OPERATORS ON HILBERT SPACES

3. Part II. Hilbert spaces of analytic functions on the unit disc

3.1. **Notation.** Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. Let $\mathbb{T}$ be the boundary of $\mathbb{D}$, let $D(a,r) = \{z : |z-a| < r\}$ denote the Euclidean disc of center a and radius r and let $\overline{D(a,r)} = \{z : |z-a| \leq r\}$ the corresponding closed disc.
Throughout these notes we shall write $f(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}(\mathbb{D})$. We will write $\|T\|_{(X,Y)}$ for the norm of an operator $T : X \to Y$, and if no confusion arises with regards to X and Y, we will simply write $\|T\|$. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \asymp E_2$, or $E_1 \lesssim E_2$, if there exists a positive constant k independent of the argument such that $\frac{1}{k}E_1 \leq E_2 \leq kE_1$, respectively $E_1 \leq kE_2$.

3.2. **Hardy spaces.** If $0 < r < 1$ and $f \in \mathcal{H}(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \, dt\right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

Whenever $0 < p \leq \infty$ the Hardy space H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

Since, $M_p(r,f)$ is an non-decreasing function of r, $\|f\|_{H^p} = \lim_{r\to 1^-} M_p(r,f)$.
We observe that $\|\cdot\|_{H^2}$ is the norm induced by the inner product

$$\begin{aligned}\langle f,g\rangle_{H^2} &= \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\, dt = \sum_{n=0}^{\infty} a_n\overline{b_n}, \quad f,g \in H^2.\end{aligned} \tag{3.1}$$

It is useful for the study of a several questions (and operators acting on H^2) to provide an equivalent norm in terms of the derivative. In this case, the classical Littlewood-Paley [20] formula says that

$$\|f\|_{H^2}^2 = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|}\, dA(z) \tag{3.2}$$

$dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. Since $\log\frac{1}{|z|} \asymp (1-|z|^2)$, $\frac{1}{2} \leq |z| < 1$, sometimes H^2 is equipped with the equivalent norm

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)\, dA(z) \tag{3.3}$$

We refer to [17, 20] for the theory of Hardy spaces.

3.3. **Weighted Bergman spaces.** A function $\omega : \mathbb{D} \to (0, \infty)$, integrable over $\mathbb{D}$, is called a *weight function* or simply a *weight*. It is *radial* if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$.
For $0 < p < \infty$ and a weight ω, the *weighted Bergman space* A^p_ω consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$.
As usual, we write A^p_α for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (1-|z|^2)^\alpha$, $-1 < \alpha < \infty$ and simply A^p for A^p_0.
It is clear that $\|\cdot\|_{A^2_\omega}$ is the norm induced by the inner product

$$\langle f, g \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(z)\overline{g(z)}\omega(z)\, dA(z), \quad f, g \in A^2_\omega \tag{3.4}$$

Mainly, we shall deal with radial continuous weights. In that case,

$$\begin{aligned} \|f\|^2_{A^2_\omega} &= 2\int_0^1 M_2^2(r,f)\omega(r)\, rdr = 2\int_0^1 \left(\sum_{n=0}^\infty |a_n|^2 r^{2n}\right)\omega(r)\, rdr \\ &= \sum_{n=0}^\infty |a_n|^2 \left(2\int_0^1 r^{2n+1}\omega(r)\, dr\right) \end{aligned}$$

and

$$\langle f, g \rangle_{A^2_\omega} = \sum_{n=0}^\infty a_n \overline{b_n}\left(2\int_0^1 r^{2n+1}\omega(r)\, dr\right) \tag{3.5}$$

There exists a Littlewood-Paley type inequality for weighted Bergman spaces A^p_α, $0 < p < \infty$ $-1 < \beta < \infty$ [19, 44]

$$\|f\|^p_{A^p_\beta} \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p\, (1-|z|)^{p+\beta}\, dA(z), \tag{3.6}$$

which gives an equivalent norm for A^p_β.
For generalizations of (3.6) see [3, 34, 40] and for the theory of the classical weighted Bergman spaces, see [18, 24, 44].
It is worth to comment that functions in weighted Bergman spaces A^p_α may have wild boundary behavior, their zero-sets are difficult to fathom, there is no obvious analogue of Blaschke products, the invariant subspaces (of $M_z(f) = zf$) need not be singly generated as they are (according to Beurling's theory) for the Hardy spaces.
Further results on weighted Bergman spaces induced by several classes of radial weights or Bekollé-Bonami weights can be found in [3, 9, 10, 35, 38].

3.4. **Classical weighted Dirichlet spaces.** For $\alpha > -1$, the weighted Dirichlet-type space $\mathcal{D}^p_\alpha$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|^p_{\mathcal{D}^p_\alpha} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha\, dA(z) < \infty.$$

We simply write $\mathcal{D}_\alpha$ for the space $\mathcal{D}^2_\alpha$. We have that $\|\cdot\|_{\mathcal{D}_\alpha}$ is the norm induced by the inner product

$$\langle f, g \rangle_{\mathcal{D}_\alpha} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\,\overline{g'(z)}(1-|z|^2)^\alpha\, dA(z), \quad f, g \in \mathcal{D}_\alpha. \tag{3.7}$$

Since for any $\alpha > -1$,

$$\int_0^1 r^{2n+1}(1-r^2)^\alpha\, dr \asymp n^{-(\alpha+1)},$$

it follows

$$\begin{aligned}||f\|_{\mathcal{D}_\alpha}^2 &= |a_0|^2 + 2\int_0^1 M_2^2(r,f')r(1-r^2)^\alpha\,dr\\ &= |a_0|^2 + 2\int_0^1 \left(\sum_{n=0}^\infty (n+1)^2|a_{n+1}|^2 r^{2n+1}\right)(1-r^2)^\alpha\,dr\\ &= |a_0|^2 + \sum_{n=0}^\infty (n+1)^2|a_{n+1}|^2\left(2\int_0^1 r^{2n+1}(1-r^2)^\alpha\,dr\right) \asymp \sum_{n=0}^\infty (n+1)^{1-\alpha}|a_n|^2,\end{aligned}$$

that is

$$||f\|_{\mathcal{D}_\alpha}^2 \asymp \sum_{n=0}^\infty (n+1)^{1-\alpha}|a_n|^2$$

which provides an equivalent norm in $\mathcal{D}_\alpha$ in terms of Taylor coefficients.

Basic inclusions or identities for Dirichlet-type spaces

Lemma 3.1. (i) $\mathcal{D}_\alpha^p \subset H^\infty$, *if* $-1 < \alpha < p-2$.
(ii) $\mathcal{D}_\alpha^p = A_{\alpha-p}^p$, *if* $p-1 < \alpha$.

Proof. (i) Take $f \in \mathcal{D}_\alpha^p$. Then, for any $0 < r < 1$

$$\begin{aligned}\infty > ||f\|_{\mathcal{D}_\alpha^p} &\gtrsim \int_r^1 M_p^p(s,f')s(1-s^2)^\alpha\,ds\\ &\gtrsim M_p^p(r,f')\int_r^1 s(1-s^2)^\alpha\,ds\\ &\asymp M_p^p(r,f')(1-r)^{\alpha+1},\end{aligned}$$

that is, $M_p(r,f') \lesssim (1-r)^{\frac{-(\alpha+1)}{p}}$.
On the other hand, for any $g \in H(\mathbb{D})$ and $0 < p \le q \le \infty$ [17, Chapter 5]

$$M_q(r,g) \lesssim M_p\left(\frac{1+r}{2}, g\right)(1-r)^{\frac{1}{q}-\frac{1}{p}}, \quad 0 < r < 1.$$

So,

$$M_\infty(r,f') \lesssim M_p\left(\frac{1+r}{2}, f'\right)(1-r)^{-\frac{1}{p}} \lesssim (1-r)^{\frac{-(\alpha+2)}{p}},$$

which gives

$$M_\infty(r,f) \lesssim |f(0)| + \int_0^r M_\infty(s,f')\,ds \lesssim |f(0)| + \int_0^r (1-s)^{\frac{-(\alpha+2)}{p}}\,ds \le C < \infty,$$

(ii) follows taking $\beta = \alpha - p$ in (3.6). This finishes the proof. □

So $\mathcal{D}_\alpha^p$ becames a "proper" Dirichlet space for the range $p-2 \le \alpha \le p-1$. In particular, for $p=2$, the interesting range is $0 \le \alpha \le 1$. We recall that $H^2 = \mathcal{D}_1^2$ (3.2). On the other hand, $\mathcal{D}_0$ is just the classical Dirichlet space and, as usual, will be simply denoted by $\mathcal{D}$. See [37, 7] for the theory of the classical Dirichlet space.

3.5. Point evaluations and uniform convergence on compact subsets.

Lemma 3.2. *If* $-1 < \alpha \leq 1$, *then*

$$\mathcal{D}^2_\alpha \subset H^2 \subset \bigcap_{\beta > -1} A^2_\beta,$$

indeed

$$||f||_{A^2_\beta} \leq ||f||_{H^2} \lesssim ||f||_{\mathcal{D}^2_\alpha}, \quad f \in H(\mathbb{D}), \quad \beta > -1.$$

Lemma 3.3. *If* ω *is a continuous weight,*

$$|f(a)|^2 \leq \frac{4}{(1-|a|)^2 \inf_{z \in D\left(0, \frac{1+|a|}{2}\right)} \omega(z)} \int_{D\left(a, \frac{1-|a|}{2}\right)} |f(z)|^2 \omega(z)\, dA(z) \leq \frac{4||f||^2_{A^2_\omega}}{(1-|a|)^2 \inf_{z \in D\left(0, \frac{1+|a|}{2}\right)} \omega(z)}$$

For any $z \in \mathbb{D}$, let us consider the point evaluation $L_z(f) = f(z)$.

