Universidad Carlos III de Madrid

Gromov Hyperbolicity of Several Products of Graphs

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A mi familia

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A todos, Gracias.

Resumen

Sea X un espacio métrico geodésico y $x_1, x_2, x_3 \in X$. Un *triángulo geodésico* $T = \{x_1, x_2, x_3\}$ es la unión de tres geodésicas $[x_1x_2]$, $[x_2x_3]$ y $[x_3x_1]$ de X. El espacio X es δ-*hiperbólico* (en el sentido de Gromov) si todo lado de T está contenido en la δ-vecindad de la unión de los otros dos lados, para todo triángulo geodésico T de X. Se denota por $\delta(X)$ la constante de hiperbolicidad óptima de X, es decir, $\delta(X) := \inf\{\delta \geq 0 :\ X \text{ es } \delta\text{-hiperbólico}\}$. El estudio de los grafos hiperbólicos es un tema interesante dado que la hiperbolicidad de un espacio métrico geodésico es equivalente a la hiperbolicidad de un grafo más sencillo asociado al espacio.

Uno de los principales objetivos de esta tesis de doctorado es obtener información cuantitativa acerca de la constante de hiperbolicidad de varios productos de grafos. Estas desigualdades permiten obtener un resultado cualitativo importante: la caracterización de la hiperbolicidad de varios productos de grafos en términos de la hiperbolicidad de sus componentes.

En este trabajo caracterizamos los productos fuertes de grafos $G_1 \boxtimes G_2$ hiperbólicos, en términos de G_1 y G_2: el producto fuerte $G_1 \boxtimes G_2$ es hiperbólico si y sólo si uno de los factores es hiperbólico y el otro está acotado. También probamos algunas relaciones óptimas entre $\delta(G_1 \boxtimes G_2)$, $\delta(G_1)$, $\delta(G_2)$ y los diámetros de G_1 y G_2 (y encontramos familias de grafos para los cuales se alcanzan las desigualdades). Obtenemos el valor exacto de la constante de hiperbolicidad para varios productos fuertes de grafos.

También caracterizamos los productos lexicográficos de grafos $G_1 \circ G_2$ hiperbólicos, en términos de G_1 y G_2: el producto lexicográfico $G_1 \circ G_2$ es hiperbólico si y sólo si G_1 es hiperbólico, a menos que G_1 sea un grafo trivial; si G_1 es trivial, entonces $G_1 \circ G_2$ es hiperbólico si y sólo si G_2 es hiperbólico. En particular, obtenemos las desigualdades $\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$ si G_1 es un grafo no trivial, y encontramos familias de grafos para las cuales se alcanzan estas desigualdades.

Además, caracterizamos las sumas cartesianas de grafos $G_1 \oplus G_2$ hiperbólicas: $G_1 \oplus G_2$ es siempre hiperbólica, a menos que G_1 ó G_2 sea el grafo trivial, y en este último caso $G_1 \oplus G_2$ es hiperbólica si y sólo si G_2 ó G_1 es hiperbólico, respectivamente. Obtenemos las desigualdades óptimas $1 \leq \delta(G_1 \oplus G_2) \leq 3/2$ para todos los grafos G_1, G_2 no triviales. Además, caracterizamos las sumas cartesianas de grafos con $\delta(G_1 \oplus G_2) = 1$, con $\delta(G_1 \oplus G_2) = 5/4$ y con $\delta(G_1 \oplus G_2) = 3/2$. También encontramos el valor exacto de la constante de hiperbolicidad para las sumas cartesianas de diversas familias de grafos.

Finalmente, probamos que si el producto directo de grafos $G_1 \times G_2$ es hiperbólico, entonces uno de los factores es hiperbólico y el otro factor está acotado. También probamos que esta condición necesaria para la hiperbolicidad es, de hecho, una caracterización en muchos casos. En otros casos, encontramos caracterizaciones que no son tan simples. Además, obtenemos

buenas cotas para la constante de hiperbolicidad del producto directo de varias clases de grafos importantes.

