r_l e^{i(t+t_l)})\right| &\geq \mathrm{Re}\big(g_l(r_l e^{i(t+t_l)})\big) = N_l^\nu \sum_{k=N_l}^{3N_l} r_l^k \widehat{G_{N_l}}(k)\cos(kt) \\ &\geq \frac{N_l^\nu r_l^{3N_l}}{2}\sum_{k=N_l}^{3N_l}\widehat{G_{N_l}}(k) \geq \kappa N_l^{\nu+1}, \end{aligned} \tag{5.11}$$

for certain $\kappa > 0$.

Using (5.8) we can estimate $M_\infty(r_l, g_n) \leq 2r_l^{N_n}N_n^{\nu+1}$. Thus,

$$\begin{aligned} \sum_{n=1}^{l-1} M_\infty(r_l, g_n) &\leq \sum_{n=1}^{l-1} 2N_n^{\nu+1} \leq 2N_l^{1+\nu}\sum_{n=1}^{l-1}\left(\frac{2}{\beta}\right)^{(l-n)(1+\nu)} \\ &\leq 4\left(\frac{2}{\beta}\right)^{\nu+1} N_l^{\nu+1} \leq \frac{\kappa N_l^{\nu+1}}{4}, \end{aligned} \tag{5.12}$$

if β is big enough.

We use $r_l^{N_l} = \exp(N_l \log r_l) \le \exp\big(N_l(r_l - 1)\big) \le e^{-1}$, to obtain, if $l < n$,

$$M_\infty(r_l, g_n) \le 2r_l^{N_n} N_n^{\nu+1} \le 2N_l^{\nu+1}\Big(\frac{N_n}{N_l}\Big)^{\nu+1} \exp\Big(-\frac{N_n}{N_l}\Big).$$

If β is big enough we have $x^{-1} \ge x^{1+\nu}e^{-x}$ for $x \ge \beta$. Thus

$$\begin{aligned}\sum_{n=l+1}^{\infty} M_\infty(r_l, g_n) &\le 2N_l^{\nu+1} \sum_{n=l+1}^{\infty} \Big(\frac{N_n}{N_l}\Big)^{\nu+1} \exp\Big(-\frac{N_n}{N_l}\Big) \le 2N_l^{\nu+1} \sum_{n=l+1}^{\infty} \frac{N_l}{N_n} \\ &\le 2N_l^{1+\nu} \sum_{n=l+1}^{\infty} \beta^{l-n} \le \frac{\kappa N_l^{\nu+1}}{4}, \qquad (5.13)\end{aligned}$$

if β is big enough.

Finally we collect all the estimates. For $t \in [-\delta(1-r_l), \delta(1-r_l)]$, by (5.10), (5.11), (5.12) and (5.13),

$$\begin{aligned}|\Phi\mathbf{a}(r_l e^{i(t+t_l)})| &\ge \frac{1}{2}\left|g_l(r_l e^{i(t+t_l)})\right| - \sum_{n=1}^{l-1} M_\infty(r_l, g_n) - \sum_{n=l+1}^{\infty} M_\infty(r_l, g_n) \\ &\ge \Big(\kappa - \frac{\kappa}{4} - \frac{\kappa}{4}\Big) N_l^{1+\nu} = \frac{\kappa}{2} N_l^{\alpha+\frac{1}{p}} \ge \frac{\kappa}{2^{\alpha+\frac{1}{p}+1}}(1-r_l)^{-\alpha-\frac{1}{p}}.\end{aligned}$$

We have proved (5.9) and thus the theorem follows for the space $H(p,\infty,\alpha)$. Since the functions g_n are polynomials, a similar argument shows that if $\mathbf{a} \in c_0$, then $\Phi\mathbf{a} \in H_0(p,\infty,\alpha)$. This finishes the proof. □

With this we are now ready to characterize the boundedness of $V_g : H(p,\infty,\alpha) \to H_0(p,\infty,\alpha)$ and the compactness of V_g on $H(p,\infty,\alpha)$ and on $H_0(p,\infty,\alpha)$. All of them are equivalent to $g \in \mathcal{B}_0$.

Corollary 5.10. *Let g be a function in the Bloch space $\mathcal{B}$. The following are equivalent:*

(a) $V_g : H(p,\infty,\alpha) \to H(p,\infty,\alpha)$ *is compact;*

(b) $V_g : H(p,\infty,\alpha) \to H(p,\infty,\alpha)$ *is weakly compact;*

(c) $V_g : H(p,\infty,\alpha) \to H_0(p,\infty,\alpha)$ *is bounded;*

(d) $V_g : H_0(p,\infty,\alpha) \to H_0(p,\infty,\alpha)$ *is compact;*

(e) $V_g : H_0(p,\infty,\alpha) \to H_0(p,\infty,\alpha)$ *is weakly compact;*

(f) $V_g : H(p,\infty,\alpha) \to H(p,\infty,\alpha)$ *does not fix a copy of* ℓ_∞*;*

(g) $V_g : H_0(p,\infty,\alpha) \to H_0(p,\infty,\alpha)$ *does not fix a copy of* c_0*;*

(h) $g \in \mathcal{B}_0$.

Proof. First we will prove that (h) implies (a) and (d). Let us assume that $g \in \mathcal{B}_0$. Take $\{f_n\}$ a sequence in the unit ball of $H(p, \infty, \alpha)$ that converges to zero uniformly on compact subsets of the unit disk. Let us fix $\varepsilon > 0$. Then there is $R < 1$ such that $|g'(z)|(1 - |z|) < \varepsilon$ whenever $|z| \geq R$. Moreover, there is $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ we have that $|f_n(z)| \leq \varepsilon/||g||_{\mathcal{B}}$ for all $|z| \leq R$. On the one hand, if $r \leq R$, then

$$\begin{aligned}(1-r)^{\alpha+1} M_p(r, g' f_n) &\leq (1-r)^{\alpha+1} \sup_{\theta \in [0,2\pi]} |g'(re^{i\theta})| M_p(r, f_n) \\ &\leq ||g||_{\mathcal{B}} (1-r)^{\alpha} M_p(r, f_n) \leq \varepsilon.\end{aligned}$$

On the other hand, if $r > R$, then

$$\begin{aligned}(1-r)^{\alpha+1} M_p(r, g' f_n) &\leq (1-r)^{\alpha+1} \sup_{\theta \in [0,2\pi]} |g'(re^{i\theta})| M_p(r, f_n) \\ &\leq ||f_n||_{p,\infty,\alpha} (1-r) \sup_{\theta \in [0,2\pi]} |g'(re^{i\theta})| \leq \varepsilon.\end{aligned}$$

Thus, $\lim ||g' f_n||_{p,\infty,\alpha+1} = 0$. This implies that $\mathbf{M}_{g'} : H(p, \infty, \alpha) \to H(p, \infty, \alpha+1)$ and $\mathbf{M}_{g'} : H_0(p, \infty, \alpha) \to H_0(p, \infty, \alpha + 1)$ are compact.

A similar argument shows that (h) implies (c). The proof of (c) $\Rightarrow$ (f) follows from the fact that, if V_g fixes a copy of ℓ_∞, say $X_0 \approx \ell_\infty$ then

$$\ell_\infty \approx V(X_0) \subseteq H_0(p, \infty, \alpha),$$

yielding a contradiction since $H_0(p, \infty, \alpha)$ is separable.

Being trivial that (a) $\Rightarrow$ (b) $\Rightarrow$ (f) and that (d) $\Rightarrow$ (e) $\Rightarrow$ (g), we only have to prove that both (f) and (g) imply (h). But this is just Theorem 5.9. □

To conclude this section on the integral operator, we give an application to the inclusion of exponential functions in the space.

Proposition 5.11. *Let X be a Banach space of analytic functions and g an analytic function on the unit disk.*

- *If $V_g : X \to X$ is bounded, then $e^{sg} \in X$ for some $s > 0$.*
- *If $V_g : X \to X$ is compact, then $e^{sg} \in X$ for every $s > 0$.*

Proof. Let g be an analytic function on $\mathbb{D}$ such that $V_g : X \to X$ is bounded and suppose $g(0) = 0$. Then, $V_g(1) = g$ and in general $V_g^n(1) = \frac{1}{n!} g^n$. Let $0 < s < 1/r(V_g)$ where $r(V_g)$ is the spectral radius of the operator V_g. Hence, $\sum_{n=0}^{\infty} s^n V_g^n$ converges in the operator norm. With $f \equiv 1$ we have that

$$\sum_{n=0}^{\infty} s^n V_g^n(1) = \sum_{n=0}^{\infty} \frac{s^n g^n}{n!} = e^{sg} \in X$$

for some $s > 0$.

If $V_g : X \to X$ is compact, its spectrum is the set of its eigenvalues and $\{0\}$. Let $\lambda \neq 0$, then $V_g f = \lambda f$ implies $f(0) = 0$. Differentiating we get $f(z)g'(z) = \lambda f'(z)$, that is, $f'(0) = 0$, and, in general, $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$. This means that if f is an eigenfunction of V_g, then $f \equiv 0$, that is, V_g has no eigenvalues $\lambda \neq 0$, and since it is compact, $r(V_g) = 0$. From the first part we have that $e^{sg} \in X$ for every $s > 0$. □

In our mixed norm spaces, the last result becomes:

Corollary 5.12. *Let g be an analytic function on the unit disk.*

- *If $g \in \mathcal{B}$, then $e^g \in H(p,\infty,\alpha)$ for some $p > 0$.*
- *If $g \in \mathcal{B}_0$, then $e^g \in H(p,\infty,\alpha)$ for every $p > 0$.*

We can also prove the converse. This result appears in [98] in the case of the Hardy space H^2 as a first step to prove that $g \in BMOA$. For g analytic on the unit disk we denote by g_ζ the function $g_\zeta(z) = g(\sigma_\zeta(z)) - g(\zeta)$, $z \in \mathbb{D}$, with $\sigma_\zeta(z) = \frac{z+\zeta}{1+\bar{\zeta}z}$ the automorphism of the disk associated with ζ.

Proposition 5.13. *Let X be a Banach space of analytic functions such that the point evaluation functionals Λ_z are bounded for every $z \in \mathbb{D}$. If $\sup_{\zeta\in\mathbb{D}} \|e^{g_\zeta}\|_X < \infty$ then $g \in \mathcal{B}$. In particular, if X is conformally invariant and $e^g \in X$, then $g \in \mathcal{B}$.*

Proof. Since X is a Banach space and every point evaluation functional is bounded, they are uniformly bounded on compact subsets of the unit disk (using the uniform boundedness principle). Therefore, every point evaluation of the derivative is bounded. Write $k(f) = f'(0)$. Taking in particular the evaluation of the derivative at zero we have

$$|k(e^{g_\zeta})| = |\sigma'_\zeta(0) g'(\sigma_\zeta(0)) e^{g_\zeta(0)}| = (1 - |\zeta|^2)|g'(\zeta)| \leq \|k\| \|e^{g_\zeta}\|_X.$$

From here,

$$\sup_{\zeta\in\mathbb{D}} (1 - |\zeta|^2)|g'(\zeta)| \leq \sup_{\zeta\in\mathbb{D}} \|k\| \|e^{g_\zeta}\|_X < \infty$$

and we conclude that $g \in \mathcal{B}$. □

5.4 Semigroups of composition operators on $H(p,\infty,\alpha)$

According to Proposition 5.4, $H_0(p,\infty,\alpha) = [\varphi_t, H_0(p,\infty,\alpha)]$, so, for every semigroup $\{\varphi_t\}$,

$$H_0(p,\infty,\alpha) \subseteq [\varphi_t, H(p,\infty,\alpha)] \subseteq H(p,\infty,\alpha).$$

In this section we will study if the previous inclusions can be equalities. The first theorem proves that the second inclusion is always strict.

Theorem 5.14. *No nontrivial semigroup induces a strongly continuous semigroup of operators on $H(p,\infty,\alpha)$. In other words,*

$$[\varphi_t, H(p,\infty,\alpha)] \subsetneq H(p,\infty,\alpha)$$

for every semigroup of analytic functions $\{\varphi_t\}$.

To prove this theorem, by Proposition 2.22 we are interested in the integral operator on $H(p,\infty,\alpha)$ and in the question of whether the subspace

$$V_\gamma(H(p,\infty,\alpha)) = \{h \in H(p,\infty,\alpha) : V_\gamma(f) = h \text{ for some } f \in H(p,\infty,\alpha)\}$$

is dense in $H(p,\infty,\alpha)$. Differentiating we have

$$V_\gamma(H(p,\infty,\alpha)) = \left\{h \in H(p,\infty,\alpha) : \frac{h'}{\gamma'} \in H(p,\infty,\alpha)\right\}.$$

If γ is the associated g-symbol of a semigroup with Denjoy-Wolff point $b = 0$ we study the density of

$$E = \left\{h \in H(p,\infty,\alpha) : P\,h' \in H(p,\infty,\alpha)\right\}$$

or

$$E = \left\{h \in H(p,\infty,\alpha) : (1-z)^2 P\,h' \in H(p,\infty,\alpha)\right\},$$

for $b = 1$, where P is the function with $\operatorname{Re} P \geq 0$ associated to the infinitesimal generator of the semigroup.

First we prove a lemma that gives us information about the functions in $\overline{E}$. For $\theta \in [0,2\pi]$ define S_θ as the Stolz region with vertex $e^{i\theta}$ and the function

$$\psi(\theta) = \inf_{z\in S_\theta} |P(z)|$$

if $b = 0$ and

$$\psi(\theta) = \inf_{z\in S_\theta} |(1-z)^2 P(z)|$$

if $b = 1$. In both cases, we have that $\psi(\theta) > 0$ a.e. $\theta \in [0,2\pi]$. We only have to prove it for the second case. Since the function $z \mapsto \frac{1}{(1-z)^2}$ belongs to the Hardy space H^β for $\beta < 1/2$ and $1/P$ belongs to H^α for $\alpha < 1$ (because its real part is positive), we have that the function $z \mapsto \frac{1}{(1-z)^2 P(z)}$ belongs to the Hardy space H^δ for $\delta < 1/3$. Consequently, for almost every θ,

$$\sup_{z\in S_\theta} \frac{1}{|(1-z)^2 P(z)|} < +\infty.$$

Lemma 5.15. *Let $h \in \overline{E}$ and $\theta \in [0,2\pi]$ such that $\psi(\theta) > 0$, then*

$$\lim_{r\to 1^-} (1-r)^\alpha \left(\int_{\theta-(1-r)}^{\theta+(1-r)} |h(re^{it})|^p dt\right)^{\frac{1}{p}} = 0.$$

Proof. We may assume that $b = 0$ and that $\theta = 0$. If $h \in E$ then for every $r \in (0,1)$ we have

$$(1-r)^\alpha \left(\int_{-(1-r)}^{1-r} \psi^p(0) |h'(re^{it})|^p dt\right)^{\frac{1}{p}}$$

$$\leq (1-r)^\alpha \left(\int_{-(1-r)}^{1-r} |P(re^{it})|^p |h'(re^{it})|^p dt\right)^{\frac{1}{p}} \leq \|Ph'\|_{p,\infty,\alpha} = M.$$

From here, since $\psi(0)>0$,

$$\left(\int_{-(1-r)}^{1-r}|h'(re^{it})|^p dt\right)^{\frac{1}{p}} \le \frac{M/\psi(0)}{(1-r)^{\alpha}}.$$

Now, writing

$$h(re^{it}) = h(0) + e^{it}\int_0^r h'(\rho e^{it})\,d\rho$$

and using Minkowski's integral inequality we have that

$$\left(\int_{-(1-r)}^{1-r}|h(re^{it})|^p dt\right)^{\frac{1}{p}} \le C + \int_0^r \frac{M/\psi(0)}{(1-\rho)^{\alpha}}d\rho = C + \frac{C'}{(1-r)^{\alpha-1}}.$$

Hence,

$$\lim_{r\to 1^-}(1-r)^{\alpha}\left(\int_{-(1-r)}^{1-r}|h(re^{it})|^p dt\right)^{\frac{1}{p}} \le \lim_{r\to 1^-}(1-r)^{\alpha}\left(C + \frac{C'}{(1-r)^{\alpha-1}}\right) = 0.$$

Now, let $h \in \overline{E}$, then $h = \lim_{n\to\infty} h_n$, with $h_n \in E$ for every n. Since

$$(1-r)^{\alpha}\left(\int_{-(1-r)}^{1-r}|h(re^{it})|^p dt\right)^{\frac{1}{p}}$$

is bounded by the norm $\|h\|_{p,\infty,\alpha}$, we have that

$$(1-r)^{\alpha}\left(\int_{-(1-r)}^{1-r}|h_n(re^{it}) - h(re^{it})|^p dt\right)^{\frac{1}{p}} \le \|h_n - h\|_{p,\infty,\alpha},$$

and the result also holds for every $h \in \overline{E}$. □

Thus, given any θ such that $\psi(\theta)>0$, the function $f(z) = \frac{1}{(1-e^{-i\theta}z)^{\alpha+\frac{1}{p}}}$, $z \in \mathbb{D}$, is an example of a function in $H(p,\infty,\alpha)$ that does not belong in $\overline{E}$, proving Theorem 5.14.