Corollary 3.4. *If* ω *is a continuous weight, point evaluations* L_z *are bounded functionals on* A^2_ω.

Corollary 3.5. *If* ω *is a continuous weight and* $\lim_{n\to\infty} ||f_n - f||_{A^2_\omega} = 0$, *then* $\{f_n\}$ *converges uniformly to* f *on compact subsets of* $\mathbb{D}$.

Corollary 3.6.
(1) *Point evaluations* L_z *are bounded functionals on* H^2 *and* $\mathcal{D}^2_\alpha$, $0 \leq \alpha < 1$.
(2) *If* $\lim_{n\to\infty} ||f_n - f||_{H^2} = 0$ *or* $\lim_{n\to\infty} ||f_n - f||_{\mathcal{D}^2_\alpha} = 0$ *then* $\{f_n\}$ *converges uniformly to* f *on compact subsets of* $\mathbb{D}$.

Corollary 3.7. *If* ω *is a continuous weight, the following* $H^2, \mathcal{D}^2_\alpha, A^2_\omega$ *are Hilbert spaces.*

A proof follows from the previous results, Fatou's lemma and standard arguments.

3.6. **Kernel functions and reproducing formulas.** Let $X = H^2, \mathcal{D}_\alpha, A^2_\omega$, where ω is a continuous weight, and let $\langle \cdot, \cdot \rangle_X$ the associated inner product. Since the point evaluations are bounded functionals on X, by the Riesz representation theorem, for each $z \in \mathbb{D}$ there exists a unique function $K^X_z \in X$ such that

$$f(z) = \langle f, K^X_z \rangle_X. \tag{3.8}$$

Theorem 3.8. *Suppose that* $\{e_n(z)\}_{n=0}^\infty$ *is an orthonormal basis of* X. *Then,*

$$K^X_z(\zeta) = \sum_{n=0}^\infty e_n(\zeta)\overline{e_n(z)}$$

and the series converges uniformly on compact subsets of $\mathbb{D} \times \mathbb{D}$. *In particular,* $K^X_z(\zeta)$ *is independent of the choice of the orthonormal basis* $\{e_n(z)\}_{n=0}^\infty$.

Proof. Bearing in mind Lemmas 3.2 and 3.3, for any compact $S \subset \mathbb{D}$, we have

$$\begin{aligned}
&\sup\left\{ \left(\sum_{n=0}^\infty |e_n(z)|^2 \right)^{\frac{1}{2}} : \quad z \in S \right\} \\
&= \sup\left\{ \left| \sum_{n=0}^\infty a_n e_n(z) \right| : \quad z \in S, \quad \sum_{n=0}^\infty |a_n|^2 = 1 \right\} \\
&= \sup\left\{ |f(z)| : \quad z \in S, \quad ||f||_X = 1 \right\} \leq C_S.
\end{aligned}$$

So, it follows from the Cauchy-Schwarz inequality that the series

$$\sum_{n=0}^{\infty} e_n(\zeta)\overline{e_n(z)}$$

converges uniformly whenever ζ and z stay in compact subsets of $\mathbb{D}$.
Now, for any $f \in X$,

$$f(z) = \sum_{n=0}^{\infty} \langle f, e_n \rangle_X \, e_n(z)$$

This series converges in X and by Corollaries 3.5 and 3.6, it also converges uniformly on compact subsets of $\mathbb{D}$. Therefore, for any $z \in \mathbb{D}$, we have

$$f(z) = \sum_{n=0}^{\infty} \langle f, e_n \rangle_X \, e_n(z) = \langle f(\cdot), \sum_{n=0}^{\infty} \overline{e_n(z)} e_n(\cdot) \rangle_X .$$

Since the series, $\sum_{n=0}^{\infty} \overline{e_n(z)} e_n(\zeta)$ converges to a function $h_z(\zeta) \in X$, the uniqueness of Riesz representation theorem shows that

$$K_z^X(\zeta) = \sum_{n=0}^{\infty} e_n(\zeta)\overline{e_n(z)}$$

. □

In particular, by (3.8)

$$||K_z^X||_X^2 = \langle K_z^X, K_z^X \rangle_{H^2} = K_z^X(z) = \sum_{n=0}^{\infty} |e_n(z)|^2, \tag{3.9}$$

for any orthonormal basis.

- **Hardy spaces.** It is clear that $\{e_n\}_{n=0}^{\infty}$, where

  $$e_n(z) = z^n, \quad n \in \mathbb{N} \cup \{0\}$$

  is a basis of H^2. So

  $$K_z^{H^2}(\zeta) = \sum_{n=0}^{\infty} (\zeta \overline{z})^n = \frac{1}{1 - \zeta \overline{z}}, \quad z, \zeta \in \mathbb{D}. \tag{3.10}$$

  Moreover

  $$f(z) = \langle f, K_z^{H^2} \rangle_{H^2} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{K_z^{H^2}(e^{it})} \, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it} z} \, dt, \quad f \in H^2, \tag{3.11}$$

  and

  $$||K_z^{H^2}||_{H^2}^2 = \frac{1}{1 - |z|^2}.$$

  The reproducing formula (3.11) remains true for any $f \in H^1$, [17, Theorem 3.6].
- **Weighted Bergman spaces.** It follows from (3.5) that $\{e_n\}_{n=0}^{\infty}$, where

  $$e_n(z) = \frac{z^n}{\sqrt{2 \int_0^1 r^{2n+1} \omega(r) \, dr}}, \quad n \in \mathbb{N} \cup \{0\}$$

  is a basis of A^2_ω whenever ω is a radial weight.
  By Theorem 3.8 and (3.8)

  $$f(z) = \langle f, K_z^{A^2_\omega} \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta) \overline{K_z^{A^2_\omega}(\zeta)} \omega(\zeta) \, dA(\zeta), \quad f \in A^2_\omega,$$

where

$$K_z^{A^2_\omega}(\zeta)=\sum_{n=0}^\infty \frac{(\zeta\overline{z})^n}{2\int_0^1 r^{2n+1}\omega(r)\,dr}. \tag{3.12}$$

It seems difficult to obtain a more concrete expression of $K_z^{A^2_\omega}$ for any radial weight, however it can be done for standard radial weight $\omega(r)=(1-r^2)^\alpha,\ \alpha>-1$

- **Classical weighted Bergman spaces.**

  First, we recall some know facts on special functions.

Lemma 3.9. (i) $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt,\ x>0.$
(ii) $\Gamma(x+1)=x\Gamma(x),\ x>0.$
(iii) $\Gamma(n+1)=n!, n\in\mathbb{N}\cup\{0\}.$
(iv) $\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ x>0, y>0.$
(v) $\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$
(vi) *If λ is neither zero nor a negative integer, then*

$$\frac{1}{(1-z)^\lambda}=\sum_{n=0}^\infty \frac{\Gamma(n+\lambda)}{\Gamma(n+1)\Gamma(\lambda)}z^n, \quad z\in\mathbb{D}.$$

If $\omega(r)=(1-r^2)^\alpha$, then

$$\begin{aligned}2\int_0^1 r^{2n+1}\omega(r)\,dr &= 2\int_0^1 r^{2n+1}(1-r^2)^\alpha\,dr=\int_0^1 t^n(1-t)^\alpha\,dt\\ &=\beta(n+1,\alpha+1)=\frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+2)}=\frac{\Gamma(\alpha+2)\Gamma(n+1)}{(\alpha+1)\Gamma(n+\alpha+2)},\end{aligned}$$

so by (3.12)

$$K_z^{A^2_\omega}(\zeta)=(\alpha+1)\sum_{n=0}^\infty \frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}(\zeta\overline{z})^n=\frac{(\alpha+1)}{(1-\zeta\overline{z})^{\alpha+2}}.$$

We also deduce

$$f(z)=(\alpha+1)\int_{\mathbb{D}}\frac{f(\zeta)}{(1-z\overline{\zeta})^{\alpha+2}}\,(1-|\zeta|^2)^\alpha\,dA(\zeta), \quad f\in A^2_\alpha, \tag{3.13}$$

and

$$||K_z^{A^2_\alpha}||^2_{A^2_\alpha}=\frac{(\alpha+1)}{(1-|z|^2)^{\alpha+2}}.$$

The reproducing formula (3.13) is valid for any $f\in A^1_\alpha$, [18, 44].

- **Classical weighted Dirichlet spaces.** Bearing in mind the reproducing formula

$$\begin{aligned}f(z)&=\langle f,K_z^{\mathcal{D}_\alpha}\rangle_{\mathcal{D}_\alpha}\\ &=f(0)\overline{K_z^{\mathcal{D}_\alpha}(0)}+\int_{\mathbb{D}}f'(\zeta)\,\overline{\frac{\partial}{\partial\zeta}K_z^{\mathcal{D}_\alpha}(\zeta)}(1-|\zeta|^2)^\alpha\,dA(\zeta), \quad f\in\mathcal{D}_\alpha,\end{aligned} \tag{3.14}$$

the fact that $f\in\mathcal{D}_\alpha\Leftrightarrow f'\in A^2_\alpha$ and (3.13), we deduce that

$$K_z^{\mathcal{D}_\alpha}(w)=1+\int_0^w\int_0^{\bar z}\frac{d\zeta}{(1-\eta\zeta)^{2+\alpha}}\,d\eta. \tag{3.15}$$

In particular, for $\alpha=0$,

$$K_z^{\mathcal{D}}(w)=1+\log\frac{1}{1-\bar z w}.$$

Also, it is easy to see that

$$\|K_z^{\mathcal{D}_\alpha}\|_{\mathcal{D}_\alpha}^2 = K_z^{\mathcal{D}_\alpha}(z) \asymp \begin{cases} \log \frac{e}{1-|z|^2} & \text{if } \alpha = 0 \\ (1-|z|^2)^{-\alpha} & \text{if } \alpha > 0. \end{cases} \tag{3.16}$$

4. Operators on Hilbert spaces of analytic functions

The main aim of this section consists of introducing classical operators on Hilbert spaces of analytic functions on $\mathbb{D}$, providing descriptions on the boundedness and compactness of these operators.