Review

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1 x_2]$, $[x_2 x_3]$ and $[x_3 x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in the δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \inf\{\delta \geq 0 :\ X \text{ is } \delta\text{-hyperbolic}\}$. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it.

One of the main aims of this PhD Thesis is to obtain quantitative information about the hyperbolicity constant of several products of graphs. These inequalities allow to obtain other main results, which characterize in a qualitative way the hyperbolicity of several products of graphs in terms of the hyperbolicity of their components.

In this work we characterize the strong product of two graphs $G_1 \boxtimes G_2$ which are hyperbolic, in terms of G_1 and G_2: the strong product graph $G_1 \boxtimes G_2$ is hyperbolic if and only if one of the factors is hyperbolic and the other one is bounded. We also prove some sharp relations between $\delta(G_1 \boxtimes G_2)$, $\delta(G_1)$, $\delta(G_2)$ and the diameters of G_1 and G_2 (and we find families of graphs for which the inequalities are attained). Furthermore, we obtain the exact values of the hyperbolicity constant for many strong product graphs.

Furthermore, we characterize the lexicographic product of two graphs $G_1 \circ G_2$ which are hyperbolic, in terms of G_1 and G_2: the lexicographic product graph $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic, unless if G_1 is a trivial graph; if G_1 is trivial, then $G_1 \circ G_2$ is hyperbolic if and only if G_2 is hyperbolic. In particular, we obtain that $\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$ if G_1 is not a trivial graph, and we find families of graphs for which the inequalities are attained.

Besides, we characterize the hyperbolic product graphs for the Cartesian sum $G_1 \oplus G_2$: $G_1 \oplus G_2$ is always hyperbolic, unless either G_1 or G_2 is the trivial graph; if G_1 or G_2 is the trivial graph, then $G_1 \oplus G_2$ is hyperbolic if and only if G_2 or G_1 is hyperbolic, respectively. We also obtain the sharp inequalities $1 \leq \delta(G_1 \oplus G_2) \leq 3/2$ for every non-trivial graphs G_1, G_2. Besides, we characterize the Cartesian sums with $\delta(G_1 \oplus G_2) = 1$, with $\delta(G_1 \oplus G_2) = 5/4$ and with $\delta(G_1 \oplus G_2) = 3/2$. Furthermore, we obtain the precise value of the hyperbolicity constant of the Cartesian sum of many graphs.

Finally, we prove that if the direct product $G_1 \times G_2$ is hyperbolic, then one factor is hyperbolic and the other one is bounded. Also, we prove that this necessary condition is, in fact, a characterization in many cases. In other cases, we find characterizations which are not so simple. Furthermore, we obtain good bounds for the hyperbolicity constant of the direct product of some important graphs.

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Introduction

Hyperbolic spaces play an important role in geometric group theory and in geometry of negatively curved spaces (see, e.g., [4, 51, 54]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, simply connected Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [4, 51, 54]).

In [116] it was proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a graph related to it; hence, it is useful to know hyperbolicity criteria for graphs from a geometrical viewpoint. Therefore, the study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [2, 3, 11, 12, 13, 19, 29, 34, 48, 68, 69, 70, 71, 72, 75, 84, 85, 92, 93, 94, 95, 103, 104, 105, 116, 117, 119].

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [54, 55] and the references therein), where it was demonstrated to have a practical importance. This theory was applied principally to the study of automatic groups (see [90]), which play an important role in the science of computation. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [113] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension (formal proofs that the distortion is related to the hyperbolicity can be found in [117]); furthermore, it is evidenced that many real networks are hyperbolic (see, e.g., [2, 3, 42, 78, 86]). A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [39, 46, 50, 77]). Another important application of these spaces is the study of the spread of viruses through the internet (see [68, 70]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [68, 70]); also to traffic flow and effective resistance of networks [38, 53, 82]. The hyperbolicity has also been used extensively in the context of random graphs (see, e.g., [109, 110, 111]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [59]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [7, 16, 60, 61, 95, 96, 97, 104, 105]. In particular, in