Now that we know

$$H_0(p,\infty,\alpha) \subseteq [\varphi_t, H(p,\infty,\alpha)] \subsetneq H(p,\infty,\alpha)$$

for every semigroup $\{\varphi_t\}$, we want to characterize the semigroups for which

$$H_0(p,\infty,\alpha) = [\varphi_t, H(p,\infty,\alpha)].$$

First, we deal with the case $b=0$. It is worth noticing that, unlike the BMOA case (see [29, Prop. 3]), the integral operator we are interested in is bounded in $H(p,\infty,\alpha)$ for every admissible function γ. Indeed, by Proposition 5.6, V_γ is bounded from $H(p,\infty,\alpha)$ to itself if and only if $\gamma \in \mathcal{B}$, that is, $\sup_{z\in\mathbb{D}}(1-|z|)\frac{1}{|P(z)|} < \infty$, and this is true since every such P induced by a semigroup of analytic functions has positive real part.

Theorem 5.16. *Let $\{\varphi_t\}$ be a semigroup with Denjoy-Wolff point $b \in \mathbb{D}$. Then*

$$H_0(p,\infty,\alpha) = [\varphi_t, H(p,\infty,\alpha)] \Leftrightarrow \gamma \in \mathcal{B}_0.$$

Moreover, if $\gamma \notin \mathcal{B}_0$ then the space $[\varphi_t, H(p,\infty,\alpha)]$ contains a subspace isomorphic to ℓ_∞.

Proof. By Proposition 2.22,

$$[\varphi_t, H(p,\infty,\alpha)] = \overline{H(p,\infty,\alpha) \cap (V_\gamma(H(p,\infty,\alpha)) \oplus \mathbb{C})},$$

where V_γ is the integral operator with symbol $\gamma(z) = \int_0^z \frac{1}{P(\zeta)} d\zeta$. Since V_γ is bounded on $H(p,\infty,\alpha)$ we have

$$V_\gamma(H(p,\infty,\alpha)) \oplus \mathbb{C} \subset H(p,\infty,\alpha)$$

and therefore

$$\overline{H(p,\infty,\alpha) \cap (V_\gamma(H(p,\infty,\alpha)) \oplus \mathbb{C})} = \overline{V_\gamma(H(p,\infty,\alpha)) \oplus \mathbb{C}}.$$

From here,

$$H_0(p,\infty,\alpha) = [\varphi_t, H(p,\infty,\alpha)] = \overline{V_\gamma(H(p,\infty,\alpha)) \oplus \mathbb{C}}$$

if and only if $V_\gamma(H(p,\infty,\alpha)) \subseteq H_0(p,\infty,\alpha)$, and, by Proposition 5.10, this is equivalent to $\gamma \in \mathcal{B}_0$. □

Notice that, since

$$\gamma(z) = -\int_0^z \frac{1}{P(\zeta)} d\zeta,$$

if $b = 0$, then $\gamma \in \mathcal{B}_0$ is equivalent to $\frac{1}{P} \in H_0(p,\infty,\alpha)$.

Now, for $b = 1$, recall that

$$E = \left\{h \in H(p,\infty,\alpha) : (1-z)^2 P\, h' \in H(p,\infty,\alpha)\right\}.$$

As before, we have that $[\varphi_t, H(p,\infty,\alpha)] = \overline{E}$ for $(1-z)^2 P(z)$ the generator of the semigroup $\{\varphi_t\}$.

Theorem 5.17. *For every semigroup of analytic functions with Denjoy-Wolff $b \in \mathbb{T}$ the set $\overline{E}$ contains a copy of ℓ_∞. Consequently, $[\varphi_t, H(p,\infty,\alpha)] \supsetneq H_0(p,\infty,\alpha)$.*

Several auxiliary results are needed to prove this theorem. First, we define the space

$$X_{\nu,\beta} = \left\{ \sum_{n=1}^{\infty} a_n f_n : \{a_n\} \in \ell_\infty \right\},$$

where $f_n(z) = \delta_n^\nu (1 + \delta_n - z)^\beta$, $\delta_n = K^{-n}$ for K big enough, $\nu > 0$ and $\beta p < -1$.

The first lemma relates the space $X_{\nu,\beta}$ with $H(p,\infty,\alpha)$ for some ν depending on p and α.

Lemma 5.18. *Let* $F(z)=\sum_{n=1}^{\infty}|f_n(z)|$ *then*

$$(1-r)^{\alpha}M_p(r,F)\le C$$

for every r *and for* $\nu=-(\alpha+\beta+\frac{1}{p})$.

Proof. Adapting the proof of [57, Lemma in Section 4.6], we can see that there exists a constant C such that

$$M_p(r,f_n)\le C\delta_n^{\nu}(1+\delta_n-r)^{\beta+\frac{1}{p}}.$$

Let $l\in\mathbb{N}$ such that $1-r\in(\delta_{l+1},\delta_l]$. Then, since $1+\delta_n-r\approx\delta_n$ for $n\le l$ and $1+\delta_n-r\approx 1-r$ for $n>l$, we have

$$\begin{aligned}(1-r)^{\alpha}M_p(r,F)&\le\sum_{n=1}^{\infty}(1-r)^{\alpha}M_p(r,f_n)\le C\sum_{n=1}^{\infty}(1-r)^{\alpha}\delta_n^{\nu}(1+\delta_n-r)^{\beta+\frac{1}{p}}\\&\lesssim C\sum_{n=1}^{l}(1-r)^{\alpha}\delta_n^{\nu+\beta+\frac{1}{p}}+C\sum_{n=l+1}^{\infty}(1-r)^{\alpha+\beta+\frac{1}{p}}\delta_n^{\nu}\\&\le C(1-r)^{\alpha}\delta_l^{-\alpha}+C(1-r)^{-\nu}\delta_{l+1}^{\nu}\le C',\end{aligned}$$

given that both α and ν are positive. □

The first property we will need about this spaces is the following:

Proposition 5.19. *Let* $\nu=-(\alpha+\beta+\frac{1}{p})$ *and define*

$$\begin{aligned}\Phi:\ \ell_\infty&\to X_{\nu,\beta}\\ \{a_n\}&\to\sum_{n=1}^{\infty}a_nf_n.\end{aligned}$$

Then Φ *is an isomorphism between* ℓ_∞ *and* $X_{\nu,\beta}$ *with the norm of* $H(p,\infty,\alpha)$.

Proof. Following the steps of Theorem 5.9 it is easy to see using the previous lemma that Φ is well defined and bounded from ℓ_∞ to $X_{\nu,\beta}$. We only need to show that, for some $C>0$, given $\mathbf{a}=\{a_n\}\in\ell_\infty$, $C\|\Phi\mathbf{a}\|_{p,\infty,\alpha}\ge\|\mathbf{a}\|_\infty$. Let $\|\mathbf{a}\|_\infty=1$ and $l\in\mathbb{N}$ such that $|a_l|>\frac{1}{2}$ and $\delta_l\approx(1-r_l)$. By definition of $\Phi\mathbf{a}$, for $|t|<\delta_l$ we have, as in the proof of the previous lemma,

$$\begin{aligned}
|\Phi\mathbf{a}(r_l e^{it})| &\geq |a_l||f_l(r_l e^{it})| - \sum_{n=1}^{l-1} |a_n||f_n(r_l e^{it})| - \sum_{n=l+1}^{\infty} |a_n||f_n(r_l e^{it})| \\
&\geq \frac{1}{2}\delta_l^\nu |1+\delta_l - r_l e^{it}|^\beta - \sum_{n=1}^{l-1} \delta_n^\nu |1+\delta_n - r_l e^{it}|^\beta - \sum_{n=l+1}^{\infty} \delta_n^\nu |1+\delta_n - r_l e^{it}|^\beta \\
&\geq \frac{C}{2}\delta_l^\nu \delta_l^\beta - C' \sum_{n=1}^{l-1} \delta_n^{\nu+\beta} - C'' \sum_{n=l+1}^{\infty} \delta_n^\nu \delta_l^\beta \\
&\geq \frac{C}{2}\delta_l^{\nu+\beta} - C'\delta_{l-1}^{\nu+\beta} - C''\delta_{l+1}^\nu \delta_l^\beta \\
&\geq \frac{C}{2}\delta_l^{\nu+\beta} - C' K^{\nu+\beta}\delta_l^{\nu+\beta} - C'' K^{-\nu}\delta_l^{\nu+\beta} \geq A\delta_l^{\nu+\beta} \geq A'(1-r_l)^{\nu+\beta}.
\end{aligned}$$

Then, since $|t| < \delta_l \approx 1 - r_l$

$$M_p^p(r_l, \Phi\mathbf{a}) \geq \int_{|t|<\delta_l} |\Phi\mathbf{a}(r_l e^{it})|^p \frac{dt}{2\pi} \geq \frac{A'}{\pi}(1-r_l)^{(\nu+\beta)p}\delta_l \approx C(1-r_l)^{(\nu+\beta)p+1}$$

it follows that

$$\begin{aligned}
\|\Phi\mathbf{a}\|_{p,\infty,\alpha} &\geq (1-r_l)^\alpha M_p(r_l, \Phi\mathbf{a}) \\
&\geq C(1-r_l)^\alpha (1-r_l)^{\nu+\beta+\frac{1}{p}} = C.
\end{aligned}$$

□

Our interest in the space $X_{\nu,\beta}$ comes from the following two propositions. In the first one we prove that, if $\alpha > 1$, the set E contains a copy of ℓ_∞.

Proposition 5.20. *Let $\nu = -(\alpha + \beta + \frac{1}{p})$ and $\alpha > 1$, then $X_{\nu,\beta} \subseteq E$.*

Proof. For $f \in X_{\nu,\beta}$ we already know that $f \in H(p,\infty,\alpha)$ for the given parameter ν (Lemma 5.18), so we only need to show $(1-z)^2 P f' \in H(p,\infty,\alpha)$. Since

$$f'(z) = -\sum_{n=1}^{\infty} a_n \delta_n^\nu \beta (1+\delta_n - z)^{\beta-1},$$

it is clear that $f' \in X_{\nu,\beta-1}$. Multiplying by $(1-z)^2$ we have that

$$|f'(z)(1-z)^2| \leq C\sum_{n=1}^{\infty} \delta_n^\nu |1+\delta_n - z|^{\beta-1}|1-z|^2 \leq C\sum_{n=1}^{\infty} \delta_n^\nu |1+\delta_n - z|^{\beta+1}.$$

Hence, by Lemma 5.18, $(1-z)^2 f' \in H(p,\infty,\alpha-1)$. Moreover, since P has positive real part, $P \in H(\infty,\infty,1)$ and then $(1-z)^2 P f' \in H(p,\infty,\alpha)$. □

The next proposition allows us to take away the condition on α by taking the closure on E. Since $\overline{E} = [\varphi_t, H(p,\infty,\alpha)]$, we finally get that, if $b = 1$, $[\varphi_t, H(p,\infty,\alpha)]$ contains a copy of ℓ_∞, and therefore it can never be $H_0(p,\infty,\alpha)$. First we need the following lemma.

Lemma 5.21. *For every $f \in X_{\nu,\beta}$ and $\theta \in (0,\alpha)$*

$$f'(z)(1-z)[(1-z)P(z)]^\theta \in H(p,\infty,\alpha).$$

Proof. As in the proof of the last proposition, we have that, for $f \in X_{\nu,\beta}$,

$$|f'(z)(1-z)^{1+\theta}| \le C\sum_{n=1}^{\infty}\delta_n^\nu|1+\delta_n - z|^{\beta-1}|1-z|^{1+\theta} \le C\sum_{n=1}^{\infty}\delta_n^\nu|1+\delta_n-z|^{\beta+\theta}$$

so, by Lemma 5.18, $(1-z)^{1+\theta}f' \in H(p,\infty,\alpha-\theta)$. Since $P^\theta \in H(\infty,\infty,\theta)$, we have $f'(z)(1-z)^{1+\theta}P(z)^\theta \in H(p,\infty,\alpha)$. □

Proposition 5.22. *If $\nu = -(\alpha+\beta+\frac{1}{p})$ then $X_{\nu,\beta} \subseteq \overline{E}$.*

Proof. For $f \in X_{\nu,\beta}$ we define

$$h_n = f(0) + V\left(\frac{nf'}{n+\psi}\right),$$

where V is the Volterra operator and $\psi(z) = [(1-z)P(z)]^{1-\theta}$ for $0<\theta<1$.

Since

$$\psi(\mathbb{D}) \subseteq \Delta = \{w : \operatorname{Arg} w \in (-\pi(1-\theta), \pi(1-\theta))\},$$

we have that there exists a constant M such that $\left|\frac{w}{n+w}\right| < M$ for every $w \in \Delta$ and $n \in \mathbb{N}$. Now, since $\frac{n}{n+\psi(z)} = 1 - \frac{\psi(z)}{n+\psi(z)}$ we have $\left|\frac{n}{n+\psi(z)}\right| < M+1$ for every $z \in \mathbb{D}$ and therefore it is a bounded function. This allows us to show that $h_n' = \frac{nf'}{n+\psi} \in H(p,\infty,\alpha+1)$, since $f' \in X_{\nu,\beta-1} \subseteq H(p,\infty,\alpha+1)$ for every $f \in X_{\nu,\beta}$, and $\frac{n}{n+\psi} \in H^\infty$. From here, $h_n \in H(p,\infty,\alpha)$.