4.1. **Preliminaries.** We shall write

$$k_z^X(\zeta) = \frac{K_z^X(\zeta)}{||K_z^X||_X}.$$

The *Carleson square* associated with an interval $I \subset \mathbb{T}$ is the set $S(I) = \{re^{it} : e^{it} \in I,\, 1-|I| \le r < 1\}$, where $|E|$ denotes the normalized Lebesgue measure of the set $E \subset \mathbb{T}$. For our purposes it is also convenient to define for each $a \in \mathbb{D}\setminus\{0\}$ the interval $I_a = \{e^{i\theta} : |\arg(ae^{-i\theta})| \le \pi(1-|a|)\}$, and denote $S(a) = S(I_a)$.
For $a \in \mathbb{D}$, define $\varphi_a(z) = (a-z)/(1-\overline{a}z)$. The automorphism φ_a of $\mathbb{D}$ is its own inverse and interchanges the origin and the point $a \in \mathbb{D}$. The *pseudohyperbolic* and *hyperbolic distances* from z to w are defined as $\varrho(z,w) = |\varphi_z(w)|$ and

$$\varrho_h(z,w) = \frac{1}{2}\log\frac{1+\varrho(z,w)}{1-\varrho(z,w)}, \quad z,w \in \mathbb{D},$$

respectively. The *pseudohyperbolic disc* of center $a \in \mathbb{D}$ and radius $r \in (0,1)$ is denoted by $\Delta(a,r) = \{z : \varrho(a,z) < r\}$. It is clear that $\Delta(a,r)$ coincides with the *hyperbolic disc* $\Delta_h(a,R) = \{z : \varrho_h(a,z) < R\}$, where $R = \frac{1}{2}\log\frac{1+r}{1-r} \in (0,\infty)$.
The following results will be used. Proofs can be found in [18, 44].

Lemma 4.1. *Let $0 < r < 1$. Then there exist positive constants C_1, C_2 C_3 and C_4 which depend only on r such that for every $z \in \Delta(a,r)$,*

$$1-|a|^2 \le C_1(1-|z|^2) \le C_2|1-\overline{a}z| \le C_3\left(A(\Delta(a,r))\right)^{1/2} \le C_4(1-|a|^2), \tag{4.1}$$

here $A(\Delta(a,r))$ denotes the Lebesgue area measure of $\Delta(a,r)$.

Lemma 4.2. *For each pseudohyperbolic radius r ($0 < r < 1$), there exist a sequence $\{a_k\}_{k=0}^\infty$ of points of $\mathbb{D}$ and an integer $N = N(r)$ such that*

$$\mathbb{D} = \bigcup_{k=1}^\infty \Delta(a_k, r)$$

and no point $z \in \mathbb{D}$ belongs to more than N of the dilated discs $\Delta(a_k,R)$, $R = \frac{1+r}{2}$.

Lemma 4.3. *Suppose that $0 < p < \infty$, $\gamma \in \mathbb{R}$ and $0 < r < 1$. Then, there is a positive constant $C = C(\gamma, r)$ such that*

$$|f(a)|^p \le \frac{C}{(1-|a|^2)^{2+\gamma}} \int_{\Delta(a,r)} |f(z)|^p(1-|z|^2)^\gamma\, dA(z).$$

4.2. **The identity operator.** For a given Banach space (or a complete metric space) X of analytic functions on $\mathbb{D}$, a positive Borel measure μ on $\mathbb{D}$ is called a *q-Carleson measure for X* if the identity operator $I_d : X \to L^q(\mu)$ is bounded. It is known that a characterization of q-Carleson measures for a space $X \subset \mathcal{H}(\mathbb{D})$ can be an effective tool, for example, in the study of different questions related to operators acting on X.
We shall focus our attention on the case $I_d : X \to L^2(\mu)$, where X is a Hilbert space of analytic functions on $\mathbb{D}$.

Theorem 4.4. *Let μ be a positive Borel measure on $\mathbb{D}$. Then the following assertions hold:*

(i) $I_d : H^2 \to L^2(\mu)$ *if and only if*

$$K_\mu = \sup_{I \subset \mathbb{T}} \frac{\mu\left(S(I)\right)}{|I|} < \infty. \tag{4.2}$$

Moreover, if the identity operator $I_d : H^2 \to L^2(\mu)$ is bounded then

$$\|I_d\|^q_{(H^2, L^2(\mu))} \asymp \sup_{I \subset \mathbb{T}} \frac{\mu\left(S(I)\right)}{|I|}.$$

(ii) *The identity operator $I_d : H^2 \to L^2(\mu)$ is compact if and only if*

$$\lim_{|I| \to 0} \frac{\mu\left(S(I)\right)}{|I|} = 0. \tag{4.3}$$

A proof of part (i) of this result hinges on the fact that $(1-|a|)^{1/2} k_a^{H^2}$ is bounded above and below on $S(a)$, on the properties of a Hörmander-type maximal function and a $(1,1)$-weak type inequality for this function. Part (ii) can be proved following these ideas, part (i) and standard techniques (some of them will appear below). In particular, the next result.

Lemma 4.5. *Let μ be a positive Borel measure on $\mathbb{D}$ satisfying* (4.3). *If*

$$d\mu_r(z) = \chi_{\{r \le |z| < 1\}} d\mu,$$

then $\lim_{r \to 1^-} K_{\mu_r} = 0$.

See [17, 12, 13, 11, 35, 29] for further details on the proofs and descriptions on q-Carleson measures for H^p, $0 < q, p < \infty$.
Now, we turn our attention to classical weighted Bergman spaces.

Theorem 4.6. *Let μ be a positive Borel measure on $\mathbb{D}$ and $\alpha > -1$. Then the following assertions are equivalent;*

(i) $I_d : A^2_\alpha \to L^2(\mu)$ *is bounded*
(ii) *the measure μ satisfies*

$$\sup_{a \in \mathbb{D}} \frac{\mu\left(\Delta(a,r)\right)}{(1-|a|)^{2+\alpha}} < \infty, \tag{4.4}$$

for some (any) $0 < r < 1$.

Moreover, if the identity operator $I_d : A^2_\alpha \to L^2(\mu)$ is bounded then

$$\|I_d\|^2_{(A^2_\alpha, L^2(\mu))} \asymp \sup_{a \in \mathbb{D}} \frac{\mu\left(\Delta(a,r)\right)}{(1-|a|)^{2+\alpha}}.$$

Proof. (i)⇒(ii). Consider the test functions $k_a^{A^2_\alpha}(z) = (\alpha+1)^{1/2}\left(\frac{(1-|a|^2)^{1/2}}{1-\overline{a}z}\right)^{2+\alpha}$. Bearing in mind Lemma 4.1,

$$\frac{\mu(\Delta(a,r))}{(1-|a|^2)^{2+\alpha}} \asymp \int_{\Delta(a,r)} |k_a^{A^2_\alpha}(z)|^2\,d\mu(z)$$
$$\leq \int_{\mathbb{D}} |k_a^{A^2_\alpha}(z)|^2\,d\mu(z) \lesssim \|I_d\|^2_{(A^2_\alpha, L^2(\mu))}\|k_a^{A^2_\alpha}(z)\|^2_{A^2_\alpha} \lesssim 1$$

for all $a \in \mathbb{D}$ and $0 < r < 1$. Thus μ satisfies (4.4) and

$$\sup_{a\in\mathbb{D}} \frac{\mu\left(\Delta(a,r)\right)}{(1-|a|)^{2+\alpha}} \lesssim \|I_d\|^2_{(A^2_\alpha, L^2(\mu))}$$

(ii)⇒(i). Fixed $0 < r < 1$, let $\{a_k\}$ the sequence given by Lemma 4.2 and $M = \sup_{a\in\mathbb{D}} \frac{\mu(\Delta(a,r))}{(1-|a|)^{2+\alpha}}$. Then, by Lemmas 4.2, 4.3 and 4.1

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^2\,d\mu(z) &\asymp \sum_k \int_{\Delta(a_k,r)} |f(z)|^2\,d\mu(z) \\
&\lesssim \sum_k \sup\left\{|f(z)|^2 : z \in \Delta(a_k,r)\right\}\mu\left(\Delta(a_k,r)\right) \\
&\leq M \sum_k \max\left\{|f(z)|^2 : z \in \overline{\Delta(a_k,r)}\right\}(1-|a_k|^2)^{2+\alpha} \\
&\lesssim M \sum_k \frac{\int_{\Delta(a_k,R)} |f(\xi)|^2\,dA(\xi)}{(1-|a_k|)^2}(1-|a_k|^2)^{2+\alpha} \\
&\asymp M(1-|a_k|^2)^{\alpha} \sum_k \int_{\Delta(a_k,R)} |f(\xi)|^2\,dA(\xi) \\
&\lesssim M \sum_k \int_{\Delta(a_k,R)} |f(\xi)|^2\,(1-|\xi|^2)^{\alpha}\,dA(\xi) \\
&\lesssim M \int_{\mathbb{D}} |f(\xi)|^2\,(1-|\xi|^2)^{\alpha}\,dA(\xi),
\end{aligned}$$

which proves (i) and implies that $\|I_d\|^2_{(A^2_\alpha, L^2(\mu))} \lesssim M = \sup_{a\in\mathbb{D}} \frac{\mu(\Delta(a,r))}{(1-|a|)^{2+\alpha}}$. □

It is not difficult to see [17, 23, 35]

$$\sup_{I\subset\mathbb{T}} \frac{\mu\left(S(I)\right)}{|I|^{2+\alpha}} \asymp \sup_{a\in\mathbb{D}} \frac{\mu\left(\Delta(a,r)\right)}{(1-|a|)^{2+\alpha}}, \quad \alpha > -1.$$

That is 2-Carleson measures for A^2_α, $\alpha > -1$, as it happens for 2-Carleson measures for H^2, can also be neatly characterized by a L^∞-type condition involving Carleson squares.