[95, 104, 105, 116] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

For a finite graph with n vertices it is possible to compute $\delta(G)$ in time $O(n^{3.69})$ [47] (this is improved in [42, 44]). Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic [91]. However, deciding whether or not a general infinite graph is hyperbolic is usually very difficult. Therefore, three main problems on the study of hyperbolic graphs are the following:

I. To characterize the hyperbolicity for important classes of graphs.

II. To obtain inequalities relating the hyperbolicity constant and other parameters of graphs.

III. To study the invariance of the hyperbolicity of graphs under appropriate transformations.

Many researches have studied the hyperbolicity of several classes of graphs: chordal graphs [9, 19, 83, 119], median graphs [114], line graphs [33, 34, 41], cubic graphs [92], complement graphs [12], regular graphs [63], planar graphs [30, 94], periodic graphs [22, 23], short graphs [99], minor graphs [31], Mycielskian graphs [52], geometric graphs [41, 101], circulant graphs [62, 100], vertex-symmetric graphs [21], bipartite and intersection graphs [43], bridged graphs [75], expanders [82], graphs with small hyperbolicity constant [10] and some products of graphs: Cartesian product [84], corona and join product [32].

Many branches of mathematics employs some notion of a product that enables the combination or decomposition of its elemental structures. In graph theory appear several kinds of products, each with its own set of applications and theoretical interpretations. The structure and applicability of these products are full of surprises. For example, large networks such as the Internet graph, with several hundred million hosts, can be efficiently modeled by subgraphs of powers of small graphs with respect to the direct product (see [81]). This is one of many examples of the dichotomy between the structure of products and that of their subgraphs.

Product of graphs occur naturally in discrete mathematics as tools in combinatorial constructions. They give rise to important classes of graphs and deep structural problems. The extensive literature on products that has evolved over the years presents a wealth of profound and beautiful results. In the beginning the emphasis was on the structure of finite and infinite products, but later it shifted to recognition algorithms for classes of isometric subgraphs of product of graphs.

Products are often viewed as a convenient language with which to describe structures, but they are increasingly being applied in more substantial ways. Computer science is one of the many fields in which graph products are becoming commonplace. As one specific example, we mention load balancing for massively parallel computer architectures.

The most usual operations in graph theory are the unitary and binary. These operations produce new graphs from one or several graphs. The unitary operations create a new graph from the original graph. Some examples of unitary operations are: adding or deleting a vertex or an edge, the contraction of an edge, line graph, graph complement or Mycielskian graph. The binary operations create a new graph from two initial graphs G_1 and G_2; the main examples of binary operations are the several kinds of products of graphs.

The different kinds of products of graphs are an important research topic. Some large graphs are composed from some existing smaller ones by using several products of graphs, and many properties of such large graphs are strongly associated with that of the corresponding smaller ones. Under reasonable and natural restrictions such as associativity, the number of different products is actually quite limited.

The product of two graphs G_1 and G_2 is another graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$ of sets. However, each product has different rules for adjacencies.

In this work, we study:

1. The hyperbolicity of the Strong product, Lexicographic product, Cartesian sum and Direct product graphs (Problem I). They are the more interesting product graphs in order to study hyperbolicity, since [84] and [32] deal with Cartesian, corona and join products.

2. Inequalities involving the hyperbolicity constants of the product graphs and the hyperbolicity constants of their components (Problems II and III).

The *strong product* $G_1 \boxtimes G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \boxtimes G_2$ are adjacent if either $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$, or $[u_1, u_2] \in E(G_1)$ and $v_1 = v_2$, or $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.

The *lexicographic product* $G_1 \circ G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \circ G_2$ are adjacent if either $[u_1, u_2] \in E(G_1)$, or $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$.

The *Cartesian sum* $G_1 \oplus G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \oplus G_2$ are adjacent if either $[u_1, u_2] \in E(G_1)$ or $[v_1, v_2] \in E(G_2)$.

Finally, the *direct product* $G_1 \times G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.

The outline of this PhD Thesis is as follows.

In Chapter 1 we give a brief introduction to hyperbolic spaces and we show some previous results which will be useful.