Moreover, $h_n \in E$ for every $n \in \mathbb{N}$. Indeed, by the definition of h_n and ψ we have

$$P(z)(1-z)^2h_n'(z) = P(z)(1-z)^2\frac{nf'}{n+\psi(z)} = f'(z)(1-z)^{1+\theta}P(z)^\theta\psi(z)\frac{n}{n+\psi(z)}.$$

By the previous Lemma, for every $f \in X_{\nu,\beta}$ and $\theta \in (0,\alpha)$

$$f'(z)(1-z)^{1+\theta}P(z)^\theta \in H(p,\infty,\alpha),$$

and recalling that $\frac{\psi}{n+\psi} \in H^\infty$, we get $P(z)(1-z)^2h_n'(z) \in H(p,\infty,\alpha)$, that is, $h_n \in E$.

To prove $X_{\nu,\beta} \subseteq \overline{E}$ we are going to show that $h_n \to f$ in $H(p,\infty,\alpha)$. Taking derivatives, this is equivalent to $h_n' \to f'$ in $H(p,\infty,\alpha+1)$, and by definition of h_n, to

$$\left(\frac{n}{n+\psi} - 1\right)f' = \left(\frac{\psi}{n+\psi}\right)f' \to 0$$

in $H(p,\infty,\alpha+1)$

Rephrasing, in what follows we want to prove that $\left(\frac{\psi}{n+\psi}\right) g \to 0$ in $H(p,\infty,\alpha)$ for $g \in X_{\nu,\beta}$. Define the sets

$$A_n = \{z = re^{it} \in \mathbb{D} : |t| \leq n(1-r), |t| \leq \pi\}$$

and

$$B_n = \{z = re^{it} \in \mathbb{D} : n(1-r) < |t| \leq \pi\}$$

and the associated functions $c_n = \frac{\psi}{n+\psi} g\chi_{A_n}$ and $d_n = \frac{\psi}{n+\psi} g\chi_{B_n}$, so

$$\frac{\psi}{n+\psi} g = c_n + d_n$$

and

$$\left\| \frac{\psi}{n+\psi} g \right\|_{p,\infty,\alpha} \leq \sup_{0\leq r<1} (1-r)^{\alpha}[M_p(r,c_n) + M_p(r,d_n)]. \tag{5.14}$$

First, suppose $z = re^{it} \in A_n$, then $|t| \leq n(1-r)$ and

$$|1-z| = \sqrt{(1-r)^2 + 4r\sin^2\frac{t}{2}} \leq \sqrt{(1-r)^2 + rt^2} \leq 1 - r + |t| \leq (n+1)(1-r).$$

Now, since $\operatorname{Re} P > 0$, let

$$Q(z) = \frac{P(z) - i \operatorname{Im} P(0)}{\operatorname{Re} P(0)},$$

then $\operatorname{Re} Q(z) > 0$ and $Q(0) = 1$. Therefore, $Q \in \mathcal{P}$ and, for the growth property (1.4),

$$|Q(z)| \leq \frac{1+|z|}{1-|z|} \leq \frac{2}{1-|z|}$$

and

$$|P(z)| \leq \frac{2\operatorname{Re} P(0) + |\operatorname{Im} P(0)|}{1-|z|}.$$

Using these last two observations,

$$|(1-z)P(z)| \leq (2\operatorname{Re} P(0) + |\operatorname{Im} P(0)|)\,(n+1),$$

so

$$|\psi(z)| = |(1-z)^{\theta} P^{\theta}(z)| \leq (2\operatorname{Re} P(0) + |\operatorname{Im} P(0)|)^{\theta}\,(n+1)^{\theta} < \frac{n}{2}$$

for n large enough. From here

$$\left|\frac{\psi}{n+\psi}\right| \leq \frac{(2\operatorname{Re} P(0) + |\operatorname{Im} P(0)|)^{\theta}\,(n+1)^{\theta}}{n - \frac{n}{2}} \leq C(n+1)^{\theta-1} = \alpha_n,$$

with $\alpha_n \to 0$ when $n \to \infty$ (recall that $0 < \theta < 1$). Therefore,

$$\lim_{n\to\infty} \sup_{0\leq r<1} (1-r)^{\alpha} M_p(r,c_n) \leq \lim_{n\to\infty} \alpha_n \|g\|_{p,\infty,\alpha} = 0.$$

Now, take $n \geq 7$. If $z = re^{it} \in B_n$, then $1 - r < \frac{\pi}{n}$, so $r > 1/2$. Recall that $d_n = \frac{\psi}{n+\psi} \left(\sum_{k=1}^{\infty} a_k f_k\right) \chi_{B_n}$ with $f_k(z) = \delta_k^{\nu}(1 + \delta_k - z)^{\beta}$, $\delta_k = K^{-k}$ and $\beta < 0$. For $z \in B_n$ we have $|1+\delta_k - z| \geq |r - z| = r|1 - e^{it}| \geq \frac{r}{\pi}|t| \geq \frac{|t|}{2\pi}$. Therefore $|f_k(z)| \leq \delta_k^{\nu} \left(\frac{|t|}{2\pi}\right)^{\beta}$ and

$$
\begin{aligned}
M_p(r, f_k \chi_{B_n}) &\leq \delta_k^{\nu} \left(2 \int_{n(1-r)}^{\pi} \left(\frac{t}{2\pi}\right)^{\beta p} \frac{dt}{2\pi} \right)^{\frac{1}{p}} \\
&\leq \delta_k^{\nu} \left(2 \int_{n(1-r)}^{\infty} \left(\frac{t}{2\pi}\right)^{\beta p} \frac{dt}{2\pi} \right)^{\frac{1}{p}} \leq C \delta_k^{\nu} \left(n(1-r)\right)^{\beta + \frac{1}{p}} .
\end{aligned}
$$

Now we take $l \in \mathbb{N}$ such that $(1-r)K^l \in \left(\frac{1}{\sqrt{n}}, \frac{K}{\sqrt{n}}\right]$ (that means, $\left((1-r)K^l\right)^{\alpha} \leq (K/\sqrt{n})^{\alpha}$ and $\left((1-r)K^l\right)^{-\nu} \leq (1/\sqrt{n})^{-\nu}$). The integral mean of d_n can be bound as

$$
\begin{aligned}
M_p(r, d_n) &\leq \left| \frac{\psi}{n+\psi} \right| \|a_n\|_{\ell_\infty} \sum_{k=1}^{\infty} M_p(r, f_k \chi_{B_k}) \\
&\leq (M+1) \|a_n\|_{\ell_\infty} \left[\sum_{k=1}^{l-1} C \delta_k^{-\alpha} + \sum_{k=l}^{\infty} C' \delta_k^{\nu} \left(n(1-r)\right)^{\beta + \frac{1}{p}} \right] \\
&\leq (M+1) \|a_n\|_{\ell_\infty} \left(C K^{l\alpha} + C' n^{\beta + \frac{1}{p}} K^{-l\nu} (1-r)^{\beta + \frac{1}{p}} \right) \\
&= \frac{(M+1) \|a_n\|_{\ell_\infty}}{(1-r)^{\alpha}} \left(C K^{l\alpha} (1-r)^{\alpha} + C' n^{\beta + \frac{1}{p}} K^{-l\nu} (1-r)^{-\nu} \right) \\
&\leq \frac{(M+1) \|a_n\|_{\ell_\infty}}{(1-r)^{\alpha}} \left(C \frac{K^{\alpha}}{n^{\alpha/2}} + C' n^{\beta + \frac{1}{p}} n^{\nu/2} \right) \\
&= \frac{(M+1) \|a_n\|_{\ell_\infty}}{(1-r)^{\alpha}} \left(C K^{\alpha} n^{-\alpha/2} + C' n^{-\alpha - \nu/2} \right).
\end{aligned}
$$

From here clearly $\sup_{0 \leq r < 1} (1-r)^{\alpha} M_p(r, d_n) \to 0$ when $n \to \infty$, and, by (5.14),

$$
\left\| \frac{\psi}{n+\psi} g \right\|_{p,\infty,\alpha} \leq \sup_{0 \leq r < 1} (1-r)^{\alpha} [M_p(r, c_n) + M_p(r, d_n)] \to 0
$$

when $n \to \infty$. This finishes the proof. □

Chapter 6

Operators on Banach spaces of analytic functions defined axiomatically

Let Ω be a domain (*i.e.,* an open and connected set) in the complex plane. We will consider general domains Ω which will sometimes be required to be bounded and sometimes will simply be the unit disk $\mathbb{D}$.

We will denote by $H(\Omega)$ the algebra of all analytic functions in Ω. Let $X \subset H(\Omega)$ be a Banach space of analytic functions in Ω. Abstract Banach spaces of analytic functions, assumed to satisfy only a handful of axioms, have been considered in the literature, *e.g.*, in [40] , [107], [3], or [37].

Recall from Chapter 1 that, given a point z in Ω, we will denote by Λ_z the *point evaluation functional* corresponding to z, defined by $\Lambda_z(f) = f(z)$, for $f \in X$. Throughout this chapter we assume that the point evaluation functionals are bounded in X. (This is the most common axiom in the literature.) Thus, they are uniformly bounded on each compact subset of Ω; indeed, given a compact set $K \subset \Omega$, we have

$$\sup_{z\in K} |\Lambda_z(f)| = \sup_{z\in K} |f(z)| < \infty$$

for each $f \in X$, hence $\sup_{z\in K} \|\Lambda_z\| < \infty$ by a direct application of the uniform boundedness principle. By a *functional Banach space* we will allude to a Banach space of analytic functions in which the point evaluation functionals are bounded and some additional axioms are satisfied.

Throughout the chapter, T will denote a general operator, while $\mathbf{T}_{F,\varphi}$ will be a weighted composition operator.

The results in this chapter can be found in [14].

6.1 Some preliminary results

6.1.1 Some consequences of a basic axiom in the disk

Let $X \subset H(\mathbb{D})$ be a functional Banach space on the disk on which the point evaluations are bounded. If $f \in X$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ then the functional Λ_0, given by $\Lambda_0(f) = a_0 = f(0)$, is bounded. This observation extends to the remaining coefficient functionals.

Proposition 6.1. *Let $X \subset H(\mathbb{D})$ be a Banach space which contains the polynomials and on which the point evaluations are bounded. The following assertions hold.*

(a) All coefficient functionals $\Lambda_n(f) = a_n$ are bounded. More precisely, for every $n \in \mathbb{N}$ and $r \in (0,1)$ there exists a constant $M_r > 0$ such that $|a_n| \le \frac{M_r}{r^n}\|f\|_X$ for all $f \in X$.

(b) $\limsup_{n\to\infty} \sqrt[n]{\|z^n\|_X} \ge 1$.

Proof. Recall that the point evaluation functionals on X are uniformly bounded on compact subsets of the disk. Thus,

$$\max_{|w|=r} |f(w)| \le M_r \|f\|_X$$

for some fixed $M_r > 0$ and all $f \in X$. Let $f \in X$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The Cauchy integral formula yields

$$|a_n| \le \frac{\max_{|w|=r} |f(w)|}{r^n} \le \frac{M_r}{r^n}\|f\|_X \,.$$

(b) Applying the conclusion of part (a) to the function $f(z) = z^n$, we get $1 \le \frac{M_r}{r^n}\|f\|_X$ for arbitrary but fixed r, $0 < r < 1$. Hence

$$r = \lim_{n\to\infty} \frac{r}{\sqrt[n]{M_r}} \le \limsup_{n\to\infty} \sqrt[n]{\|z^n\|_X} \,.$$

Since this holds for arbitrary $r \in (0,1)$, the statement follows. □

In the next section a relevant condition in some statements will be the assumption that $\limsup_{n\to\infty} \sqrt[n]{\|z^n\|_X} \le 1$ and, thus, actually $\limsup_{n\to\infty} \sqrt[n]{\|z^n\|_X} = 1$. It is not difficult to see that in most spaces of interest to us the above lim sup is actually equal to one. This is readily verified for the Hardy, Bergman, and mixed-norm spaces but also in the Bloch, analytic Besov spaces, and weighted Banach spaces H_v^{∞}.

The next example may not seem so natural but will be relevant for further discussion.

Example 1. *Consider the space X whose elements $f|_{\mathbb{D}}$ are the restrictions to the disk of the functions f in the Fock space F^2, defined as the space of entire functions [133] which are square integrable in the complex plane $\mathbb{C}$ with respect to the Gaussian measure:*

$$\|f\|_F^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z) < \infty \,.$$

Clearly, if $f|_{\mathbb{D}} \in X$*, its extension* f *to the entire plane is unique by the principle of isolated zeroes, hence there is a one-to-one correspondence between the members of* X *and the functions in* F^2*. If* X *is equipped by the norm of the extension to the plane:* $\|f|_{\mathbb{D}}\|_X = \|f\|_{F^2}$*, it is clear that it is a Banach space (actually, Hilbert) of analytic functions in the unit disk. It is also well-known that the point evaluations (at all points in the plane) are bounded on* F^2 *[133, Theorem 2.7] and the polynomials are dense [133, Proposition 2.9], hence the point evaluations (at the points of* $\mathbb{D}$*) are bounded on* X *and the polynomials are dense in* X*. So, the space* X *defined in this fashion is one more in our collection of spaces; however, it does not satisfy the condition* $\limsup_{n\to\infty} \sqrt[n]{\|z^n|_{\mathbb{D}}\|_X} = 1$*. In fact, one easily calculates that*

$$\|z^n|_{\mathbb{D}}\|_X^2 = \|z^n\|_{F^2}^2 = 2\int_0^\infty r^{2n+1}e^{-r^2}dr = \Gamma(n+1) = n!$$

and Stirling's formula shows that $\lim_{n\to\infty} \sqrt[n]{\|z^n\|_X} = \infty$*.*

6.1.2 Pointwise multipliers and the domination property

As in Chapter 2, a function F analytic in $\mathbb{D}$ is said to be a pointwise multiplier of a Banach space of analytic functions X into itself if $Ff \in X$ for every $f \in X$. For any such F we can define the multiplication operator $\mathbf{M}_F$ in a natural way:

$$\mathbf{M}_F : X \to X\,, \qquad \mathbf{M}_F f = Ff\,, \quad f \in X\,.$$

Under the assumption that each point-evaluation functional $\Lambda_\zeta(f) = f(\zeta)$ is bounded on X, a standard normal family argument shows that $\mathbf{M}_F$ has closed graph and is, thus, a bounded operator.

We also denote by $\mathbf{M}(X)$ the space of all (bounded) multipliers from X into itself, a closed subspace of the space of bounded operators on X. By Proposition 2.11 (a simple consequence of the boundedness of point evaluations), $\mathbf{M}(X) \subseteq H^\infty$ and $\|F\|_\infty \le \|\mathbf{M}_F\|$. We also saw in Chapter 2 that, in general, $H^\infty \ne \mathbf{M}(X)$; for example, in the Bloch and Dirichlet spaces. Thus, a most natural question comes to mind: for which "reasonable" Banach spaces X of analytic functions is $H^\infty = \mathbf{M}(X)$ true? We have not been able to find an answer in the literature and we give here a very simple answer in terms of what we call the domination property. Note the similarity with a property seen earlier in the study of semigroups (Sec. 5.2, Prop 5.2).