Theorem 4.7. *Let μ be a positive Borel measure on $\mathbb{D}$ and $\alpha > -1$. Then the following assertions are equivalent,*

(i) $I_d : A^2_\alpha \to L^2(\mu)$ *is compact*

(ii) *the measure μ satisfies*

$$\lim_{|a|\to 1^-} \frac{\mu\left(\Delta(a,r)\right)}{(1-|a|)^{2+\alpha}} = 0, \tag{4.5}$$

for some (any) $0 < r < 1$.

Proof. (i)⇒(ii). Since $\sup_{a\in\mathbb{D}} ||k_a^{A_a^2}||_{A_a^2} = 1 < \infty$ and

$$\lim_{|a|\to 1^-} k_a^{A_a^2}(z) = 0$$

uniformly on compact subsets of $\mathbb{D}$, then an standard argument gives that

$$\lim_{|a|\to 1^-} ||k_a^{A_a^2}||_{L^2(\mu)} = 0$$

so reasoning as in the proof of Theorem 4.6 we deduce (4.5).
(ii)⇒(i) follows bearing in mind Corollary 3.5 and the fact that the sequence $\{a_k\}_{k=0}^\infty$ constructed in Lemma 4.2 has the property $\lim_{k\to\infty} |a_k| = 0$. □

We also have that that for any $\alpha > -1$, (4.5) is equivalent to

$$\lim_{|I|\to 0} \frac{\mu\left(S(I)\right)}{|I|^{2+\alpha}} = 0. \tag{4.6}$$

Going further, since $K_\alpha^{A_a^2}$ are analytic zero free functions on $\mathbb{D}$, taking as test functions $\left(K_\alpha^{A_a^2}\right)^{\frac{2}{p}}$ and using the same techniques, it can be proved that (4.4) and (4.5) describe the boundedness and respectively the compactness of the operator $I_d : A_\alpha^p \to L^p(\mu)$, $0 < p < \infty$, $\alpha > -1$, which in particular says that p-Carleson measures for A_α^p do not depend on p. This is also true for p-Carleson measures for H^p, a result which was proved by L. Carleson [12, 13]. Hastings, Luecking, Oleinik and Pavlov [23, 27, 30, 32], among others, have characterized q-Carleson measures for A_α^p. Constantin [14] have given an extension of these classical results to the case when $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to the class $B_{p_0}(\eta)$ of Bekollé-Bonami weights. We recall that for $1 < p_0, p_0' < \infty$ such that $\frac{1}{p_0} + \frac{1}{p_0'} = 1$ and $\eta > -1$, a weight $\omega : \mathbb{D} \to (0,\infty)$ satisfies the *Bekollé-Bonami $B_{p_0}(\eta)$-condition* denoted by $\omega \in B_{p_0}(\eta)$, if there exists a constant $C = C(p_0, \eta, \omega) > 0$ such that

$$\begin{aligned} &\left(\int_{S(I)} \omega(z)(1-|z|)^\eta\, dA(z)\right)\left(\int_{S(I)} \omega(z)^{\frac{-p_0'}{p_0}}(1-|z|)^\eta\, dA(z)\right)^{\frac{p_0}{p_0'}} \\ &\le C|I|^{(2+\eta)p_0} \end{aligned} \tag{4.7}$$

for every interval $I \subset \mathbb{T}$.
A description for q-Carleson measures for A_ω^p, where $0 < q, p < \infty$ and ω is a rapidly decreasing weight can be found in [33]. The case $0 < p \le q$ for rapidly increasing weights has been recently solved in [35].

Finally, we introduce the modified Carleson box

$$\tilde{S}(a) = \left\{z \in \mathbb{D} : 1 - |z| \le 2(1-|a|),\ \left|\frac{\arg(a\bar{z})}{2\pi}\right| \le \frac{1-|a|}{2}\right\}.$$

Theorem 4.8. *Let $-1 < \alpha < 1$. Then μ is a Carleson measure for $\mathcal{D}_\alpha^2$ if and only if there is a positive constant C such that for all $a \in \mathbb{D}$*

$$\int_{\tilde{S}(a)} \left(\mu(S(z) \cap S(a))\right)^2 \frac{dA(z)}{(1-|z|^2)^{2+\alpha}} \le C\,\mu(S(a)).$$

Theorem 4.8 is a particular case a more general result proved in [6] which is proved using "measures on trees".

4.3. **Integral operators.** Let us consider the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta, \quad g \in \mathcal{H}(\mathbb{D}).$$

Pommerenke was probably one of the first authors to consider the operator T_g. He used it in [36] to study the space BMOA, which consists of those functions in the Hardy space H^1 that have *bounded mean oscillation* on the boundary $\mathbb{T}$ [8, 21], that is if $m_I(f) = \frac{1}{|I|}\int_I f(e^{i\theta})\, d\theta$

$$\sup_{I\subset\mathbb{T}} \frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\, d\theta < \infty.$$

The space BMOA can be equipped with several different equivalent norms [21]. We will use the one given by

$$\|g\|^2_{\mathrm{BMOA}} = \sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |g'(z)|^2(1-|z|^2)\, dA(z)}{1-|a|} + |g(0)|^2.$$

The space VMOA consists of those functions in the Hardy space H^1 that have *vanishing mean oscillation* on the boundary $\mathbb{T}$, that is,

$$\lim_{|I|\to 0^+} \frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\, d\theta = 0.$$

It is known that this space is the closure of polynomials in BMOA and is characterized by the condition

$$\lim_{|a|\to 1^-} \frac{\int_{S(a)} |g'(z)|^2(1-|z|^2)\, dA(z)}{1-|a|} = 0.$$

By using the Litlewood-Paley formula (3.3),

$$||T_g(f)||^2_{H^2} \asymp \int_{\mathbb{D}} |f(z)g'(z)|^2\,(1-|z|^2)\, dA(z),$$

so $T_g : H^2 \to H^2$ is bounded if and only if

$$\int_{\mathbb{D}} |f(z)g'(z)|^2\,(1-|z|^2)\, dA(z) \le C||f||^2_{H^2},$$

that is, $\mu_g = |g'(z)|^2\,(1-|z|^2)\, dA(z)$ is a 2-Carleson measure for H^2, which together Theorem 4.4 gives the first part of the next result.

Theorem 4.9. *Let $g \in \mathcal{H}(\mathbb{D})$. Then,*
(i) $T_g : H^2 \to H^2$ is bounded if and only if $g \in$ BMOA.
(ii) $T_g : H^2 \to H^2$ is compact if and only if $g \in$ VMOA.

In order to present a detailed proof of Theorem 4.9 (ii), we shall use the following well-known result [42, 22].

Lemma A. *Let X and Y be two Banach spaces (or complete metric spaces) of analytic functions on $\mathbb{D}$, and let $T : X \to Y$ be a linear operator. Suppose that the following conditions are satisfied:*

(a) *The point evaluation functionals on Y are bounded.*
(b) *For every bounded sequence in X, there is a subsequence which converges uniformly to an element of X on compact subsets of $\mathbb{D}$.*
(c) *If $\{f_n\} \subset X$ converges uniformly to zero on compact subsets of $\mathbb{D}$, then $\{T(f_n)\}$ converges uniformly to zero on compact subsets of $\mathbb{D}$.*

Then T is a compact operator from X to Y if and only if for any bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, the sequence $\{T(f_n)\}$ converges to zero in the norm (or in the metric) of Y.

We note that $\Rightarrow$ only uses (a) and (c). $\Leftarrow$ only uses (b).

