Open Problem

STABILITY OF PROPERTIES OF RIEMANN SURFACES UNDER QUASI-ISOMETRIES

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An interesting problem in geometric function theory is to consider general transformations between manifolds and to study what geometric properties they preserve. Among these transformations, quasi-isometries are of special interest since they preserve Gromov hyperbolicity of geodesic metric spaces (see e.g. [3], [4]).

By nature, quasi-isometries represent a flexible class of maps that behave well on a global scale, but that produce a large distortion on the local properties of the manifolds involved. Along the former line, in [6], [7], [8], M. Kanai studied several geometric properties (such as isoperimetric inequalities, Ponicaré-Sobolev inequalities, recurrence and transience of the Brownian motion, growth rate of the volume of balls, and Liouville type theorems) for a large class of Riemannian manifolds whose Ricci curvature is bounded from below, and proved that they were preserved under quasi-isometries. Kanai was working in the general setting of Riemannian manifolds, but due to the local distortion that quasi-isometries produce, a hypothesis about the injectivity radius to be bounded from below needed to be added. Subsequently, other authors such as Holopainen and Soardi (see [5], [9]) proved that the existence of non-trivial solutions of a wide class of partial differential equations is also preserved.

A natural context to apply Kanai's results are Riemann surfaces with their Poincaré metrics, since they have constant negative curvature. (Recall that the Poincaré metric plays a main role in geometric function theory, since if R, S are Riemann surfaces with their Poincaré metrics, then $d_S(f(w), f(z)) \le d_R(w, z)$ for every holomorphic map $f : R \to S$ and every $z, w \in R$.) However, these surfaces usually have isolated singularities which are cusps in the domain. Since every Riemann surface with at least one cusp has injectivity radius equal to zero, unfortunately, it is not possible to apply Kanai's results in this case.

Hence, a natural open problem consists of formulating different hypotheses that allow to obtain similar conclusions to Kanai's results, but that could be applied to Riemann surfaces with cusps. In particular, the linear isoperimetric inequality is preserved on such

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surfaces with genus zero by quasi-isometries (recall that plane domains are the most important class of Riemann surfaces) without hypothesis on the injectivity radius; besides, the topological hypotheses on the genus can be weakened (see [1] and [2]).

But there are still many properties left to study.

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