We will say that a Banach space X of analytic functions has the *Domination Property* (DP) if there is a universal constant $C > 0$ such that if $f \in H(\Omega)$, $g \in X$ and $|f(z)| \le |g(z)|$ holds for all $z \in \mathbb{D}$, then $f \in X$ and $\|f\|_X \le C\|g\|_X$.

It is readily checked that all Hardy, Bergman, mixed-norm spaces $H(p,q,\alpha)$, and weighted Banach spaces H_v^∞ have this property.

Theorem 6.2. *Let* $X \subset H(\Omega)$ *be a functional Banach space in which the point evaluations are bounded. Then the following conditions are equivalent:*

(a) $H^\infty(\Omega) \subset \boldsymbol{M}(X)$*; that is,* $H^\infty(\Omega) = \boldsymbol{M}(X)$*.*

(b) X has (DP).

Moreover, the least constant possible in the inequality that defines the property (DP) is $C = \|J\|$, where J is the correspondence operator $J :\ H^\infty \to \boldsymbol{M}(X)$, $J(F) = \boldsymbol{M}_F$.

In the special case when $\Omega = \mathbb{D}$, both conditions above are equivalent to the following:

(c) The set BP of all Blaschke products satisfies $BP \subset \boldsymbol{M}(X)$ and $\sup_{B\in BP} \|\boldsymbol{M}_B\|_{X\to X} < \infty$.

Proof. (b)⇒(a) Suppose X has (DP). Let $F \in H^\infty(\Omega)$. Then the function $\|F\|_\infty f \in X$ and

$$|F(z)f(z)| \le \|F\|_\infty |f(z)|$$

holds for all $z \in \mathbb{D}$, hence by (DP) we have $Ff \in X$ and $\|Ff\|_X \le C\|F\|_\infty \|f\|_X$. This shows that $F \in \mathbf{M}(X)$.

(a)⇒(b) Now assume that $H^\infty(\Omega) \subset \mathbf{M}(X)$ and let us show that the space X has (DP).

First of all, observe that the correspondence operator $J :\ H^\infty \to \mathbf{M}(X)$, given by $J(F) = \mathbf{M}_F$, is bounded. To verify this, it suffices to see that it has closed graph by an application of a normal families argument in the usual way: suppose that $F_n \to F$ in H^∞ and $J(F_n) = \mathbf{M}_{F_n} \to T$ in the space of bounded operators on X. In order to show that $T = \mathbf{M}_F$, we note that

$$\|F_n f - Tf\|_X \le \|\mathbf{M}_{F_n} - T\| \cdot \|f\|_X \to 0\,, \qquad n \to \infty\,.$$

In view of the boundedness of point evaluations, it follows that $F_n \to F$ pointwise (actually, uniformly on compact subsets) and also $F_n f \to Tf$ pointwise, hence $F_n f \to Ff$ pointwise and therefore $Tf = Ff$. This shows that $T = \mathbf{M}_F$.

Knowing that J is a bounded operator, there exists a universal constant $C > 0$ such that for each $F \in H^\infty(\Omega)$ we have $\|\mathbf{M}_F\| \le C\|F\|_\infty$. Thus, whenever $\|F\|_\infty \le 1$, we have that $\|\mathbf{M}_F\| \le C$ for some fixed constant C.

Let $f \in H(\Omega)$ and $g \in X$ be such that $|f(z)| \le |g(z)|$ holds for all $z \in \Omega$. The trivial case $g \equiv 0$ yields $f \equiv 0$ which clearly presents no problem. When g is not identically zero in Ω, every zero of g is a zero of f (of at least the same order) and is isolated so one easily extends the function $F = f/g$ to be analytic in the whole domain Ω. This function $F \in H^\infty(\Omega)$ and $\|F\|_\infty \le 1$. By the above observation based on the Closed Graph Theorem, there is a fixed constant C (independent of F) such that $\|\mathbf{M}_F\| \le C$. Hence, we obtain

$$\|f\|_X = \|Fg\|_X \le \|\mathbf{M}_F\| \|g\|_X \le C\|g\|_X\,.$$

This shows that X has (DP). It is also clear from the proof that the smallest possible value of C is precisely $\|J\|$.

(a), (b) ⇔ (c) It is readily seen that any of the assumptions (a), (b) implies (c). The converse follows from Marshall's Theorem 1.2 on the density of the convex hull of Blaschke products in the unit ball of H^∞. □

It is worth remarking that it does not seem easy to find a natural and satisfactory analogue of Theorem 6.2 for composition operators. That is, we have not been able to identify any property that would be equivalent to the statement: Every $\varphi \in H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ induces a bounded composition operator $\mathbf{C}_\varphi$ on X.

6.2 Weighted composition operators on the disk

In this section we only consider spaces of analytic functions in the disk that satisfy a fixed set of five natural axioms. We will prove two results: one characterizing all weighted composition operators among the bounded operators on such a space, and another, characterizing the invertible weighted composition operators among the bounded ones.

6.2.1 A characterization of weighted composition operators on spaces of the disk

The unilateral shift operator S, defined by $Sf(z) = zf(z)$, for $z \in \mathbb{D}$ and $f \in X$, makes sense and is often bounded on many functional Banach spaces $X \subset H(\mathbb{D})$. Clearly, $S = \mathbf{M}_z$.

Throughout the whole section, we will consider Banach spaces $X \subset H(\mathbb{D})$ that satisfy the following set of axioms:

Ax1 All point evaluation functionals Λ_z are bounded on X.

Ax2 The set of all algebraic polynomials of z is contained in X and dense in it (in $\|\cdot\|_X$).

Ax3 The shift operator $S = \mathbf{M}_z$ is bounded on X.

Ax4 $\limsup_{n\to\infty} \|z^n\|^{1/n} = 1$.

Ax5 Every disk automorphism induces a bounded composition operators on X.

Each one of these axioms appears in a most natural way in one context or another. The Hardy spaces H^p and the Bergman spaces A^p, $1 \le p < \infty$, satisfy all five axioms. More generally, the spaces $H(p, q, \alpha)$, $1 \le q < \infty$ and $H_0(p, \infty, \alpha)$ verify them whenever $1 \le p \le \infty$ and $\alpha > 0$. Also, the little Bloch space $\mathcal{B}_0$ and analytic Besov spaces B^p, $1 < p < \infty$, satisfy all of these axioms (see [131, Chapter 5], [76], or [42] and Chapter 1 and 2).

As we saw in Chapter 1, the weighted spaces H^2_β satisfy Axiom [**Ax1**] and clearly also Axiom [**Ax2**]. They satisfy Axiom [**Ax3**] if and only if $\sup_n \beta(n+1)/\beta(n) < \infty$ by [50, Proposition 2.7] and Axiom [**Ax4**] if and only if $\limsup_n \beta(n)^{1/n} = 1$ by a trivial calculation; a simple example of a coefficient weight that satisfies both is $\beta(n) = n^\alpha$, $\alpha \ge 0$ or $\beta(n) = (n+1)^\alpha$, $\alpha < 0$. Axiom [**Ax5**] is discussed in Chapter 2. Theorem 2.7 proves that all the weighted spaces H^2_β with $\beta(n) = (n+1)^\gamma$, $\gamma < 2$, satisfy [**Ax5**].

The non-separable spaces such as $\mathcal{B}$, H^∞, $H(p,\infty,\alpha)$ and also H_v^∞ fail to satisfy Axiom [**Ax2**].

Theorem 6.4, to be proved in this subsection, provides a generalization of the well-known fact that in Hardy spaces or Bergman spaces the only operators that commute with the shift are the pointwise multiplication operators. It also generalizes a well-known characterization of composition operators on H^2 which states that they are the only multiplicative operators on this space, in the sense that $T(fg) = Tf \cdot Tg$ for all f, $g \in H^2$ such that also $fg \in H^2$; see [91, Corollary 5.1.14, p. 170].

Before stating and proving Theorem 6.4 we will need a simple auxiliary statement, easy to prove and valid for arbitrary planar domains [59, Lemma 11, p. 57]; however, we will state it only for the disk.

Lemma 6.3. *Let $g \in H(\mathbb{D})$, $g \not\equiv 0$, let H be meromorphic in $\mathbb{D}$, and assume that $gH^n \in H(\mathbb{D})$ for all $n \geq 1$. Then $H \in H(\mathbb{D})$.*

It will become obvious from the proof that the extra assumption on the norms of the monomials, together with a minor technical assumption, forces the analytic function $\varphi = Tz/(T1)$ to become a self-map of the disk, which is the key to the validity of the result. It should be noted that Axiom [**Ax5**] is not required in the statement.

Theorem 6.4. *Let $X \subset H(\mathbb{D})$ be a functional Banach space in which the axioms [**Ax1**] - [**Ax4**] are fulfilled. Let T be a continuous operator on X with the property that $Tz \neq \lambda \cdot T1$ for every unimodular number λ. Then the following conditions are equivalent:*

(a) T is a weighted composition operator;

(b) There exists $\varphi \in H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\mathbf{M}_\varphi T = TS$;

(c) There exists φ, meromorphic in $\mathbb{D}$, such that $\mathbf{M}_\varphi T = TS$;

(d) There exists φ, meromorphic in $\mathbb{D}$, such that $\mathbf{M}_{\varphi^n} T = TS^n$ for all integers $n \geq 0$;

(e) There exists φ, meromorphic in $\mathbb{D}$, such that $\varphi^n \cdot T1 = T(z^n)$ for all integers $n \geq 0$;

(f) There exists $\varphi \in H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi^n \cdot T1 = T(z^n)$ for all integers $n \geq 0$;

(g) $T1 \cdot T(fg) = Tf \cdot Tg$ holds for all functions f, $g \in X$ for which $fg \in X$ as well.

Whenever any of the conditions (a)–(g) is fulfilled, then $\varphi = Tz/(T1)$ is the composition symbol of T.

Proof. It suffices to prove the chain of implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a) and then also (a) $\Rightarrow$ (g) $\Rightarrow$ (e).

(a) $\Rightarrow$ (b) Let $T = \mathbf{M}_F \mathbf{C}_\varphi$. Then for any f in X we have $\mathbf{M}_\varphi Tf = F\varphi(f \circ \varphi) = TSf$, hence $\mathbf{M}_\varphi T = TS$ holds. (Note that this step involves using [**Ax3**]).

(b) ⇒ (c) This implication is trivial.

(c) ⇒ (d) The statement is obvious for $n = 0$ and we also have it for $n = 1$ by (c). Now $\mathbf{M}_\varphi T = TS$ implies $TS^2 = \mathbf{M}_\varphi TS = \mathbf{M}_\varphi \mathbf{M}_\varphi T = \mathbf{M}_{\varphi^2} T$. Proceeding by induction, we obtain $TS^n = \mathbf{M}_{\varphi^n} T$ for all $n \geq 0$.

(d) ⇒ (e) Follows trivially by applying both sides of the equality in (d) to the constant function one.

(e) ⇒ (f) First, we have to distinguish between two possibilities: $T1 \equiv 0$ and $T1 \not\equiv 0$. If $T1 \equiv 0$, then it follows from (e) that $T(z^n) \equiv 0$ for all $n \geq 1$. But this contradicts the property that $Tz \neq \lambda \cdot T1$ for all λ of modulus one.

Knowing that $T1 \not\equiv 0$, it follows from the assumption (e) for $n = 1$ that $\varphi = Tz/(T1)$, a meromorphic function. Since $T(z^n) = T1 \cdot \varphi^n$ holds for all integers $n \geq 1$, it follows from Lemma 6.3 that actually $\varphi \in H(\mathbb{D})$.

It is only left to show that $|\varphi(\zeta)| < 1$ for all $\zeta \in \mathbb{D}$. Since $T1 \cdot \varphi^n \in X$ for all $n \geq 1$, by the boundedness of the point-evaluation functionals for an arbitrary but fixed ζ in $\mathbb{D}$ (by virtue of **[Ax1]**) we get

$$|(T1 \cdot \varphi^n)(\zeta)| \leq \|\Lambda_\zeta\| \cdot \|T1 \cdot \varphi^n\| = \|\Lambda_\zeta\| \cdot \|T(z^n)\| \leq \|\Lambda_\zeta\| \cdot \|T\| \cdot \|z^n\| .$$

Taking the n-th roots of both sides and then $\limsup_{n\to\infty}$, using condition **[Ax4]** we first get that $|\varphi(\zeta)| \leq 1$ for all $\zeta \in \mathbb{D}$. Next, it is immediate that $|\varphi(\zeta)| = 1$ for some $\zeta \in \mathbb{D}$ is impossible in view of the maximum modulus principle and the assumption that $Tz \neq \lambda \cdot T1$ for λ with $|\lambda| = 1$. Thus, φ is an analytic function in $\mathbb{D}$ which maps $\mathbb{D}$ into itself.

(f) ⇒ (a) Suppose that $\varphi^n \cdot T1 = T(z^n)$ for some analytic self-map φ of $\mathbb{D}$ and for all integers $n \geq 0$. We need to show that T is a weighted composition operator, namely, $Tf = F \cdot (f \circ \varphi)$ for this same mapping φ and $F = T1$.

By assumption, $T(z^n) = T1 \cdot \varphi^n = F \cdot \varphi^n$ for all integers $n \geq 0$, hence by linearity $Tp = F \cdot (p \circ \varphi)$ for every polynomial p. Now let f be an arbitrary function in X and $\{p_n\}_n$ a sequence of polynomials convergent to f in the norm of X (which exists by **[Ax2]**). Then, by continuity of T, we have $Tp_n \to Tf$ in X and since $Tp_n = F \cdot (p_n \circ \varphi)$, it follows that $F \cdot (p_n \circ \varphi) \to Tf$ in X, and hence pointwise as well. That is, $Tf(z) = F(z) \cdot (f \circ \varphi)(z)$ at every point z in $\mathbb{D}$.

(a) ⇒ (g) Let $Tf = F(f \circ \varphi)$. It is readily checked that (g) holds as both sides of the equality $T1 \cdot T(fg) = Tf \cdot Tg$ are equal to $F^2(f \circ \varphi)(g \circ \varphi)$.

(g) ⇒ (e) Assume that $T1 \cdot T(fg) = Tf \cdot Tg$ holds whenever f, g, $fg \in X$. It is clear that the identity in (e) holds trivially for $n = 0$ and that it holds for $n = 1$ for $\varphi = Tz/(T1)$, clearly a meromorphic function in $\mathbb{D}$ since Tz, $T1 \in X$. We now prove that $\varphi^n \cdot T1 = T(z^n)$ by induction on $n \geq 1$. Assuming that $T(z^n) = \varphi^n \cdot T1 = \frac{(Tz)^n}{(T1)^{n-1}}$ for some $n \geq 1$, we get from (g) with $f(z) = z^n$ and $g(z) = z$ and from the inductive hypothesis that

$$T(z^{n+1}) = \frac{T(z^n)T(z)}{T1} = \frac{(Tz)^{n+1}}{(T1)^n} = \varphi^{n+1} \cdot T1\,,$$

which completes the inductive proof. □

An inspection of the proof shows that we have effectively used all assumptions on T (linearity and boundedness), on φ, and on X. It should be clear that all of our assumptions are really needed but we still include some examples to illustrate this.