Lemma 4.10. *Let $X = H^2, A^2_\beta$ or $\mathcal{D}_\alpha$, $\beta, \alpha > -1$. Assume that $g \in H(\mathbb{D})$ and $T_g : X \to X$ is bounded. Then, (a), (b) and (c) of Lemma A hold.*

Proof. (a) and (b) follow from Corollaries 3.5, 3.6 and 3.7. Now, take $\{f_n\} \subset X$ a sequence converging uniformly to zero on compact subsets of $\mathbb{D}$, and fix K a compact subset of $\mathbb{D}$. Since there exists $r_0 \in (0,1)$ such that $K \subset \overline{D(0,r_0)}$. For any $\varepsilon > 0$ there is and $n_0 = n_0(r_0, \varepsilon)$ such that

$$|f_n(z)| < \varepsilon, \quad \text{for any } z \in \overline{D(0,r_0)} \text{ and } n \geq n_0.$$

So, if $n \geq n_0$ and $z \in K$,

$$|T_g f_n(z)| \leq \int_0^{|z|} |f_n(\zeta)||g'(\zeta)||d\zeta| \leq M_\infty(r_0, g') \int_0^{|z|} |f_n(\zeta)||d\zeta| \leq M_\infty(r_0, g')\varepsilon,$$

which implies (c). This finishes the proof. □

Proof of Theorem 4.4. Part (ii)
Observe that Lemmas A and 4.10 imply that, $T_g : H^2 \to H^2$ is compact if and only if for any bounded sequence $\{f_n\}$ in H^2 such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then

$$\lim_{n\to\infty} ||T_g(f_n)||^2_{H^2} \asymp \lim_{n\to\infty} \int_{\mathbb{D}} |f_n(z)g'(z)|^2 \, (1-|z|^2) \, dA(z) = 0, \tag{4.8}$$

where in the last equivalence we have used (3.3).
Assume that $T_g : H^2 \to H^2$ is compact. Since $\sup_{a\in\mathbb{D}} ||k_a^{H^2}||_{A^2_a} = 1 < \infty$, $\lim_{|a|\to 1^-} k_a^{H^2}(z) = 0$ uniformly on compact subsets of $\mathbb{D}$ and

$$|k_a^{H^2}(z)| \asymp \frac{(1-|a|^2)^{1/2}}{|1-\overline{a}z|} \asymp \frac{1}{(1-|a|^2)^{1/2}}, \quad z \in S(a) \tag{4.9}$$

it follows from (4.8)

$$\begin{aligned} \lim_{|a|\to 1^-} \frac{\int_{S(a)} |g'(z)|^2(1-|z|^2)\, dA(z)}{1-|a|} &\asymp \lim_{|a|\to 1^-} \int_{S(a)} |g'(z)k_a^{H^2}(z)|^2(1-|z|^2)\, dA(z) \\ &\leq \lim_{|a|\to 1^-} \int_{\mathbb{D}} |g'(z)k_a^{H^2}(z)|^2(1-|z|^2)\, dA(z) = 0, \end{aligned}$$

so $g \in$ VMOA.
On the other hand, assume that $g \in VMOA$ and $\{f_n\}$ is a bounded sequence in H^2 such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$.
Fix $\varepsilon > 0$, and let us write $\mu_g = |g'(z)|^2 \, (1-|z|^2) \, dA(z)$. By Lemma 4.5, there is r_0 such that $K_{(\mu_g)_r} < \varepsilon^2$ for any $r \in [r_0, 1)$. Moreover, there is $n_0 = n_0(r_0, \varepsilon)$ such that

$$|f_n(z)| < \varepsilon, \quad \text{for any } z \in \overline{D(0,r_0)} \text{ and } n \geq n_0.$$

So, bearing in mind Theorem 4.4, for $n \geq n_0$

$$\begin{aligned}&\int_{\mathbb{D}} |f_n(z)g'(z)|^2 \, (1-|z|^2) \, dA(z)\\ &\leq \varepsilon^2 \int_{|z|\leq r_0} |g'(z)|^2 \, (1-|z|^2) \, dA(z) + ||f_n||^2_{L^2((\mu_g)_{r_0})}\\ &\leq \varepsilon^2 ||g||^2_{H^2} + CK_{\mu_{g_{r_0}}} ||f_n||^2_{H^2}\\ &\leq C\varepsilon^2 (||g||^2_{BMOA} + \sup_n ||f_n||^2_{H^2}) \lesssim \varepsilon^2,\end{aligned}$$

that is $\lim_{n\to\infty} ||T_g(f_n)||^2_{H^2} = 0$. This finishes the proof. □
We recall that the Bloch space $\mathcal{B}$ [5] consists of those $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2) \, |f'(z)| < \infty.$$

and $f \in \mathcal{B}_0$ (the little Bloch space) if

$$\lim_{|z|\to 1^-} (1-|z|^2) \, |f'(z)| = 0.$$

See [5, 44] for theory of these spaces.

Lemma 4.11. *Assume that $g \in H(\mathbb{D})$, $\gamma \in \mathbb{R}$ and $0 < r < 1$. Then*

(i) *$g \in \mathcal{B}$ if and only*

$$\sup_{a\in\mathbb{D}} \frac{1}{(1-|a|^2)^\gamma} \int_{\Delta(a,r)} |g'(z)|^2 (1-|z|^2)^\gamma \, dA(z) < infty.$$

(ii) *$g \in \mathcal{B}_0$ if and only*

$$\lim_{|a|\to 1^-} \frac{1}{(1-|a|^2)^\gamma} \int_{\Delta(a,r)} |g'(z)|^2 (1-|z|^2)^\gamma \, dA(z) = 0$$

With these tools in our hands, and reasoning as in Theorem 4.9 we can prove the following.

Theorem 4.12. *Let $g \in \mathcal{H}(\mathbb{D})$ and $\alpha > -1$. Then,*
(i) $T_g : A^2_\alpha \to A^2_\alpha$ is bounded if and only if $g \in \mathcal{B}$.
(ii) $T_g : A^2_\alpha \to A^2_\alpha$ is compact if and only if $g \in \mathcal{B}_0$.

Finally, as a direct byproduct of Theorem 4.8 we obtain a description of those symbols $g \in \mathcal{H}(\mathbb{D})$ such that the integral operator is bounded on $\mathcal{D}_\alpha$.

Theorem 4.13. *Let $g \in \mathcal{H}(\mathbb{D})$ and $-1 < \alpha < 1$. Then, $T_g : \mathcal{D}^2_\alpha \to \mathcal{D}^2_\alpha$ is bounded if and only if*

$$\begin{aligned}&\int_{\tilde{S}(a)} \left(\int_{S(z)\cap S(a)} |g'(\zeta)|^2 (1-|\zeta|^2)^\alpha dA(\zeta) \right)^2 \frac{dA(z)}{(1-|z|^2)^{2+\alpha}}\\ &\leq C \int_{S(a)} |g'(\zeta)|^2 (1-|\zeta|^2)^\alpha dA(\zeta).\end{aligned} \tag{4.10}$$

We refer to [1, 3, 41, 35] and the references therein for the theory of these operators.

4.4. **Multiplication operators.** For $g \in \mathcal{H}(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f)(z) = g(z)f(z), \quad f \in \mathcal{H}(\mathbb{D}),\ z \in \mathbb{D}.$$

If X and Y are two spaces of analytic function in $\mathbb{D}$ (which will always be assumed to be Banach or F-spaces continuously embedded in $\mathcal{H}(\mathbb{D})$) and $g \in \mathcal{H}(\mathbb{D})$, then g is said to be a multiplier from X to Y if $M_g : X \to Y$ is bounded. The space of all multipliers from X to Y will be denoted by $M(X,Y)$ and $M(X)$ will stand for $M(X,X)$.
Firstly, we shall prove an easy but useful lemma [43, Lemma 1.10].

Lemma 4.14. *Assume that X is an space of analytic functions on $\mathbb{D}$ such that the point evaluations are bounded on X. Then $M(X) \subset H^\infty$ and*

$$||g||_{H^\infty} \le ||M_g||_{(X,X)}.$$

Proof. Take $f \in X$, $f \neq 0$ and $a \notin Z(f) = \{z \in \mathbb{D} : f(z) = 0\}$. Then, there is $C_a > 0$ such that

$$|f(a)| \le C_a ||f||_X.$$

So, for each $n \in \mathbb{N}$ and $g \in M(X)$,

$$|g^n(a)f(a)| \le C_a||g^n f||_X \le C_a||M_g||_{(X,X)} \cdot ||g^{n-1}f||_X \le C_a||M_g||^n_{(X,X)} \cdot ||f||_X,$$

that is

$$|g(a)|f(a)|^{1/n} \le ||M_g||_{(X,X)}(C_a||f||_X)^{1/n},$$

which implies

$$|g(a)| \le ||M_g||_{(X,X)}.$$

Since g is continuous on $\mathbb{D}$ and $Z(f)$ is a discrete set, the proof follows. □

The previous result and trivial calculations give that

$$M(H^p) = H^\infty, \quad 0 < p \le \infty$$

and $M(A^p_\omega) = H^\infty$ if ω is a continuous weight and $0 < p < \infty$.
A description of those symbols $g \in \mathcal{H}(\mathbb{D})$ of the operator $M_g : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$, $0 < \alpha < 1$, is a bit more complicated. With this aim, we introduce the operator I_g, defined as follows:

$$I_g(f)(z) = \int_0^z g(\xi)f'(\xi)d\xi,\ \ f \in \mathcal{H}(\mathbb{D}),\ \ z \in \mathbb{D}.$$

We remark also that

$$I_g(f) + J_g(f) = M_g(f) - f(0)g(0). \tag{4.11}$$

Thus if two of the operators I_g, J_g, M_g are bounded from X to Y, so is the third one.