In Chapter 2 we study several inequalities involving the distance in the strong product of graphs and we obtain the exact value of its diameter. Furthermore, we also study the relations between the geodesics of $G_1 \boxtimes G_2$ and geodesics in G_1 and G_2; it is not a trivial issue as Example 2.1.7 will show.

Besides, we prove several lower and upper bounds for the hyperbolicity constant of $G_1 \boxtimes G_2$, involving $\delta(G_1)$, $\delta(G_2)$ and the diameters of G_1 and G_2. One of the main results of this work is Theorem 2.2.11, which characterizes the hyperbolic strong product graphs $G_1 \boxtimes G_2$ in terms of G_1 and G_2: the graph $G_1 \boxtimes G_2$ is hyperbolic if and only if one of its factors is hyperbolic and the other one is bounded. We also find families of graphs for which many of the inequalities of this section are attained. Another main result in this Chapter is Theorem 2.2.7 which provides the precise value of $\delta(G_1 \boxtimes G_2)$ for a large class of graphs G_1, G_2; we also obtain the exact values of the hyperbolicity constant for many strong product graphs; this kind of result is not usual at all in the theory of hyperbolic graphs.

In Chapter 3 we characterize the hyperbolic lexicographic product of two graphs $G_1 \circ G_2$, in terms of G_1 and G_2: if G_1 has at least two vertices, then $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic; besides, if G_1 has a single vertex, then $G_1 \circ G_2$ is hyperbolic if and only if G_2 is hyperbolic (see Theorem 3.2.14 and Remark 3.2.15). We also prove the sharp inequalities $\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$ if G_1 is not a trivial graph (the graph with a single vertex), see Theorems 3.2.1 and 3.2.10; Example 4.2.3 provides a family of graphs for which the first inequality is attained; besides, Theorems 3.2.16 and 3.2.20 characterize the graphs for which the second inequality is attained.

Furthermore, we obtain the precise value of the hyperbolicity constant for many lexicographic products (see Examples 3.2.2, 4.2.3 and Theorem 3.2.21). In particular, Theorem 3.2.21 allows to compute, in a simple way, the hyperbolicity constant of the lexicographic product of any tree and any graph.

In Chapter 4 we characterize the hyperbolic Cartesian sum graphs $G_1 \oplus G_2$ (see Theorem 4.2.2): $G_1 \oplus G_2$ is always hyperbolic unless either G_1 or G_2 is the trivial graph; if G_1 or G_2 is the trivial graph, then $G_1 \oplus G_2$ is hyperbolic if and only if G_2 or G_1 is hyperbolic, respectively. Besides, we characterize the Cartesian sums with $\delta(G_1 \oplus G_2) = 1$ and with $\delta(G_1 \oplus G_2) = 3/2$ (see Theorems 4.2.6 and 4.2.20, respectively). Also, we have proved many inequalities involving $\delta(G_1 \oplus G_2)$, as Lemma 4.2.9 and Corollaries 4.2.10 and 4.2.12. Furthermore, we obtain simple formulae for the hyperbolicity constant of many Cartesian sum graphs (see Examples 4.2.4, 4.2.3 and 4.2.5, Theorems 4.2.7, 4.2.13, 4.2.14, 4.2.17 and 4.2.19 and Corollaries 4.2.11 and 4.2.15). We want to remark that it is not usual at all to obtain explicit formulae for the hyperbolicity constant of large classes of graphs. Finally, Theorem 4.3.4 provides precise bounds for the hyperbolicity constant of the complement graph of many Cartesian sums: $\frac{3}{2} \leq \delta(\overline{G_1 \oplus G_2}) \leq 2$.