The following example shows that the result is false if the assumption [**Ax4**] is not fulfilled. In this case, the remaining axioms [**Ax1**] - [**Ax3**] hold in X and, in spite of the fact that the representation $Tf = F(f \circ \varphi)$ in view of (d) holds on a dense subset of X, namely for all polynomials, T cannot be represented in this fashion on the whole space.

Example 2. *Let X be the space from Example 1 formed by restrictions of functions in the Fock space to the disk and let T be the linear operator given by*

$$Tf(z) = f\left(\frac{z}{2} + 1\right), \quad f \in X, \quad z \in \mathbb{D}\,.$$

As some very simple integral estimates show, this is a bounded operator on X. (A more general result characterizing the symbols of all bounded composition operators on the Fock space can be found in [45].)

It is trivial to check that T satisfies condition (c) of Theorem 6.4 with $\varphi(z) = Tz/(T1) = \frac{z}{2} + 1$. However, φ is not a self-map of $\mathbb{D}$ so other conditions do not hold. Thus, even though the formal relation $T = \boldsymbol{M}_1 \boldsymbol{C}_\varphi = \boldsymbol{C}_\varphi$ holds, T is not a weighted composition operator on the disk in the sense considered here -in view of the fact that φ does not map $\mathbb{D}$ into itself.

It is important to note that there does not exist any other representation of T as a weighted composition operator with an admissible pair of symbols. Indeed, if we had $T = \boldsymbol{C}_\varphi = \boldsymbol{M}_F \boldsymbol{C}_\psi$ for some $F \in H(\mathbb{D})$ and some analytic self-map ψ of $\mathbb{D}$, then taking first $f \equiv 1$ and then $f(z) = z$ in $f \circ \varphi = F(f \circ \psi)$, this would imply $F \equiv 1$ and $\psi = \varphi$ respectively.

Our next example shows that the assumption [**Ax2**] is also essential. Indeed, if X does not contain the constant functions (even if the polynomials lacking the constant term are dense in the space), the conclusion of the result fails.

Example 3. *Let $X = A^2_0 = \{f \in A^2 : f(0) = 0\}$, the subspace of the Bergman space A^2 of codimension one equipped with the inherited norm. It is clear that this is a Hilbert space and [**Ax1**], [**Ax3**], and [**Ax4**] are satisfied. Also, X contains all polynomials that vanish at the origin and it is easy to see that they are actually dense in X. However, the constant function one does not belong to X.*

If we define the operator T by $Tf(z) = f'(0)z$, it is clear that this is a linear rank one operator from X into itself and satisfies the relation $\boldsymbol{M}_\varphi T = TS = 0$ for the *constant function* $\varphi \equiv 0$ *(which trivially maps* $\mathbb{D}$ *into itself). However, the operator* T *defined in this fashion cannot be written as a weighted composition operator in any way. In fact, if we had* $F(z) \cdot f(\varphi(z)) = f'(0)z$ *for some* F *analytic in* $\mathbb{D}$ *and some analytic mapping* φ *of* $\mathbb{D}$ *into itself, after substituting* $f(z) = z$ *we would get* $F(z)\varphi(z) = z$ *while substituting* $f(z) = z^2$ *would yield* $F\varphi^2 \equiv 0$*, hence either* $F \equiv 0$ *or* $\varphi \equiv 0$*, immediately leading to a contradiction.*

6.2.2 Invertibility of weighted composition operators in spaces on the disk

We already know that in certain "natural" spaces the inverse of an invertible composition operator is again a composition operator though this does not hold in general; *cf.* [50, Exercises 2.1.14 and 3.1.6]. We now address the following question: when is a weighted composition operator invertible? A rule of thumb should suggest that the composition symbol should be an automorphism of the domain and the multiplication symbol should be invertible (bounded from above and below) or, alternatively, a self-multiplier of the space whose multiplicative inverse is also a self-multiplier, but we already mentioned in the Introduction that it is possible to have a weighted composition operator which is bounded, surjective, and invertible and whose individual multiplication and composition operator are both unbounded operators. We will see an example of this type in this section.

Statements of this exact type have already appeared in the recent literature. Gunatillake [70] studied the spectrum of weighted composition operators with automorphic symbols acting on the Hardy space H^2. As a motivation for this, in the same paper he characterized the invertibility of the operator on H^2.

Theorem 6.5 (Gunatillake)**.** *The operator* $\boldsymbol{T}_{F,\varphi}$ *on* H^2 *is invertible if and only if* F *is both bounded and bounded away from zero on the unit disk and* φ *is an automorphism of the unit disk. The inverse operator is the weighted composition operator* $\boldsymbol{T}_{1/F\circ\varphi^{-1},\varphi^{-1}}$.

Two generalizations were obtained subsequently. Bourdon [37] obtained a version of the last statement in the context of sets of analytic functions in the disk (with no linear structure at all) that satisfy certain axioms.

Theorem 6.6 (Bourdon)**.** *Suppose that* X *is a set of functions analytic on* $\mathbb{D}$ *such that*

(i) $\boldsymbol{T}_{F,\varphi}$ *maps* X *to* X.

(ii) X *contains a nonzero constant function.*

(iii) X *contains a function of the form* $z \mapsto z + c$ *for some constant* c.

(iv) There is a dense subset S *of the unit circle such that for each point in* S *there is function in* X *that does not extend analytically to a neighborhood of that point.*

If $\boldsymbol{T}_{F,\varphi} : X \to X$ *is invertible, then* φ *is an automorphism of* $\mathbb{D}$.

Moreover, if X *is conformally invariant, then* $\boldsymbol{T}_{F,\varphi}$ *is invertible on* X *if and only if* φ *is an automorphism of* $\mathbb{D}$ *and* F *as well as* $\frac{1}{F}$ *are multipliers of* X.

He then applied his findings to certain specific spaces of the disk. Hyvärinen, Lindström, Nieminen and Saukko [77] obtained a generalization to other spaces of analytic functions on the disk with certain growth control.

Theorem 6.7 (Hyvärinen, Lindström, Nieminen and Saukko)**.** *Let X satisfy*

- *There is a positive constant s such that for each $f \in X$ and each $z \in \mathbb{D}$ we have $|f(z)| \lesssim \|f\|_X(1-|z|^2)^{-s}$ and for any $z \in \mathbb{D}$ there is some $f_z \in X$ with $\|f_z\|_X \le 1$ such that $f_z(z)(1-|z|^2)^s = 1$.*
- *There is a positive constant s such that $\|C_\varphi\| \lesssim (1-|\varphi(0)|^2)^{-s}$ whenever φ is an automorphism of $\mathbb{D}$.*
- *Polynomials are dense in X.*

The operator $\boldsymbol{T}_{F,\varphi}$ is invertible on X if and only if F is bounded and bounded away from zero on the unit disk and φ is an automorphism of the unit disk. The inverse operator of $T_{F,\varphi} : X \to X$ is also a weighted composition operator and it has the form

$$\boldsymbol{T}_{F,\varphi}^{-1} = \frac{1}{F \circ \varphi^{-1}} \boldsymbol{C}_{\varphi^{-1}}.$$

In this subsection we consider a different set of axioms and prove similar invertibility results which complement the earlier results. It is readily verified that every space of the disk that satisfies the hypotheses of [77] also satisfies our axioms **[Ax1] - [Ax5]**. Thus, our next result is more general. If compared with Bourdon's result [37, Theorem 2.2] which applies in a very general context of sets of functions without linear structure but with certain boundary properties, our result below gives an impression of being less general. However, we have neither been able to prove rigorously that our axioms **[Ax1] - [Ax4]** imply Bourdon's from [37, Theorem 2.2] nor to give an example of a space that satisfies our axioms but does not satisfy his. The main interest of the theorem may reside in the method of proof.

Theorem 6.8. *Let $X \subset H(\mathbb{D})$ be a functional Banach space in which the axioms **[Ax1] - [Ax4]** are satisfied, as in Theorem 6.4, and suppose that a weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is bounded in X.*

(a) If $\boldsymbol{T}_{F,\varphi}$ is invertible in X then its composition symbol φ is an automorphism of $\mathbb{D}$, the multiplication symbol F does not vanish in the disk, and the inverse operator $\boldsymbol{T}_{F,\varphi}^{-1}$ is another weighted composition operator $\boldsymbol{T}_{G,\psi}$, whose symbols are:

$$G = \frac{1}{F \circ \varphi^{-1}}, \qquad \psi = \varphi^{-1}. \tag{6.1}$$

*(b) Assuming that Axiom **[Ax5]** also holds, we have the following characterization.*

The weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is invertible on X if and only if its composition symbol φ is an automorphism of $\mathbb{D}$, the multiplication symbol F does not vanish in the disk, and $1/F \in \boldsymbol{M}(X)$. If this is the case, then F is also a self-multiplier of X and the inverse operator is $\boldsymbol{T}_{G,\psi}$, with the symbols given by (6.1).

Note that the condition $F \in \mathbf{M}(X)$ is not needed in the proof of (b), hence it is not listed among the hypotheses. It actually follows easily from the remaining assumptions. This may seem paradoxical but can be explained by the fact that we are assuming from the start that $\mathbf{T}_{F,\varphi}$ acts boundedly so in certain multiplications the boundedness of $\mathbf{M}_F$ is not really required.

Proof. (a) To simplify the notation, write $\mathbf{T}$ for our operator $\mathbf{T}_{F,\varphi}$ and U for its inverse. Then $\mathbf{T}U = U\mathbf{T} = I$, where I is the identity operator on X.

We first make sure that the possibility $Uz = \lambda \cdot U1$ (for some fixed λ with $|\lambda| = 1$) is ruled out: indeed, if this happens then $z = \mathbf{T}Uz = \lambda \mathbf{T}U1 = \lambda$ for all $z \in \mathbb{D}$, which is absurd. This shows the hypotheses of Theorem 6.4 are all satisfied so we can apply the statement.

We want to use part (e) of Theorem 6.4 to conclude that U is also a weighted composition operator. To this end, we ought to show that

$$U(z^n) = U1 \cdot \left(\frac{Uz}{U1}\right)^n , \quad n \geq 2 . \tag{6.2}$$

First observe that

$$1 = \mathbf{T}U1 = F(U1 \circ \varphi)$$

which shows that neither F nor $U1 \circ \varphi$ can vanish in $\mathbb{D}$ and

$$F = \frac{1}{U1 \circ \varphi} . \tag{6.3}$$

Now, let $g = Uz$ and $h = U(z^n)$. Then $\mathbf{T}g = z$ and

$$F(h \circ \varphi) = \mathbf{T}h = z^n = (\mathbf{T}g)^n = F^n(g^n \circ \varphi) .$$

It follows from (6.3) that

$$(h \circ \varphi)((U1)^{n-1} \circ \varphi) = g^n \circ \varphi .$$

Since φ is not identically constant, we deduce from the uniqueness principle for analytic functions that $h(U1)^{n-1} = g^n$ holds throughout $\mathbb{D}$. Thus, in view of the removable singularities,

$$h = \frac{g^n}{(U1)^{n-1}} = \left(\frac{g}{U1}\right)^n \cdot U1$$

also holds in all of $\mathbb{D}$. By the way g and h were defined, this means that (6.2) holds. Now Theorem 6.4 implies that U is a weighted composition operator.

Next, knowing that $U = \mathbf{T}_{G,\psi}$ for some $G \in H(\mathbb{D})$ and some analytic self-map ψ of $\mathbb{D}$, we find an explicit formula for U. Starting from

$$1 = U\mathbf{T}1 = U(F) = G(F \circ \psi)$$

it is clear that G does not vanish in $\mathbb{D}$. Using also the representation of $\mathbf{T} = \mathbf{T}_{F,\varphi}$ and the fact that $f(z) = z$ is also a function in X, we obtain

$$z = U\mathbf{T}z = U(F \cdot \varphi) = G(F \circ \psi)(\varphi \circ \psi) = U(F) \cdot (\varphi \circ \psi) = \varphi \circ \psi$$

so $\varphi \circ \psi = id$, the identity mapping of $\mathbb{D}$. By a similar reasoning, we also deduce that $\psi \circ \varphi = id$. Therefore both φ and ψ are bijective mappings of $\mathbb{D}$, hence they are disk automorphisms and mutually inverse.

As for the symbols of $U = \mathbf{T}^{-1}$, the above reasoning shows that $\psi = \varphi^{-1}$ and from (6.3) we get

$$G = U1 = (U1 \circ \varphi) \circ \psi = \frac{1}{F \circ \psi} = \frac{1}{F \circ \varphi^{-1}} \, .$$

(b) $\boxed{\Rightarrow}$ First suppose that $\mathbf{T} = \mathbf{T}_{F,\varphi}$ is invertible. By what we have already proved in part (a) using only axioms **[Ax1] - [Ax4]**, we know that F does not vanish in $\mathbb{D}$, that $U = \mathbf{T}^{-1}$ is also a weighted composition operator and we know that its symbols are given by (6.1). So, it is only left to show that $1/F \in \mathbf{M}(X)$, assuming also Axiom **[Ax5]**.

To this end, we first prove an auxiliary fact that $F \in \mathbf{M}(X)$. Let f be an arbitrary function in X. By our assumptions, every disk automorphism induces a bounded composition operator so $\mathbf{C}_\psi$ is bounded on X. Therefore $f \circ \psi \in X$ and also $\mathbf{T}(f \circ \psi) \in X$. But

$$\mathbf{T}(f \circ \psi) = F(f \circ \psi \circ \varphi) = Ff \, ,$$

a function clearly in X. This shows that $F \in \mathbf{M}(X)$.

Similarly, considering $\mathbf{C}_\varphi$ instead of $\mathbf{C}_\psi$, $U = \mathbf{T}^{-1}$ instead of $\mathbf{T}$ and $G = U1$ instead of $F = \mathbf{T}1$, we see that $G \in \mathbf{M}(X)$ as well.

In order to show that $1/F \in \mathbf{M}(X)$, let $f \in X$ be arbitrary. Since $\mathbf{C}_\psi$ is bounded on X it follows that $f \circ \psi \in X$, hence

$$G(f \circ \psi) = \frac{f \circ \psi}{F \circ \psi} \in X \, .$$

Finally, since $\mathbf{C}_\varphi$ is bounded on X, it follows that

$$\frac{f \circ \psi}{F \circ \psi} \circ \varphi = \frac{f}{F} \in X \, .$$

This shows that $1/F \in \mathbf{M}(X)$.

$\boxed{\Leftarrow}$ Now assume that $\mathbf{T}_{F,\varphi} = \mathbf{T}$ is bounded on X, its composition symbol φ is an automorphism of $\mathbb{D}$, the multiplication symbol F does not vanish in the disk, and $1/F \in \mathbf{M}(X)$. We want to show that $\mathbf{T}$ is invertible by checking that the operator $U = \mathbf{T}_{G,\psi}$, with G and ψ given by (6.1), acts boundedly on X and $\mathbf{T}U = U\mathbf{T} = I$.