Lemma 4.15. *Assume that $g \in H(\mathbb{D})$ and $0 < \alpha < \infty$. Then $I_g : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is bounded if and only if $g \in H^\infty$.*

Proof. If $g \in H^\infty$, then it is clear that $I_g : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is bounded. On the other hand, assume that $I_g : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is bounded. If $\alpha > 0$, let us consider the family of test functions

$$f_a(z) = \frac{(1-|a|^2)^{\frac{\alpha}{2}}}{(1-\overline{a}z)^\alpha}, \quad z \in \mathbb{D}.$$

Bearing in mind our previous results on reproducing kernels, direct calculations or [44, Lemma 3.10], we have that

$$\sup_{a\in\mathbb{D}} ||f_a||_{\mathcal{D}_\alpha} < \infty \quad \text{and } |f_a'(a)| \asymp \frac{1}{(1-|a|^2)^{1+\frac{\alpha}{2}}},\ \text{if } |a| \ge \frac{1}{2}.$$

Consequently, by Lemmas 4.3 and 4.1, we deduce

$$\begin{aligned}\frac{|g(a)|^2}{(1-|a|)^{2+\alpha}} &\asymp |g(a)f_a'(a)|^2 \lesssim \frac{1}{(1-|a|^2)^2}\int_{\Delta(a,r)} |g(z)f_a'(z)|^2\,dA(z)\\ &\asymp \frac{1}{(1-|a|^2)^{2+\alpha}}\int_{\Delta(a,r)} |g(z)f_a'(z)|^2(1-|z|^2)^\alpha\,dA(z)\\ &\lesssim \frac{1}{(1-|a|^2)^{2+\alpha}}\int_{\mathbb{D}} |g(z)f_a'(z)|^2(1-|z|^2)^\alpha\,dA(z)\\ &\asymp \frac{1}{(1-|a|^2)^{2+\alpha}}||I_g(f_a)||^2_{\mathcal{D}_\alpha}\\ &\asymp \frac{1}{(1-|a|^2)^{2+\alpha}}||I_g||^2_{(\mathcal{D}_\alpha,\mathcal{D}_\alpha)}||(f_a)||^2_{\mathcal{D}_\alpha}\\ &\lesssim \frac{1}{(1-|a|^2)^{2+\alpha}}||I_g||^2_{(\mathcal{D}_\alpha,\mathcal{D}_\alpha)},\end{aligned}$$

which implies that $g \in H^\infty$ and $||g||_{H^\infty} \lesssim ||I_g||_{(\mathcal{D}_\alpha,\mathcal{D}_\alpha)}$. This finishes the proof. □

Theorem 4.16. *Assume that $g \in H(\mathbb{D})$ and $0 < \alpha < 1$. Then $M_g : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is bounded if and only if $g \in H^\infty$ and (4.10) holds.*

Proof. A proof follows from Theorem 4.12, Lemma 4.15, (4.11) and Lemma 4.14. □

4.5. **Composition operators.** Every analytic self-map ψ of $\mathbb{D}$ induce a composition operator $C_\psi(f) = f \circ \psi$ acting on $\mathcal{H}(\mathbb{D})$. With regards to the theory of composition operators we refer to [15, 39], see also [26] and the references therein for recent further results.
It follows from the Littlewood's subordination theorem that every analytic self-map ψ of $\mathbb{D}$ such that $\psi(0) = 0$, induces a bounded composition operator C_ψ on H^2 or A^2_α, $\alpha > -1$. This does not remain true for Dirichlet spaces $\mathcal{D}_\alpha$, $-1 < \alpha < 1$.
On the other hand, a simple change variables gives that any automorphism of $\mathbb{D}$, $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, $a \in \mathbb{D}$, induces a bounded composition operator on H^2, A^2_α and $\mathcal{D}_\alpha$, $\alpha > -1$.
Finally, if $\psi(0) \neq 0$, we write

$$\psi = \varphi_{\psi(0)} \circ T, \quad \text{where } T = \varphi_{\psi(0)} \circ \psi\ ,$$

that is, $C_\psi = C_T C_{\varphi_{\psi(0)}}$ where $T(0) = 0$. Therefore we conclude the following.

Theorem 4.17. *Every analytic self-map ψ of $\mathbb{D}$ induces a bounded composition operator C_ψ on H^2 or A^2_α, $\alpha > -1$.*

In order to give some light on the compactness of C_ψ on these spaces, let us recall the definition of *finite angular derivative* and the Julia-Carathéodory theorem.

We say that ψ has finite angular derivative at ξ on the unit circle if there exists $\eta \in \mathbb{T}$ such that $\frac{\psi(z)-\eta}{z-\xi}$ has finite nontangential limit as $z \to \xi$. When, it exists (as a finite complex number), the limit is denoted $\psi'(\xi)$.

Theorem 4.18. *(Julia-Carathéodory theorem)*
Assume that ψ is a analytic self-map of $\mathbb{D}$ and $\xi \in \mathbb{T}$. Then, the following assertions are equivalent;

(i) $d(\xi) = \liminf_{z\to\xi} \frac{1-|\psi(z)|}{1-|z|} < \infty$, *where the limit is taken as z approaches ξ unrestrictedly in $\mathbb{D}$.*
(ii) *ψ has finite angular derivative $\psi'(\xi)$ at ξ.*
(iii) *Both ψ and ψ' have (finite) nontangential limits at ξ, with $|\eta| = 1$ for $\eta = \lim_{r\to 1^-} \psi(r\xi)$.*

Moreover, when these conditions hold, we have $\lim_{r\to 1^-}\psi'(r\xi) = \psi'(\xi) = d(\xi)\bar{\xi}\eta$*, and* $d(\xi)$ *is the nontangential limit* $\lim_{z\to\xi}\frac{1-|\psi(z)|}{1-|z|}$.

Proposition 4.19. *Let* ψ *be an analytic self-map of* $\mathbb{D}$*. If* $C_\psi : X \to X$ $(X = H^2, A^2_\alpha)$ *is compact, then* φ *has no finite angular derivative at any point of* $\mathbb{T}$.

Proof. We have seen that in both cases ($X = H^2$ and $X = A^2_\alpha$), the reproducing kernels $\{k_a^X\}_{a\in\mathbb{D}}$, satisfies that

$$\sup_{a\in\mathbb{D}}||k_a^X||_X = 1 < \infty \quad \text{and} \lim_{|a|\to 1^-}|k_a^X(z)| = 0 \text{ uniformly on compact subsets of } \mathbb{D},$$

this together with the fact that the adjoint operator $C_\psi^\star$ is compact, implies that (this step needs some calculation, see [39, p. 44])

$$0 = \lim_{|a|\to 1^-}||C_\psi^\star(k_a^X)||_X^2 = \lim_{|a|\to 1^-}\frac{||K^X_{\psi(a)}||_X^2}{||K_a^X||_X^2}.$$

So bearing in mind our previous results on kernels, we deduce that

$$\lim_{|a|\to 1^-}\frac{1-|a|}{1-|\psi(a)|} = 0,$$

then by Julia-Carathéodory Theorem, ψ has no finite angular derivative at any point of $\mathbb{T}$. □

With some more effort, it can be proved the following.

Theorem 4.20. *Let* $-1 < \alpha < \infty$*. Then* $C_\psi : A^2_\alpha \to A^2_\alpha$ *is compact if and only if* ψ *has no finite angular derivative at any point of* $\mathbb{T}$.

Theorem 4.21. *Let* ψ *be a bounded valent analytic self-map of* $\mathbb{D}$*. Then* $C_\psi : H^2 \to H^2$ *is compact if and only if* ψ *does not have finite angular derivative at any point of* $\mathbb{T}$.

There are analytic self-maps ψ of $\mathbb{D}$, that has an angular derivative at no point of $\mathbb{T}$, but induces a non-compact operator C_ψ on H^2 (see [39, Section 10.2]).
In order to describe those symbols ψ such that $C_\psi : H^2 \to H^2$ is compact, the following change of variable formula, which proof pass through (3.2), will be used.

$$||C_\psi(f)||^2_{H^2} = |f(\psi(0))|^2 + 2\int_{\mathbb{D}}|f'(\omega)|^2 N_\psi(\omega)\,dA(\omega), \tag{4.12}$$

where

$$N_\psi(\omega) = \sum_{z\in\psi^{-1}(\omega)}\log\frac{1}{|z|}$$

is the classical Nevanlinna counting function of ψ.
This function is not subharmonic on $\mathbb{D}$, however it has nice properties.

Lemma 4.22. *Let* ψ *be a analytic self-map of* $\mathbb{D}$*. Then,*

(i) $N_\psi(\omega) = \mathrm{O}\left(\log\frac{1}{|\omega|}\right)$*, as* $|\omega| \to 1^-$.

(ii) *If* $\psi(0) \neq 0$*, then*

$$N_\psi(0) \le \frac{1}{R^2}\int_{D(0,R)}N_\psi(\omega)\,dA(\omega), \quad 0 < R < |\psi(0)|.$$

(iii) *For each* $a\in\mathbb{D}$*,* $N_\psi(\varphi_a(z)) = N_{\varphi_a\circ\psi}(z)$.

Theorem 4.23. *Let* ψ *be a analytic self-map of* $\mathbb{D}$*. Then* $C_\psi : H^2 \to H^2$ *is compact if and only if* $N_\psi(\omega) = \mathrm{o}\left(\log\frac{1}{|\omega|}\right)$*, as* $|\omega| \to 1^-$.