In Chapter 5 we characterize in many cases the hyperbolic direct product of graphs. Here the situation is more complex than with the Cartesian or the strong product, which is in part due to the facts that the direct product of two bipartite graphs is already disconnected and that the formula for the distance in $G_1 \times G_2$ is more complicated that in the case of other products of graphs. Theorem 5.1.19 proves that if $G_1 \times G_2$ is hyperbolic, then one factor is hyperbolic and the other one is bounded. Also, we prove that this necessary condition is, in fact, a characterization in many cases. If G_1 is a hyperbolic graph and G_2 is a bounded graph, then we prove that $G_1 \times G_2$ is hyperbolic when G_2 has some odd cycle (Theorem 5.1.9) or G_1 and G_2 do not have odd cycles (Theorem 5.1.10). Otherwise, the characterization is a more difficult task; if G_1 has some odd cycle and G_2 do not have odd cycles, Theorems 5.1.20 and 5.1.22 provide sufficient conditions for non-hyperbolicity and hyperbolicity, respectively; besides, Theorems 5.1.31 and Corollary 5.1.32 characterize the hyperbolicity of $G_1 \times G_2$ under some additional conditions. Furthermore, we obtain good bounds for the hyperbolicity constant of the direct product of some important graphs.

The results in this work appear in [24, 25, 27, 28]; these papers have been published or submitted to international mathematical journals which appear in the Journal Citation Reports.

Besides, these results were presented in the following international and national conferences:

- IX Encuentro Andaluz de Matemática Discreta, in October 2015, at Universidad de Almería, Spain.
- IX Workshop of Young Researchers in Mathematics, in September 2015, at Universidad Complutense de Madrid, Spain.
- III Congreso de Jóvenes Investigadores de la Real Sociedad Matemática Española, in September 2015, at Universidad de Murcia, Spain.
- VIII Workshop of Young Researchers in Mathematics, September 2014, Universidad Complutense de Madrid, Spain.
- IX Jornadas de Matemática Discreta y Algorítmica, in July 2014, at Universidad de Tarragona, Spain.
- VIII Encuentro Andaluz de Matemática Discreta, in October 2013, at Universidad de Sevilla, Spain.
- VII Workshop of Young Researchers in Mathematics, in September 2013, at Universidad Complutense de Madrid, Spain.

The work presented in IX Jornadas de Matemática Discreta y Algorítmica appears in the Proceedings of the Conference, published in a good international mathematical journal (see [26]).

One of these results was presented in the GAMA[1] Seminar, in March 2016, at Universidad Carlos III de Madrid, Spain.

[1]Group of Applied Mathematical Analysis

Chapter 1

Hyperbolic spaces

Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. We say that γ is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where L denotes the length of a curve. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If the metric space X is a graph, we use the notation $[u, v]$ for the edge joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, we must identify any edge $[u, v] \in E(G)$ with the interval $[0, 1]$; therefore, any point in the interior of any edge is a point of G and, if we consider the edge $[u, v]$ as a graph with just one edge, then it is isometric to $[0, 1]$. A connected graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G. Then, we see G as a metric graph.

Throughout this work we just consider non-oriented connected simple (without loops and multiple edges) graphs with edges of length 1; these properties guarantee that the graphs are geodesic metric spaces (since we consider that every point in any edge of a graph G is a point of G, whether or not it is a vertex of G). We want to remark that by [13] the study of the hyperbolicity of graphs with loops and multiple edges (non-simple graphs) can be reduced to the study of the hyperbolicity of simple graphs (see Theorems 1.3.8 and 1.3.9).

1.1 Definition of hyperbolic spaces and examples

The concept of hyperbolicity offers a global approach to spaces like the hyperbolic plane, simply-connected Riemannian manifolds with negative sectional curvature, metric trees and others classical hyperbolic spaces. Several of their properties were introduced by Mikhael Gromov in the context of finitely generated groups but its generality reached new horizons.

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geq 0 : \ J \ \text{ is } \delta\text{-thin}\}$.

Definition 1.1.1. *Given $x_1, x_2, x_3 \in X$. A* geodesic triangle $T = \{x_1, x_2, x_3\}$ *is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is δ-*hyperbolic (*or satisfies the Rips condition with constant δ) if every geodesic triangle in X is δ-thin.*

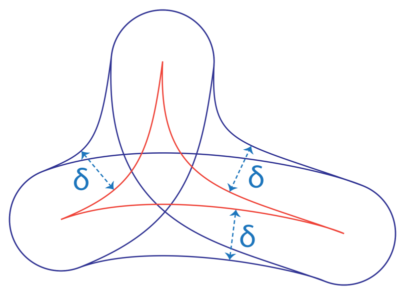

Figure 1.1: δ-thin triangle.