In view of Axiom **[Ax5]**, both operators $\mathbf{C}_\varphi$ and $\mathbf{C}_\psi$ where $\psi = \varphi^{-1}$, are bounded on X. Thus, if $f \in X$ then $f \circ \varphi \in X$ and then also $(f \circ \varphi)/F \in X$ since $1/F \in \mathbf{M}(X)$. But $\mathbf{C}_\psi$ maps X to itself and therefore we also have that

$$\frac{f}{F \circ \psi} = \frac{f \circ \varphi}{F} \circ \psi \in X \, .$$

Since this holds for arbitrary $f \in X$, if follows that $G = 1/(F \circ \psi) \in \mathbf{M}(X)$. Now it is clear that the compositions

$$U\mathbf{T} = \mathbf{M}_G \mathbf{C}_\psi \mathbf{M}_F \mathbf{C}_\varphi \, , \qquad \mathbf{T}U = \mathbf{M}_F \mathbf{C}_\varphi \mathbf{M}_G \mathbf{C}_\psi$$

make sense as operators mapping X to itself, and a short and direct computation shows that both coincide with the identity operator I on X. □

EXAMPLE 4. *Consider the space $X = H^\infty_v$ with the non-radial weight $v(z) = |1 - z|$. This space contains H^∞ and also some unbounded functions such as $f(z) = \frac{1}{1-z}$. It is immediate that the point evaluations are bounded on X (and uniformly bounded on compact sets), with*

$$\frac{1}{2} \le \|\Lambda_z\| \le \frac{1}{|1-z|}$$

*for all z in $\mathbb{D}$ (for the first inequality, consider any non-zero constant function). By a standard normal family argument, it follows that X is a Banach space. The shift operator is trivially bounded on X, so the space satisfies the axioms [**Ax1**], [**Ax3**], and [**Ax4**]. It can also be checked that it fails to satisfy [**Ax2**] and [**Ax5**] so we cannot expect Theorem 6.8 to hold automatically. In fact, some of the conclusions in part (b) do not follow, and even more can be said.*

It is easy to check that the rotations do not necessarily generate bounded operators on X. If $\varphi(z) = -z$, then $\boldsymbol{C}_\varphi$ does not map X into itself: the function given by $f(z) = \frac{1}{1-z}$ is in X but $\boldsymbol{C}_\varphi f$ is not:

$$\|\boldsymbol{C}_\varphi f\| = \|f(-z)\| = \sup_{z\in\mathbb{D}} \left|\frac{1-z}{1+z}\right| = \infty \,.$$

Consider also the function $F(z) = \ell(z) = \frac{1+z}{1-z}$. Since it is unbounded, F cannot generate a bounded pointwise multiplier $\boldsymbol{M}_F$ from X into itself. Nonetheless, the weighted composition operator $\boldsymbol{T}_{F,\varphi}$ with

$$F(z) = \frac{1+z}{1-z}\,, \quad \varphi(z) = -z$$

is bounded on X. While such examples have appeared earlier in the literature, what seems remarkable about this situation is that our operator $\boldsymbol{T}_{F,\varphi}$ is even a surjective isometry and an involution:

$$\|\boldsymbol{T}_{F,\varphi}\| = \sup_{z\in\mathbb{D}} |1+z||f(-z)| = \sup_{w\in\mathbb{D}} |1-w||f(w)| = \|f\|\,, \qquad \boldsymbol{T}_{F,\varphi}^{-1} = \boldsymbol{T}_{F,\varphi}\,.$$

6.3 Invertibility in axiomatic spaces on general domains

In this section we work in a more general context of Banach spaces of analytic functions on a general bounded planar domain (without any connectivity assumptions). It should be noted that while the axioms in [37] are quite general they refer necessarily to the disk so they are not suited for this general context. Our axioms -when applied to the spaces on the disk- are still weaker than those assumed in [77] and hence our result is more general. The main idea is to avoid the use of Carleson measures and the technicalities typical of any individual space while relying on the properties of functions which are near extremal for the point evaluations.

It should be noted that in this section we only prove a theorem on invertibility, different from Theorem 6.8. Since the monomials are very special and so is the disk as a domain, it is not at all clear how an analogue of Theorem 6.4 would look like on a general bounded domain.

In what follows we consider Banach spaces $X \subset H(\Omega)$ that satisfy the following set of axioms (ordered following certain similarity with the previous section):

A1 All point evaluation functionals Λ_z are bounded on X.

A2 $f_0 \in X$, where $f_0(z) \equiv 1$.

A3 The shift operator is bounded on X.

A4 For every function $f \in X$ we have $\dfrac{|f(z)|}{\|\Lambda_z\|} \to 0$ as $\operatorname{dist}(z, \partial\Omega) \to 0$.

A5 Each automorphism of Ω induces a bounded composition operator in X.

One easily notices the similarities between the odd-numbered axioms in the above list with the corresponding ones from the previous section. Note that Axiom **(A2)** is much weaker than the second axiom required earlier while Axiom **(A4)** is different. It generalizes the well-known fact that in Hilbert spaces the normalized reproducing kernels tend to zero weakly: if we denote by K_z the reproducing kernel at z, then

$$\frac{|f(z)|}{\|\Lambda_z\|} = \frac{|\langle f, K_z\rangle|}{\|\Lambda_z\|} = \left|\left\langle f, \frac{K_z}{\|\Lambda_z\|}\right\rangle\right| \to 0\,.$$

All of the spaces in Chapter 1 satisfy Axiom **(A1)**, and Axioms **(A2)** and **(A3)** are easy to check in most of them. Axiom **(A4)** is satisfied by the Hardy spaces (see the remark after Proposition 1.3), the Bergman spaces (see Theorem 1.4), the Dirichlet space (Proposition 1.5), the weighted Hilbert spaces H^2_β with $\sum_{n=0}^\infty \frac{1}{\beta(n)^2} = \infty$ (Theorem 1.6), the Besov spaces (Theorem 1.8), the little Bloch space by definition, and the "little-oh" weighted Banach space H^0_v. The boundedness of the composition operators was discussed in Chapter 2.

As we mentioned in Section 6.2.1, the weighted Hilbert spaces spaces may or may not satisfy Axioms **(A3)** and **(A5)** depending on the sequence $\{\beta(n)\}$. Likewise, by Theorem 2.6 the boundedness of composition operators induced by automorphisms of the disk on the "little-oh" weighted Banach space H^0_v depends on the weight.

The mixed norm spaces $H(p, q, \alpha)$ satisfy all five axioms when $0 < q < \infty$, since **(A1)**, **(A2)** and **(A3)** are easy to check, Axiom **(A4)** is Proposition 4.9 and Axiom **(A5)** is Proposition 5.1.

It is quite routine to check from our axioms that if a weighted composition operator acts on a space:

$$\mathbf{T}_{F,\varphi} : X \to X\,, \qquad \mathbf{T}_{F,\varphi} f = F \cdot (f \circ \varphi),$$

where F is analytic in Ω and φ is an analytic self-map of Ω, then the operator is actually bounded. This follows from Axiom **(A1)** and the usual argument involving normal families and the closed graph theorem.

Note that in this context it is not clear at all what result, if any, should constitute an analogue of Theorem 6.4 as the geometry of the disk and the role played by the monomials z^n are quite special while here we are working with general bounded domains. Thus, we prove only one result in this section: a theorem on invertibility of a weighted composition operator based on the five axioms above.

We first need a simple topological lemma which states essentially that when we delete part of a domain we inevitably add some boundary (of course, we may lose some but that is not of interest here). As is usual, ∂D denotes the boundary of the set D while $D(z;r)$ will denote the open disk centered at z of radius r.

Lemma 6.9. *Let Ω be a bounded planar domain and D a non-empty domain contained in Ω, $D \neq \Omega$. Then $\partial D \cap \Omega \neq \emptyset$.*

Proof. Suppose that, on the contrary, $\partial D \cap \Omega = \emptyset$. Then for every z in Ω we can find a positive r such that either $D(z;r) \subset D$ or $D(z;r) \subset \mathbb{C} \setminus \overline{D}$.

Now there are three possible scenarios:

(1) If it happens that for every z in Ω there exists $r > 0$ such that $D(z;r) \subset D$ then $\Omega \subset D$, hence $D = \Omega$ which is absurd.

(2) If it turns out that for every z in Ω there exists $r > 0$ such that $D(z;r) \subset \mathbb{C} \setminus \overline{D}$ then $D \subset \Omega \subset \mathbb{C} \setminus \overline{D}$. But this means that $D = \emptyset$, which again contradicts our assumptions.

(3) In the remaining case, we can find $z_1 \in \Omega$ and $r_1 > 0$ such that $D(z_1;r_1) \subset D$ and also $z_2 \in \Omega$ and $r_2 > 0$ such that $D(z_2;r_2) \subset \mathbb{C} \setminus \overline{D}$. Since Ω is a planar domain, it is path connected so we can find a simple curve γ connecting z_1 with z_2 and entirely contained in Ω. Since $z_1 \in D$ and $z_2 \in \mathbb{C} \setminus \overline{D}$, the curve γ can neither be contained entirely in D nor in $\mathbb{C} \setminus \overline{D}$ there must exist a point z on the curve γ such that $z \in \partial D$. But then $z \in \Omega \cap \partial D = \emptyset$, which is again absurd.

This completes the proof. □

The following purely complex analysis statement may be of interest by itself. It should be compared with [99, Corollary 2.10, p. 25], a known statement with much more restrictive hypotheses and a bit stronger conclusion. It shows that for nicely behaved self-maps of a domain the behavior inside is somehow controlled by the behavior near the boundary. As is usual, we will denote by $d(z, \partial\Omega)$ the distance of the point z to the boundary of Ω and will often write $z \to \partial\Omega$ to denote the fact that $d(z, \partial\Omega) \to 0$.

Theorem 6.10. *Let Ω be a bounded planar domain and φ a non-constant analytic self-map of Ω with the property that $d(\varphi(z), \partial\Omega) \to 0$ whenever $d(z, \partial\Omega) \to 0$. Then $\varphi(\Omega) = \Omega$.*

Proof. Clearly, $\varphi(\Omega)$ is a domain contained in Ω. Suppose that $\varphi(\Omega) \neq \Omega$. Then by Lemma 6.9, $\Omega \cap \partial\varphi(\Omega) \neq \emptyset$, and we can find a point $w_0 \in \Omega \cap \partial\varphi(\Omega)$. Note that $w_0 \notin \varphi(\Omega)$. Hence there exists a sequence of points $w_n \in \varphi(\Omega)$ such that $w_n \to w_0$ and therefore also a sequence of points $z_n \in \Omega$ such that $w_n = \varphi(z_n) \to w_0 \in \Omega$, as $n \to \infty$. Since Ω is a bounded domain, some subsequence $z_{n_k} \to z_0 \in \overline{\Omega}$. Of course, $w_{n_k} \to w_0 \in \Omega$ and this will force a contradiction in both possible cases: $z_0 \in \Omega$ and $z_0 \in \partial\Omega$.

Indeed, if $z_0 \in \Omega$ then $w_0 = \lim \varphi(z_{n_k}) = \varphi(z_0)$, which is contradiction with the fact that $w_0 \notin \varphi(\Omega)$. And if $z_0 \in \partial\Omega$ then $z_{n_k} \to z_0$ means that $\lim_{k\to\infty} \operatorname{dist}(z_{n_k}, \Omega) = 0$ hence, by assumption, $\lim_{k\to\infty} \operatorname{dist}(\varphi(z_{n_k}), \Omega) = 0$. But since $\varphi(z_{n_k}) = w_0$, this means that $w_0 \in \partial\Omega$, which is in contradiction with $w_0 \in \Omega$. □

We are now ready to prove a theorem on invertibility for our set of five axioms. Since the axioms assumed here are much weaker than the ones in [77], our result is more general even in the case when $\Omega = \mathbb{D}$. The result, of course, also generalizes a theorem for H^2 from [70] whose proof has partially served as an inspiration although some entirely new techniques were required here.

The reader will undoubtedly notice that the statement below resembles Theorem 6.8 to some extent. Just like in Theorem 6.8, the condition $F \in \mathbf{M}(X)$ is not needed in the proof of (b), and thus it is not listed among the assumptions (it will again follow from the remaining ones).

Theorem 6.11. *Let $X \subset H(\Omega)$ be a Banach space which satisfies the set of axioms **(A1) - (A4)** and suppose that the weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is bounded in X.*

(a) If $\boldsymbol{T}_{F,\varphi}$ is invertible in X then its composition symbol φ is an automorphism of Ω and the multiplication symbol F does not vanish in Ω.

*(b) If, in addition to the axioms listed, the space X also satisfies Axiom **(A5)** we have the following characterization.*

The weighted composition operator $\boldsymbol{T}_{F,\varphi}$ is invertible on X if and only if its composition symbol φ is an automorphism of Ω, the multiplication symbol F does not vanish in Ω, and $1/F \in \boldsymbol{M}(X)$. If this is the case, then F is also a self-multiplier of X and the inverse operator is $\boldsymbol{T}_{G,\psi}$, with the symbols given by the formula (6.1)*, which formally reads as in the case of the disk.*

Proof. (a) Suppose that $\mathbf{T}_{F,\varphi}$ is invertible. We divide the proof into a few steps.

- φ is injective:

  First of all, the function F is in X since $f_0 \in X$ with $f_0(z) \equiv 1$ by Axiom **(A2)** and $F = \mathbf{T}_{F,\varphi} f_0 \in X$. Then, by Axiom **(A3)**, $f_1 \cdot F \in X$, with $f_1(z) = z$, $z \in \Omega$. By assumption, $\mathbf{T}_{F,\varphi}$ is invertible on X and hence surjective. Therefore, there exists a function $g \in X$ such that

  $$\mathbf{T}_{F,\varphi} g = F(g \circ \varphi) = f_1 \cdot F,$$

  that is, $g(\varphi(z)) = z$ for $z \in \Omega$. From here it follows that φ is injective: if α, $\beta \in \Omega$ are such that $\varphi(\alpha) = \varphi(\beta)$, then

  $$\alpha = g(\varphi(\alpha)) = g(\varphi(\beta)) = \beta\,.$$

- φ is onto and, thus, an automorphism of Ω, and also F does not vanish in Ω:

  Let $z \in \mathbb{D}$ be an arbitrary point and let $f_z \in X$ be a function in X such that $\|f_z\| = 1$ and $|f_z(z)| \geq C\|\Lambda_z\|$. Since the operator $\mathbf{T}_{F,\varphi}$ is surjective, there exists a function $g \in X$ such that $\mathbf{T}_{F,\varphi} g = F \cdot (g \circ \varphi) = f_z$. Taking into account that the operator $\mathbf{T}_{F,\varphi}$ is invertible, there is a constant $m > 0$ such that

$$m\|g\| \le \|\mathbf{T}_{F,\varphi} g\| = \|f_z\| = 1.$$

In view of Axiom **(A1)**, we have

$$|g(\varphi(z))| \le \|\Lambda_{\varphi(z)}\| \|g\| \le \frac{\|\Lambda_{\varphi(z)}\|}{m}.$$

From here it follows that

$$|F(z)| = \frac{|f_z(z)|}{|g(\varphi(z))|} \ge \frac{C\, m \|\Lambda_z\|}{\|\Lambda_{\varphi(z)}\|} > 0\,.$$

Equivalently,

$$0 < \frac{C\, m}{\|\Lambda_{\varphi(z)}\|} \le \frac{|F(z)|}{\|\Lambda_z\|}. \tag{6.4}$$

Since z was arbitrary, we conclude that this holds for all $z \in \mathbb{D}$.