Proof. It is clear that H^2 and C_ψ satisfies the hypotheses (a), (b) and (c) of Lemma A. Therefore, $C_\psi : H^2 \to H^2$ is a compact operator if and only if for any bounded sequence $\{f_n\}$ in H^2 such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, the sequence $\{f_n \circ \psi\}$ converges to zero in the norm of H^2.
Assume that $N_\psi(\omega) = \mathrm{o}\left(\log \frac{1}{|\omega|}\right)$, as $|\omega| \to 1^-$, and let $\{f_n\}$ be a bounded sequence $\{f_n\}$ in H^2 such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Let $\varepsilon > 0$ be given, then by hypotheses, there is $r_0 \in (0,1)$ such that

$$N_\psi(\omega) < \varepsilon \log \frac{1}{|\omega|}, \quad \text{whenever } r_0 < |\omega| < 1.$$

Moreover, we can choose $n_0 = n_0(r_0, \varepsilon, \psi)$ such that

$$|f_n(\omega)| < \varepsilon^{1/2} \quad \text{if } n_0 \le n \text{ and } \omega \in \overline{D(0, r_0)} \cup \{\psi(0)\}.$$

Thus, by the change of variable formula (4.12),

$$\begin{aligned}
||C_\psi(f_n)||^2_{H^2} &= |f_n(\psi(0))|^2 + 2 \int_{|\omega| \le r_0} + \int_{r_0 < |\omega| < 1} |f_n'(\omega)|^2 N_\psi(\omega)\, dA(\omega) \\
&\le \varepsilon + 2\varepsilon \int_{|\omega| \le r_0} N_\psi(\omega)\, dA(\omega) + 2\varepsilon \int_{r_0 < |\omega| < 1} |f_n'(\omega)|^2 \log \frac{1}{|\omega|}\, dA(\omega) \\
&\le \varepsilon + 2\varepsilon \int_{\mathbb{D}} N_\psi(\omega)\, dA(\omega) + 2\varepsilon \int_{\mathbb{D}} |f_n'(\omega)|^2 \log \frac{1}{|\omega|}\, dA(\omega) \\
&\le \varepsilon + \varepsilon ||z||^2_{H^2} + \varepsilon ||f_n||^2_{H^2} \\
&\lesssim \varepsilon,
\end{aligned}$$

that is, $C_\psi : H^2 \to H^2$ is compact.
Reciprocally, assume that $C_\psi : H^2 \to H^2$ is compact and consider the family of normalized reproducing kernels,

$$k_a^{H^2}(z) = \frac{(1 - |a|^2)^{1/2}}{1 - \bar{a} z}, \quad a, z \in \mathbb{D}.$$

By, our first observation

$$\lim_{|a| \to 1^-} ||C_\psi(k_a^{H^2})||^2_{H^2} = 0. \tag{4.13}$$

Next, by (4.12), a new change of variable and Lemma 4.22, we deduce that whenever $|a| \geq \frac{1}{2}$ and $|\varphi_a(\psi(0))| > \frac{1}{2}$

$$\begin{aligned}
||C\psi(k_a^{H^2})||_{H^2}^2 &\geq 2\int_{\mathbb{D}} |(k_a^{H^2})'(\omega)|^2 N_\psi(\omega)\, dA(\omega) \\
&= 2\int_{\mathbb{D}} \frac{|a|^2(1-|a|^2)}{|1-\bar{a}\omega|^4} N_\psi(\omega)\, dA(\omega) \\
&= \frac{2|a|^2}{1-|a|^2}\int_{\mathbb{D}} |(\varphi_a)'(\omega)|^2 N_\psi(\omega)\, dA(\omega) \\
&= \frac{2|a|^2}{1-|a|^2}\int_{\mathbb{D}} N_\psi(\varphi_a(\omega))\, dA(\omega) \\
&= \frac{2|a|^2}{1-|a|^2}\int_{\mathbb{D}} N_{\varphi_a\circ\psi}(\omega)\, dA(\omega) \\
&\geq \frac{2|a|^2}{1-|a|^2}\int_{|z|<\frac{1}{2}} N_{\varphi_a\circ\psi}(\omega)\, dA(\omega) \\
&\geq \frac{8|a|^2}{1-|a|^2} N_{\varphi_a\circ\psi}(0) \\
&= \frac{8|a|^2}{1+|a|}\frac{N_\psi(a)}{1-|a|} \\
&\geq \frac{4}{3}\frac{N_\psi(a)}{1-|a|},
\end{aligned}$$

which together (4.13), implies

$$\lim_{|a|\to 1^-} \frac{N_\psi(a)}{1-|a|} = 0.$$

This finishes the proof. □

4.6. **Schatten Classes.** Let H and K be separable Hilbert spaces. Given $0 < p < \infty$, let $\mathcal{S}_p(H,K)$ denote the Schatten p-class of operators from H to K. If $H = K$ we simply shall write $\mathcal{S}_p(H)$. The class $\mathcal{S}_p(H,K)$ consists of those compact operators T from H to K with its sequence of singular numbers

$$\lambda_n(T) = \inf\{\|T-R\| : R \in \mathcal{L}(H,K), \text{rank of } R < n\}.$$

belonging to ℓ^p, the p-summable sequence space. We recall that the singular numbers of a compact operator T are the square root of the eigenvalues of the positive operator T^*T, where T^* denotes the Hilbert adjoint of T. Finite rank operators belong to every $\mathcal{S}_p(H)$, and the membership of an operator in $\mathcal{S}_p(H)$ measures in some sense the size of the operator. We remind the reader that $T \in \mathcal{S}_p(H)$ if and only if $T^*T \in \mathcal{S}_{p/2}(H)$. Also, the compact operator T admits a decomposition of the form

$$T = \sum_n \lambda_n \langle \cdot, e_n\rangle_H\, \sigma_n,$$

where $\{\lambda_n\}$ are the singular numbers of T, $\{e_n\}$ is an orthonormal set in H, and $\{\sigma_n\}$ is an orthonormal set in H.

For $p \geq 1$, the class $\mathcal{S}_p(H,K)$ is a Banach space equipped with the norm

$$\|T\|_{\mathcal{S}_p} = \left(\sum_n |\lambda_n|^p\right)^{1/p},$$

while for $0 < p < 1$ one has the inequality $\|S+T\|^p_{\mathcal{S}_p} \leq \|S\|^p_{\mathcal{S}_p} + \|T\|^p_{\mathcal{S}_p}$. We refer to [16] or [44, Chapter 1] for a brief account on the theory of Schatten p-classes.

Toeplitz operators

For a positive Borel measure μ on $\mathbb{D}$ and a Hilbert space X of analytic functions with reproducing kernel K^X_z let us consider the operator

$$T_\mu(f)(w) = \int_{\mathbb{D}} f(z) K^X_z(w)\, d\mu(z), \quad f \in X \tag{4.14}$$

Some calculations based on Fubini's theorem and the reproducing formulas (3.11), (3.13) and (3.14), give that

$$\langle T_\mu(g), f\rangle_X = \langle g, f\rangle_{L^2(\mu)} \tag{4.15}$$

where $X = H^2, A^2_\omega$, or $\mathcal{D}^2_\alpha$. Consequently,

Theorem 4.24. *T_μ is bounded on X if and only if $I_d : X \to L^2(\mu)$ is bounded.*

Next, we shall offer a description obtained by Luecking [28] of those measures μ such that $T_\mu \in \mathcal{S}_p(X)$. Let Υ denote the family of all dyadic arcs of $\mathbb{T}$. Every dyadic arc $I \subset \mathbb{T}$ is of the form

$$I_{n,k} = \left\{e^{i\theta} : \frac{2\pi k}{2^n} \leq \theta < \frac{2\pi(k+1)}{2^n}\right\},$$

where $k = 0, 1, 2, \dots, 2^n - 1$ and $n = 0, 1, 2, \dots$. For each $I \subset \mathbb{T}$, set

$$R(I) = \left\{z \in \mathbb{D} : \frac{z}{|z|} \in I,\ 1 - |I| \leq |z| < 1 - \frac{|I|}{2}\right\}.$$

Then the family $\{R(I) :\ I \in \Upsilon\}$ consists of pairwise disjoint sets whose union covers $\mathbb{D}$. For $I_j \in \Upsilon \setminus \{I_{0,0}\}$, we will write z_j for the unique point in $\mathbb{D}$ such that $z_j = (1 - |I_j|)a_j$, where $a_j \in \mathbb{T}$ is the midpoint of I_j. For convenience, we associate the arc $I_{0,0}$ with the point $1/2$. For simplicity, we shall write R_j for $R(I_j)$.

Theorem 4.25. *Let $0 < p < \infty$ and $0 < \alpha$ such that $p(1-\alpha) < 1$. Let μ be a positive Borel measure on $\mathbb{D}$. If*

$$\sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{(1-|z_j|)^\alpha}\right)^p < \infty, \tag{4.16}$$

then $T_\mu \in \mathcal{S}_p(\mathcal{D}_\alpha)$, and there exists a constant $C > 0$ such that

$$|T_\mu|^p_p \leq C \sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{(1-|z_j|)^\alpha}\right)^p. \tag{4.17}$$

Conversely, if μ is a positive Borel measure on $\mathbb{D}$ and $T_\mu \in \mathcal{S}_p(\mathcal{D}_\alpha)$, then (4.16) is satisfied.