By Axiom **(A4)**, we have

$$\frac{|F(z)|}{\|\Lambda_z\|} \to 0 \quad \text{as} \quad z \to \partial\Omega\,,$$

hence by (6.4):

$$\|\Lambda_{\varphi(z)}\| \to \infty \quad \text{as} \quad z \to \partial\Omega\,.$$

We will now see that this means that $\varphi(z) \to \partial\Omega$ as $z \to \partial\Omega$. If this was not the case, we could find a sequence $\{z_n\} \subset \Omega$ such that $z_n \to \partial\Omega$, $\|\Lambda_{\varphi(z_n)}\| \to \infty$ and $\{\varphi(z_n)\}$ is contained in a compact subset of Ω. For every function $f \in X$ and a compact subset K of Ω, the set of values $\{|\Lambda_{\varphi(z)} f| : z \in K\}$ is bounded as a consequence of Axiom **(A1)**. Hence by the uniform boundedness principle there exists a constant C_K such that $\|\Lambda_{\varphi(z_n)}\| \le C_K$, which is absurd.

Thus, φ has the property claimed and this implies that it is a surjective self-map of Ω by Theorem 6.10.

(b) As before, some of the conclusions follow from part (a) already proved. We prove the rest in two steps.

- F is a self-multiplier of X.

  Since φ is an automorphism of Ω, by Axiom **(A5)** the operator $\mathbf{C}_{\varphi^{-1}}$ is bounded and

$$\|\mathbf{M}_F f\| = \|\mathbf{T}_{F,\varphi} \mathbf{C}_{\varphi^{-1}} f\| \le M\|f\|$$

  for all $f \in X$. In other words, $\mathbf{M}_F$ is a bounded operator on X.

- $\frac{1}{F}$ is a self-multiplier of X.

  By (6.4), we have
$$|F(z)| \geq Cm \frac{\|\Lambda_z\|}{\|\Lambda_{\varphi(z)}\|}.$$
  In order to bound F from below, it suffices to see that
$$\|\Lambda_{\varphi(z)}\| = \|\Lambda_z \mathbf{C}_\varphi\| \leq \|\Lambda_z\| \|\mathbf{C}_\varphi\| . \tag{6.5}$$
  It follows from here that
$$|F(z)| \geq Cm \frac{\|\Lambda_z\|}{\|\Lambda_{\varphi(z)}\|} \geq \frac{Cm}{\|\mathbf{C}_\varphi\|} > 0 .$$
  for every $z \in \Omega$ since the composition operator with symbol φ is bounded by Axiom **(A5)**. Note that $\|\mathbf{C}_\varphi\| > 0$ by virtue of Axiom **(A2)**. Hence $\frac{1}{F}$ is analytic in Ω. Since the operator $\mathbf{T}_{F,\varphi}$ is surjective, for each $f \in X$ there exists $g \in X$ such that
$$f = \mathbf{T}_{F,\varphi} g = F \cdot (g \circ \varphi).$$
  Thus,
$$\frac{1}{F} f = g \circ \varphi \in X,$$
  since $\mathbf{C}_\varphi$ is a bounded operator on X. It follows that $\mathbf{M}_{\frac{1}{F}}$ is also bounded on X.

Conversely, if the following three assumptions are satisfied: the composition symbol φ is an automorphism of Ω, the multiplication symbol F does not vanish in Ω, and $1/F \in \mathbf{M}(X)$, we will show that $\mathbf{T}_{F,\varphi}$ is invertible and its inverse is given by the expected formula.

If φ is an automorphism of Ω, so is its inverse φ^{-1}. By Axiom **(A5)** the composition operator $\mathbf{C}_{\varphi^{-1}}$ is bounded on X. The multiplication operator $\mathbf{M}_{\frac{1}{F}}$ is bounded on X by assumption, hence the operator $\mathbf{C}_{\varphi^{-1}} \mathbf{M}_{\frac{1}{F}}$ is bounded. Moreover, for each function f in X we have
$$\mathbf{T}_{F,\varphi} \mathbf{C}_{\varphi^{-1}} \mathbf{M}_{\frac{1}{F}} f = \mathbf{T}_{F,\varphi} \left(\frac{f}{F} \circ \varphi^{-1} \right) = F \cdot \left(\left(\frac{f}{F} \circ \varphi^{-1} \right) \circ \varphi \right) = f$$
and also
$$\mathbf{C}_{\varphi^{-1}} \mathbf{M}_{\frac{1}{F}} \mathbf{T}_{F,\varphi} f = \mathbf{C}_{\varphi^{-1}} \mathbf{M}_{\frac{1}{F}} \left(F \cdot (f \circ \varphi) \right) = (f \circ \varphi) \circ \varphi^{-1} = f .$$
In other words, the operator $\mathbf{T}_{F,\varphi}$ is invertible and
$$\mathbf{T}_{F,\varphi}^{-1} = \mathbf{C}_{\varphi^{-1}} \mathbf{M}_{\frac{1}{F}} = \mathbf{M}_{\frac{1}{F \circ \varphi^{-1}}} \mathbf{C}_{\varphi^{-1}} = \mathbf{T}_{\frac{1}{F \circ \varphi^{-1}}, \varphi^{-1}} .$$
□

The space from Example 4 is again relevant here. This time it satisfies the axioms **(A1)**, **(A2)**, and **(A3)**. It can also be checked that it fails to satisfy **(A4)** and **(A5)**; for example, in view of our observations in Example 4, we see that in general

$$\frac{|f(z)|}{\|\Lambda_z\|} \geq 2|f(z)| \not\to 0\,, \qquad |z| \to 1^-\,.$$

Thus, we cannot expect Theorem 6.11 to hold automatically. In fact, we already know from Example 4 that not all conclusions in part (b) can hold since $1/F$ is an unbounded function and therefore cannot multiply X into itself.

Conclusions

The results of this thesis are contained in the following research papers:

- I. Arévalo, A characterization of the inclusions between mixed norm spaces, *J. Math. Anal. Appl.* **429** (2015), 942–955.
- I. Arévalo, R. Hernández, M. J. Martín and D. Vukotić, On weighted compositions preserving the Carathéodory class, arXiv preprint arXiv:1608.04577 (2016).
- I. Arévalo, M. D. Contreras and L. Rodríguez-Piazza, Semigroups of composition operators and integral operators on mixed norm spaces, arXiv preprint arXiv:1610.08784 (2016).
- I. Arévalo, D. Vukotić, Weighted composition operators in functional Banach spaces: an axiomatic approach, arXiv preprint arXiv:1706.07133 (2017).

There is another related paper which does not form part of this thesis:

- I. Arévalo, M. Oliva, Semigroups of weighted composition operators in spaces of analytic functions.

The topics studied in this thesis suggest further related and natural questions. Among them, we mention the following.

1) It is a question of interest to study the compactness of weighted composition operators assuming only certain axioms or properties such as (DP) or similar. In particular, it would be interesting to continue the study began in [32] and determine a general criterion for compactness.

2) Also on spaces defined by axioms, it would be of interest to describe when a weighted composition operators is a Fredholm operator, thus generalizing some of the results from [77].

We have already obtained several partial results on these topics but there is no time to finish our study and also be able to include all of our findings into this thesis. Hence, such investigation could form part of further research projects or, alternatively, some results could be included in a later version of the paper [14] (still unpublished).

Conclusiones

Los resultados de esta tesis están contenidos en los siguientes artículos de investigación:

- I. Arévalo, A characterization of the inclusions between mixed norm spaces, *J. Math. Anal. Appl.* **429** (2015), 942–955.
- I. Arévalo, R. Hernández, M. J. Martín y D. Vukotić, On weighted compositions preserving the Carathéodory class, arXiv preprint arXiv:1608.04577 (2016).
- I. Arévalo, M. D. Contreras y L. Rodríguez-Piazza, Semigroups of composition operators and integral operators on mixed norm spaces, arXiv preprint arXiv:1610.08784 (2016).
- I. Arévalo, D. Vukotić, Weighted composition operators in functional Banach spaces: an axiomatic approach, arXiv preprint arXiv:1706.07133 (2017).

El siguiente es otro artículo de investigación relacionado pero que no forma parte de esta tesis:

- I. Arévalo, M. Oliva, Semigroups of weighted composition operators in spaces of analytic functions.

Los temas estudiados en esta tesis sugieren nuevas preguntas relacionadas. Entre ellas, mencionamos las siguientes.

1) Es una pregunta de interés estudiar la compacidad de operadores de composición ponderados asumiendo sólo ciertos axiomas o propiedades como (DP) o similares. En particular, sería interesante continuar el estudio iniciado en [32] y determinar un criterio general para la compacidad.

2) También en espacios definidos por axiomas, sería de interés poder describir cuándo un operador de composición ponderado es un operador de Fredholm, generalizando así algunos de los resultados de [77].

Ya hemos obtenido algunos resultados parciales sobre estos temas, pero no ha habido tiempo de terminar nuestro estudio y de incluirlo en la tesis. Por lo tanto, dicha investigación podría formar parte de proyectos de investigación posteriores o, alternativamente, algunos resultados podrían ser incluidos en una versión posterior del artículo [14] (por publicar).

Bibliography

[1] P. Ahern and M. Jevtić, Duality and multipliers for mixed norm spaces, *Michigan Math. J.* **30** (1983), 53–64.

[2] A. Aleman, A class of integral operators on spaces of analytic functions, *Topics in complex analysis and operator theory*, 3–30, Univ. Málaga, Málaga, 2007.

[3] A. Aleman, P. L. Duren, M. J. Martín, and D. Vukotić, Multiplicative isometries and isometric zero-divisors of function spaces in the disk, *Canad. J. Math.* **62** (2010), 961–974.

[4] A. Aleman and A. G. Siskakis, An integral operator on H^p, *Complex Variables Theory Appl.* **28** (1995), 149–158.

[5] A. Aleman and A. G. Siskakis, Integration operators on Bergman spaces, *Indiana Univ. Math. J.* **46** (1997), 337–356.

[6] J. M. Anderson, J. Clunie, and Ch. Pommerenke, On Bloch functions and normal functions, *J. Reine Angew. Math.* **270** (1974), 12–37.

[7] J. M. Anderson and J. Duncan, Duals of Banach spaces of entire functions, *Glasgow Math. J.* **32** (1990), 215–220.

[8] J. Arazy and S. Fisher, The uniqueness of the Dirichlet space among Möbius-invariant Hilbert spaces, *Illinois J. Math.* **29** (1985), 449–462.

[9] J. Arazy, S. Fisher, and J. Peetre, Möbius invariant function spaces, *J. Reine. Angew. Math.* **363** (1985), 110–145.

[10] N. Arcozzi, R. Rochberg, E. T. Sawyer, and B. D. Wick, The Dirichlet space: a survey, *New York J. Math.* **17A** (2011), 45–86.

[11] I. Arévalo, A characterization of the inclusions between mixed norm spaces, *J. Math. Anal. Appl.* **429** (2015), 942–955. Corrigendum: *J. Math. Anal. Appl.* **433** (2016), 1904–1905.

[12] I. Arévalo, M. D. Contreras and L. Rodríguez-Piazza, Semigroups of composition operators and integral operators on mixed norm spaces, arXiv preprint arXiv:1610.08784 (2016).

[13] I. Arévalo, R. Hernández, M. J. Martín and D. Vukotić, On weighted compositions preserving the Carathéodory class, arXiv preprint arXiv:1608.04577 (2016).

[14] I. Arévalo and D. Vukotić, Weighted composition operators in functional Banach spaces: an axiomatic approach, arXiv preprint arXiv:1706.07133 (2017).

[15] R. Aron and M. Lindström, Spectra of weighted composition operators on weighted Banach spaces of analytic functions, *Israel J. Math.* **141** (2004), 263–276.

[16] M. Arsenović and R. Shamoyan, On some new theorems on multipliers in harmonic function spaces in higher dimension II, *Bull. Korean Math. Soc.* **50** (2013), 1451–1469.

[17] L. Attele, Analytic multipliers of Bergman spaces, *Michigan Math. J.* **31** (1984), 307–319.

[18] S. Axler, Zero multipliers of Bergman spaces, *Canad. Math. Bull.* **28** (1985), 237–242.

[19] S. Axler, Toeplitz operators *A glimpse at Hilbert space operators*, 125–133, Oper. Theory Adv. Appl., **207**, Birkhäuser Verlag, Basel, 2010.

[20] K. L. Avetisyan, A note on mixed norm spaces of analytic functions, *Aust. J. Math Anal. Appl.* **9** (2012), 1–6.

[21] K. L. Avetisyan, Sharp inclusions and lacunary series in mixed-norm spaces on the polydisc, *Complex Var. Elliptic Equ.* **58** (2013), 185–195.

[22] M. Basallote, M. D. Contreras, C. Hernández-Mancera, M. J. Martín, and P. J. Paúl, Volterra operators and semigroups in weighted Banach spaces of analytic functions, *Collect. Math.* **65** (2014), 233–249.

[23] A. Benedek and R. Panzone, The space L^p, with mixed norm, *Duke Math. J.* **28** (1961), 301–324.

[24] S. Bergman, *The Kernel Function and Conformal Mapping* Mathematical Surveys, No. 5. American Mathematical Society, New York, NY, 1950.

[25] E. Berkson and H. Porta, Semigroups of analytic functions and composition operators, *Michigan Math. J.* **25** (1978), 101–115.

[26] A. Beurling, On two problems concerning linear transormation on Hiblert space, *Acta Math.* **81** (1949), 239–255.

[27] K. A. Bierstedt, J. Bonet and J. Taskinen, Associated weights and spaces of holomorphic functions, *Studia Math.* **127** (1998), 137–168.

[28] O. Blasco, Multipliers on spaces of analytic functions, *Canad. J. Math.* **47** (1995), 44–64.

[29] O. Blasco, M. D. Contreras, S. Díaz-Madrigal, J. Martínez, M. Papadimitrakis, and A. G. Siskakis, Semigroups of composition operators and integral operators in spaces of analytic functions, *Ann. Acad. Sci. Fenn. Math.* **38** (2013), 67–89.

[30] O. Blasco, M. D. Contreras, S. Díaz-Madrigal, J. Martínez, and A. G. Siskakis, Semigroups of composition operators in BMOA and the extension of a theorem of Sarason, *Integral Equations Operator Theory* **61** (2008), 45–62.

[31] A. Bloch, Les théorèmes de M. Valiron sur les fonctions entières et la théorie de l'uniformisation, *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.* **3** (1925), 1–22.

[32] J. Bonet, P. Domański, M. Lindström, and J. Taskinen, Composition operators between weighted Banach spaces of analytic functions, *J. Austral. Math. Soc. Ser. A* **64** (1998), 101–118.

[33] J. Bonet, M. Lindstrom, and E. Wolf, Isometric weighted composition operators on weighted Banach spaces of type H^∞, *Proc. Amer. Math. Soc.* **136** (2008), 4267–4273.

[34] J. Bonet, M. Lindstrom, and E. Wolf, Norm-attaining weighted composition operators on weighted Banach spaces of analytic functions, *Arch. Math. (Basel)* **99** (2012), 537–546.

[35] A. Böttcher and Y.I. Karlovich, Carleson curves, Muckenhoupt weights, and Toeplitz operators, *Progress in Mathematics* **154**, Birkhäuser Verlag, Basel, 1997.

[36] A. Böttcher and B. Silbermann, Analysis of Toeplitz operators, Springer- Verlag, Berlin, 1990.

[37] P.S. Bourdon, Invertible weighted composition operators, *Proc. Amer. Math. Soc.*, **142** (2014), no. 1, 289–299.

[38] P. Bourdon and J.H. Shapiro, Riesz composition operators, *Pacific J. Math.* **181** (1997), 231–246.

[39] J. E. Brennan, The integrability of the derivative in conformal mapping, *J. London Math. Soc.* **18** (1978), 261–272.

[40] L. Brown and A. L. Shields, Multipliers and cyclic vectors in the Bloch space, *Michigan Math. J.* **38** (1991), 141–146.

[41] S. M. Buckley, P. Koskela and D. Vukotić, Fractional integration, differentiation, and weighted Bergman spaces, *Math. Proc. Cambridge Philos. Soc.* **126** (1999), 369–385.

[42] S.M. Buckley and D. Vukotić, Univalent interpolation in Besov spaces and superposition into Bergman spaces, *Potential Anal.* **29** (2008), 1–16.

[43] C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen, *Math. Ann.*, **64** (1907), 95–115

[44] C. Carathéodory, *Theory of Functions. II*, Chelsea, New York, NY 1960.

[45] B. Carswell, B. MacCluer, and A.P. Schuster, Composition operators on the Fock space, *Acta Sci. Math. (Szeged)* **69** (2003), no. 3–4, 871–887.

[46] J. Cima, A.G. Siskakis, Cauchy transforms and Cesàro averaging operators, *Acta Sci. Math. (Szeged)* **65** (1999), no. 3-4, 505–513.

[47] F. Colonna and M. Tjani, Operator norms and essential norms of weighted composition operators between Banach spaces of analytic functions, *J. Math. Anal. Appl.* **434** (2016), 93–124.

[48] M. D. Contreras and A.G. Hernández-Díaz, Weighted composition operators in weighted Banach spaces of analytic functions, *J. Austral. Math. Soc. Ser. A* **69** (2000), 41–60.

[49] M. D. Contreras and A.G. Hernández-Díaz, Weighted composition operators on Hardy spaces, *J. Math. Anal. Appl.* **263** (2001), no. 1, 224–233.

[50] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Ratón 1995.

[51] Ž. Čučkovic and R. Zhao, Weighted composition operators on the Bergman space, *J. London Math. Soc. (2)* **70** (2004), 499–511.

[52] Ž. Čučkovic and R. Zhao, Weighted composition operators between different weighted Bergman spaces and different Hardy spaces, *Illinois J. Math.* **51** (2007), 479–498.

[53] E. Diamantopoulos and A. G. Siskakis, Composition operators and the Hilbert matrix, *Studia Math.* **140** (2000), 191–198.

[54] M. M. Djrbashian, On the representability of certain classes of functions meromorphic in the unit disk, *Akad. Nauk Armyan. SSR Dokl.* **3** (1945), 3–9 (in Russian).

[55] M. M. Djrbashian, On the problem of representing analytic functions, *Soobshch. Inst. Mat. Mekh. Akad. Nauk Armyan. SSR* **2** (1948), 3–40 (in Russian).

[56] M. Dostanić, M. Jevtić, and D. Vukotić, Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type, *J. Funct. Anal.* **254** (2008), 2800–2815.

[57] P. L. Duren, *Theory of H^p Spaces*, Dover Publications, Mineola, NY 2000.

[58] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, NY 1983.

[59] P. L. Duren and A. Schuster, *Bergman Spaces*, American Mathematical Society, Providence, RI 2004.

[60] T. Eklund, P. Galindo, M. Lindström, and I. Nieminen, Norm, essential norm and weak compactness of weighted composition operators between dual Banach spaces of analytic functions, *J. Math. Anal. Appl.* **451** (2017), 1–13.

[61] O. El-Fallah, K. Kellay, J. Mashreghi, and T. Ransford, *A primer on the Dirichlet space*, Cambridge Tracts in Mathematics **203** Cambridge University Press, Cambridge, 2014.

[62] N. S. Feldman, Pointwise multipliers from the Hardy space to the Bergman space, *Illinois J. Math.* **43** (1999), 211–221.

[63] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.* **38** (1972), 746–765.

[64] T. M. Flett, Lipschitz spaces of functions on the circle and the disk, *J. Math. Anal. Appl.* **39** (1972), 125–158.

[65] F. Forelli, The isometries of Hp, *Canad. J. Math.* **16** (1964), 721–728.

[66] S. Gadbois, Mixed-norm generalizations of Bergman spaces and duality, *Proc. Amer. Math. Soc.* **104** (1988), 1171–1180.

[67] P. Galanopoulos, D. Girela, and M. J. Martín, Besov spaces, multipliers and univalent functions, *Complex Anal. Oper. Theory*, **7** (2013), 1081–1116.

[68] J. B. Garnett, *Bounded analytic functions*, Graduate Texts in Mathematics, 236, Springer, New York, NY 2007.

[69] J. B. Garnett and D. E. Marshall, *Harmonic Measure*, Cambridge University Press, Cambridge 2005.

[70] G. Gunatillake, Invertible weighted composition operators, *J. Funct. Anal.* **261** (2011), 831–860.

[71] G. H. Hardy, The mean value of the modulus of an analytic function, *Proc. London Math. Soc.* **14** (1915), 269–277.

[72] G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals. II., *Math. Z.* **34** (1932), 403–439.

[73] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, 199, Springer-Verlag, New York, NY 2000.

[74] E. Hille, *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, **31**. American Mathematical Society, New York, 1948.

[75] K. Hoffman, *Banach spaces of analytic functions*, reprint of the 1962 original, Dover Publications, Inc., New York, NY 1988.

[76] F. Holland and D. Walsh, Growth estimates for functions in the Besov spaces A_p, *Proc. Roy. Irish Acad. Sect. A* **88** (1988), 1–18.

[77] O. Hyvärinen, M. Lindström, I. Nieminen, and E. Saukko, Spectra of weighted composition operators with automorphic symbols, *J. Funct. Anal.* **265** (2013), 1749–1777.

[78] M. Jevtić, D. Vukotić, and M. Arsenović, *Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-Type Spaces*, RSME Springer Series, Vol. **2**, Springer, Cham, Switzerland 2016.

[79] H. Kamowitz, Compact operators of the form uC_φ, *Pacific J. Math.* **80** (1979), no. 1, 205–211.

[80] A.K. Kitover, The spectrum of operators in ideal spaces (Russian), Investigations on linear operators and the theory of functions, VII, *Zap. Naucn. Sem. Leningrad. Otdel Mat. Inst. Steklov (LOMI)* **65** (1976), 196–198, 209–210.

[81] G. Koenigs, Recherches sur les intégrales de certaines équations fonctionnelles, *Ann. Sci. École Norm. Sup.* 3, **1** (1884), 3–41.

[82] C. J. Kolaski, Isometries of weighted Bergman spaces, *Canad. J. Math.* **34** (1982), 910–915.

[83] P. Koosis, *Introduction to H_p Spaces* (Second Edition), Cambridge Univ. Press, Cambridge 1999.

[84] S. Kumar and K. J. Singh, Weighted composition operators between Besov spaces, *Extracta Math.* **23** (2008), 217–234.

[85] J. E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.* **23** (1925), 481–519.

[86] H. P. Lotz, Semigroups on L^∞ and H^∞, in *One-parameter semigroups of positive operators, Lecture Notes Math.* **1184**, Springer Verlag, Berlin, Heidelberg, New York, 1986, 54–58

[87] D. H. Luecking, Multipliers of Bergman spaces into Lebesgue spaces, *Proc. Edinburgh Math. Soc.* **29** (1986), 125–131.

[88] W. Lusky, On generalized Bergman spaces, *Studia Math.* **119** (1996), 77–95.

[89] M. Mateljević, The isoperimetric inequality and some extremal problems in H^1, *Lecture Notes in Math.* **798** (1980), 364–369.

[90] M. Mateljević and M. Pavlović, L^p-behavior of power series with positive coefficients and Hardy spaces, *Proc. Amer. Math. Soc.* **87** (1983), 309–316.

[91] R.A. Martínez-Avendaño and P. Rosenthal, *An introduction to operators on the Hardy-Hilbert space*, Graduate Texts in Mathematics, **237**, Springer, New York 2007.

[92] N.K. Nikol'skii, editor, *Toeplitz operators and spectral function theory*, Essays from the Leningrad Seminar on Operator Theory. Operator Theory: Advances and Applications, **42**, Birkhäuser Verlag, Basel, 1989.

[93] E. A. Nordgren, Composition operators, *Canad. J. Math.* **20** (1968), 442–449.

[94] S. Ohno, R. Zhao, Weighted composition operators on the Bloch space, *Bull. Austral. Math. Soc.* **63** (2001), 177–185.

[95] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, **44**. Springer-Verlag, New York, 1983.

[96] Ch. Pommerenke, On Bloch functions, *J. London Math. Soc.* **2** (1970), 689–695.

[97] Ch. Pommerenke, On the angular derivative and univalence, *Anal. Math.* **3** (1977), 291–297.

[98] Ch. Pommerenke, Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation, *Comment. Math. Helv.* **52** (1977), 591–602.

[99] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin 1992.

[100] L. A. Rubel and A. L. Shields, The second duals of certain spaces of analytic functions, *J. Austral. Math. Soc.* **11** (1970), 276–280.

[101] L. A. Rubel and R. M. Timoney, An extremal property of the Bloch space, *Proc. Amer. Math. Soc.* **75** (1979), 45–49.

[102] J. V. Ryff, Subordinate Hp functions, *Duke Math. J.* **33** (1966), 347–354.

[103] D. Sarason, Holomorphic spaces: a brief and selective survey. Holomorphic spaces (Berkeley, CA, 1995), 1–34, *Math. Sci. Res. Inst. Publ.*, **33**, Cambridge Univ. Press, Cambridge, 1998.

[104] E. Schöder, Über iterierte Funktionen, *Math. Ann.* **3** (1871), 296–322.

[105] H.J. Schwartz, Composition Operators on Hp. Ph.D. Thesis, The University of Toledo, OH 1969.

[106] W. Seidel and J. L. Walsh, On the derivatives of functions analytic in the unit circle and their radii of univalence and p-valence, *Trans. Amer. Math. Soc.* **52** (1942), 128–216.

[107] J.H. Shapiro, Compact composition operators on spaces of boundary-regular holomorphic functions, *Proc. Amer. Math. Soc.* **100** (1987), 49–57.

[108] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, NY 1993.

[109] A.L. Shields, Weighted shift operators and analytic function theory, in: *Topics in operator theory*, pp. 49–128, Math. Surveys, No. **13**, Amer. Math. Soc., Providence, RI 1974.

[110] A. L. Shields and D. L. Williams, Bounded projections, duality, and multipliers in spaces of harmonic functions, *J. Reine Angew. Math.* **299/300** (1978), 256–279.

[111] A. L. Shields and D. L. Williams, Bounded projections and the growth of harmonic conjugates in the disc, *Michigan Math. J.* **29** (1982), 3–25

[112] S. Shimorin, Weighted composition operators associated with conformal mappings, *Quadrature Domains and Their Applications*, 217–237, Oper. Theory Adv. Appl., **156**, Birkhauser, Basel, 2005.

[113] R. K. Singh, A relation between composition operators on $H2(D)$ and $H2(P+)$, *Pure Appl. Math. Sci.* **1** (1974/75), 1–5.

[114] A. Siskakis, Semigroups of composition operators in Bergman spaces, *Bull. Austral. Math. Soc.* **35** (1987), 397–406.

[115] A. Siskakis, Semigroups of composition operators on the Dirichlet space, *Results Math.* **30** (1996), 165–173.

[116] A. Siskakis, Semigroups of composition operators on spaces of analytic functions, a review, *Contemp. Math.* **232** (1990), 229–252.

[117] A. Siskakis, Weighted composition semigroups on Hardy spaces, *Linear Algebra Appl.* **84** (1986), 359–371.

[118] W. T. Sledd, Some results about spaces of analytic functions introduced by Hardy and Littlewood, *J. London Math. Soc.* **9** (1974/75), 328–336.

[119] W. Smith, Brennan's conjecture for weighted composition operators, *Recent Advances in Operator-Related Function Theory*, 209–214, Contemp. Math., **393**, Amer. Math. Soc., Providence, RI, 2006.

[120] D. A. Stegenga, Multipliers of the Dirichlet space, *Illinois J. Math.* **24** (1980), 113–139.

[121] M. Tjani, Compact composition operators on Besov spaces, *Trans. Amer. Math. Soc.* **355** (2003), 4683–4698.

[122] G. Valiron, Sur l'iteration des fonctions holomorphes dans un demi-plan, *Bull. Sci. Math.* 55 (1931), 105–128.

[123] D. Vukotić, A sharp estimate for A^p_α functions in C^n, *Proc. Amer. Math. Soc.* **117** (1993), 753–756.

[124] D. Vukotić, Pointwise multiplication operators between Bergman spaces on simply connected domains, *Indiana Univ. Math. J.* **48** (1999), 793–803.

[125] D. Vukotić, Analytic Toeplitz operators on the Hardy space H^p: a survey, *Bull. Belg. Math. Soc. Simon Stevin* **10** (2003), 101–113.

[126] D. Vukotić, The isoperimetric inequality and a theorem of Hardy and Littlewood, *Amer. Math. Monthly* **110** (2003), 532–536.

[127] D. Vukotić, On norms of composition operators acting on Bergman spaces, *J. Math. Anal. Appl.* **291** (2004), 189–202. Corrigendum: *J. Math. Anal. Appl.* **311** (2005), 377–380.

[128] K. Yosida, On the differentiability and the representation of one-parameter semigroups of linear operators, *J. Math. Soc. Japan* **1** (1948), 15–21.

[129] K. Yosida, *Functional Analysis*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[130] R. Zhao, Pointwise multipliers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces, *Ann. Acad. Sci. Fenn. Math.* **29** (2004), 139–150.

[131] K. Zhu, *Operator Theory on Function Spaces*, Marcel Dekker, Inc., New York and Basel, 1990.

[132] K. Zhu, Analytic Besov spaces, *J. Math. Anal. Appl.* **157** (1991), 318–336.

[133] K. Zhu, *Analysis on Fock Spaces*, Springer, New York 2012.

[134] N. Zorboska, Composition operators on S_a spaces, *Indiana Univ. Math. J.* **39** (1990), 847–857.

[135] N. Zorboska, Multiplication and Toeplitz operators on the analytic Besov spaces, Proceedings of the 5th International ISAAC Congress held at the University of Catania, Catania, July 25?30, 2005. Edited by H. G. W. Begehr and F. Nicolosi. *World Scientific Publishing Co. Pte. Ltd.*, Hackensack, NJ, 2009.

Index