We recall that $\mathcal{D}^2_1 = H^2$ and $\mathcal{D}^2_\alpha = A^2_{\alpha-2}$, $\alpha > 1$. The above deep result is of independent interest and going further it can be applied to characterize those symbols such that the integral operators T_g and composition operators C_ψ belong to $\mathcal{S}_p(\mathcal{D}_\alpha)$. Indeed, let us observe that

$$\begin{aligned} T_g^\star T_g f(z) &= \langle T_g^\star T_g f, K_z^{\mathcal{D}_\alpha}\rangle_{\mathcal{D}_\alpha} \\ &= \langle T_g f, T_g K_z^{\mathcal{D}_\alpha}\rangle_{\mathcal{D}_\alpha} = \int_{\mathbb{D}} f(\omega)\overline{K_z^{\mathcal{D}_\alpha}(\omega)}|g'(\omega)|^2(1-|\omega|^2)^\alpha\, dA(\omega) \\ &= \int_{\mathbb{D}} f(\omega)K_\omega^{\mathcal{D}_\alpha}(z)|g'(\omega)|^2(1-|\omega|^2)^\alpha\, dA(\omega) = T_{\mu_g}(f)(z) \end{aligned}$$

where $d\mu_g(\omega) = |g'(\omega)|^2(1-|\omega|^2)^\alpha\, dA(\omega)$. So by Luecking's theorem,

Theorem 4.26. *Let $g \in H(\mathbb{D})$, $0 < p < \infty$ and $0 < \alpha$ such that $p(1-\alpha) < 2$. Then $T_g \in \mathcal{S}_p(\mathcal{D}_\alpha)$ if and only if*

$$\sum_{R_j \in \Upsilon} \left(\frac{\int_{R_j} |g'(\omega)|^2(1-|\omega|^2)^\alpha\, dA(\omega)}{(1-|z_j|)^\alpha} \right)^{p/2} < \infty, \tag{4.18}$$

We recall that, for $p > 1$, the Besov space B_p is the space of all analytic functions g in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |g'(z)|^p(1-|z|^2)^p\, d\lambda(z) < \infty,$$

where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ is the hyperbolic measure on $\mathbb{D}$.
Using Theorem 4.26, the properties of the net Υ, the properties of the pseudohyperbolic metric, subharmonocity of $|g'|^p$ and Hölder's inequality, the following result can be proved.

Theorem A. *Let $g \in H(\mathbb{D})$. We have the following:*

(a) *Let $0 < \alpha$ and $p > 1$ with $p(1-\alpha) < 2$. Then $T_g \in \mathcal{S}_p(\mathcal{D}_\alpha)$ if and only if g belongs to B_p.*
(b) *If $0 < p \le 1$ and $0 < \alpha$, then $T_g \in \mathcal{S}_p(\mathcal{D}_\alpha)$ if and only if g is constant.*

In particular, this result characterizes those $g \in \mathcal{H}(\mathbb{D})$ such that $T_g \in \mathcal{S}_p(H^2)$ or $T_g \in \mathcal{S}_p(A^2_\beta)$ for any $0 < p < \infty$ and $\beta > -1$.
Finally, it is worth to mention that using a similar approach based on Theorem 4.25 and the good properties of the classical Nevanlinna counting function, Luecking and Zhu [31] proved

Theorem 4.27. *Let $0 < p < \infty$, and ψ an analytic self-map of $\mathbb{D}$. Then the following assertions are equivalent:*

(1) $C_\psi \in \mathcal{S}_p(H^2)$;
(2) $\frac{N_\psi}{1-|z|^2} \in L^{\frac{p}{2}}\left(\frac{1}{(1-|z|)^2}\right)$

References

[1] A. Aleman, A class of integral operators on spaces of analytic functions. Topics in complex analysis and operator theory, 3—30, Univ. Málaga, Málaga, 2007.
[2] A. Aleman, J. A. Cima, An integral operator on H^p and Hardy's inequality, J. Anal. Math. 85 (2001), 157-–176.
[3] A. Aleman and O. Constantin, Spectra of integration operators on weighted Bergman spaces, J. Anal. Math. 109 (2009), 199-–231.
[4] A. Aleman and A. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. 46 (1997), 337–356.
[5] J. M. Anderson, J. Clunie and Ch. Pommerenke, On Bloch functions and normal functions, J. Reine Angew. Math. 270 (1974), 12-–37.
[6] N. Arcozzi, R. Rochberg and E. Sawyer, Carleson measures for analytic Besov spaces, Rev. Mat. Iberoamericana **18** (2002), 443–510.

[7] N. Arcozzi, R. Rochberg, E. Sawyer and B. Wick, *The Dirichlet space: A survey*, New York J. Math. **17a** (2011), 45–86.

[8] A. Baernstein II, Analytic functions of bounded mean oscillation, in Aspects of Contemporary Complex Analysis, D. Brannan and J. Clunie (editors), Academic Press (1980), 3–36.

[9] D. Bekollé and A. Bonami, Inégalités á poids pour le noyau de Bergman, (French) C. R. Acad. Sci. Paris Sér. A-B 286 (1978), no. 18, 775—778.

[10] A. Borichev, R. Dhuez and K. Kellay *Sampling and interpolation in large Bergman and Fock spaces,* J. Funct. Anal. **242** (2007), 563–606.

[11] O. Blasco and H. Jarchow, A note on Carleson measures for Hardy spaces, Acta Sci. Math. (Szeged) 71 (2005), 371–389.

[12] L. Carleson, An interpolation problem for bounded analytic functions, Amer. J. Math. 80 1958 921-–930.

[13] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Ann. of Math. (2) 76 1962 547-–559.

[14] O. Constantin, Carleson embeddings and some classes of operators on weighted Bergman spaces, J. Math. Anal. Appl. 365 (2010), no. 2, 668—682

[15] C. C. Cowen, and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL 1995.

[16] N. Dunford and J. T. Schwartz, Linear operators I, Wiley, New York, 1988.

[17] P. Duren, Theory of H^p Spaces, Academic Press, New York-London 1970.

[18] P. L. Duren and A. P. Schuster, Bergman Spaces, Math. Surveys and Monographs, Vol. 100, American Mathematical Society: Providence, Rhode Island, 2004.

[19] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, J. Math. Anal. Appl. **38** (1972), 756–765.

[20] J. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

[21] D. Girela, Analytic functions of bounded mean oscillation, in Complex functions spaces, (R. Aulaskari, editor), Univ. Joensuu Dept. Math. Report Series No. 4, 61–171 (2001).

[22] D. Girela, J. A. Peláez, F. Pérez-González and J. Rättyä, Carleson measures for the Bloch space. Int. Equ. Oper. Th. **61**(4), 511-547 (2008).

[23] W. W. Hastings, A Carleson measure theorem for Bergman spaces, Proc. Amer. Math. Soc. 52 (1975), 237–241.

[24] H. Hedenmalm, B. Korenblum and K. Zhu, Theory of Bergman Spaces, Graduate Texts in Mathematics, Vol. 199, Springer, New York, Berlin, etc. 2000.

[25] L. Hörmander, L^p estimates for (pluri-)subharmonic functions, Math. Scand. (20) (1967), 65–78.

[26] P. Lefévre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Composition operators on Hardy-Orlicz spaces, Memoirs Amer. Math. Soc., Vol. 207, n. 974.

[27] D. H. Luecking, Forward and reverse inequalities for functions in Bergman spaces and their derivatives, Amer. J. Math. 107, (1985), 85–111.

[28] D. H. Luecking, Trace ideal criteria for Toeplitz operators, J. Funct. Anal. 73 (1987), 345–368.

[29] D. H. Luecking, Embedding derivatives of Hardy spaces into Lebesgue spaces, Proc. London Math. Soc. **63** (1991), no. 3, 595–619.

[30] D. H. Luecking, Embedding theorems for spaces of analytic functions via Khinchine's inequality, Michigan Math. J. 40 (1993), no. 2, 333–358.

[31] D. H. Luecking and K. Zhu, Composition operators belonging to the Schatten ideals, Amer. J. Math. 114 (1992), 1127–1145.

[32] V. L. Oleinik and B. S. Pavlov, Embedding theorems for weighted classes of harmonic and analytic functions, J. Soviet Math. Vol. 2, (1974) 135–142; translation in: Zap. Nauchn. Sem. LOMI Steklov 22 (1971).

[33] J. Pau and J. A. Peláez, Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights, J. Funct. Anal. 259 n. 10, (2010), 2727–2756.

[34] M. Pavlović and J. A. Peláez, An equivalence for weighted integrals of an analytic function and its derivative. Math. Nachr. 281 (2008), no. 11, 1612–1623.

[35] J. A. Peláez and J. Rättyä, *Weighted Bergman spaces induced by rapidly increasing weights*, Mem. Amer. Math. Soc. (to appear). Available at http://arxiv.org/abs/1210.3311

[36] C. Pommerenke, Schlichte funktionen und analytische funktionen von beschränkter mittlerer oszillation, Comment. Math. Helv. 52 (1977), 591–602.

[37] W. T. Ross, The classical Dirichlet space, Recent advances in operator- related function theory, Contemp. Math., vol. 393, Amer. Math. Soc., Provi- dence, RI, (2006), pp. 171–197.

[38] K. Seip, Interpolation and sampling in small Bergman spaces, to appear in Coll. Math., arXiv:1106.3847v2.

[39] J. H. Shapiro, Composition Operators and Classical Function Theory, Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[40] A. Siskakis, Weighted integrals of analytic functions, Acta Sci. Math. (Szeged) 66 (2000), no. 3-4, 651—664.

[41] A. Siskakis, Volterra operators on spaces of analytic functions, Proceedings of the First Advanced Course in Operator Theory and Complex Analysis, 51—68, Univ. Sevilla Secr. Publ., Seville, 2006.

[42] M. Tjani, Compact composition operators on Besov spaces, Trans. Amer. Math. Soc. **355**, (2003), n. 11, 4683–4698.

[43] S. A. Vinogradov, Multiplication and division in the space of analytic functions with area integrable derivative, and in some related spaces (in russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **222** (1995), Issled. po Linein. Oper. i Teor. Funktsii **23**, 45–77, 308; translation in J. Math. Sci. (New York) **87**, no. 5 (1997), 3806–3827.

[44] K. Zhu, Operator Theory in Function Spaces, Second Edition, Math. Surveys and Monographs, Vol. 138, American Mathematical Society: Providence, Rhode Island, 2007.