We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Sometimes we write the geodesic triangle T as $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$.

Remark 1.1.2. *If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$.*

One can check that every geodesic polygon in X with n sides is $(n-2)\delta(X)$-thin; in particular, any geodesic quadrilateral is $2\delta(X)$-thin. The above result is obtained by dividing the polygon into $n-2$ triangles.

A geodesic bigon is a geodesic triangle $\{x_1, x_2, x_3\}$ with $x_2 = x_3$. Therefore, every bigon in a δ-hyperbolic geodesic metric space is δ-thin.

There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' which just depends on δ (see, e.g., [17, 51]). We have chosen this definition since it has a deep geometric meaning (see, e.g., [51]).

The following are interesting examples of hyperbolic spaces.

Example 1.1.3. *Any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore $\mathbb{R}$ is 0-hyperbolic.*

Example 1.1.4. *The Euclidean plane $\mathbb{R}^2$ is not hyperbolic: it is clear that equilateral triangles can be drawn with arbitrarily large diameter.*

The argument in Example 1.1.4 can be generalized to higher dimensions:

$$\textit{a normed vector space } E \textit{ is hyperbolic if and only if } \dim\ E = 1.$$

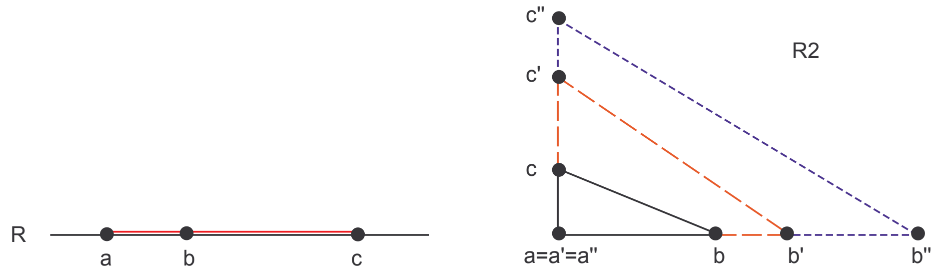

Figure 1.2: $\mathbb{R}$ and $\mathbb{R}^2$ as examples of hyperbolic spaces.

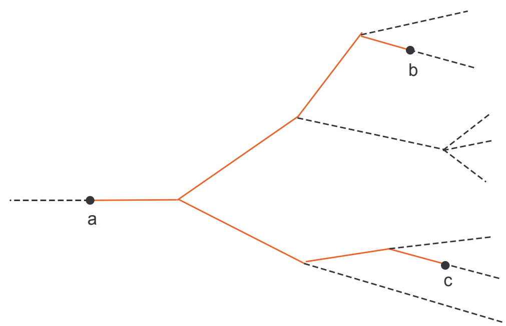

Figure 1.3: Any metric tree T verifies $\delta(T) = 0$.

Example 1.1.5. *Every metric tree is* 0*-hyperbolic: in fact, every point of a geodesic triangle in a tree belongs simultaneously to two sides of the triangle (see Figure 1.3).*

Example 1.1.6. *Every bounded metric space* X *is* $(\operatorname{diam} X/2)$*-hyperbolic: in fact, the distance from any point of a geodesic triangle to the endpoints of its geodesic is at most* $\operatorname{diam}(X)/2$.

Example 1.1.7. *Every simply connected complete Riemannian manifold with sectional curvature verifying* $K \le -c^2$*, for some positive constant* c*, is hyperbolic.*

The following example is an exercise in [102, p.191] (it is a particular case of Example 1.1.7).

Example 1.1.8. *The open unit disk in the complex plane with its Poincaré metric is* $\log(1+\sqrt{2})$*-hyperbolic.*

We refer to [17, 51] for more background and further results.

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [36]).

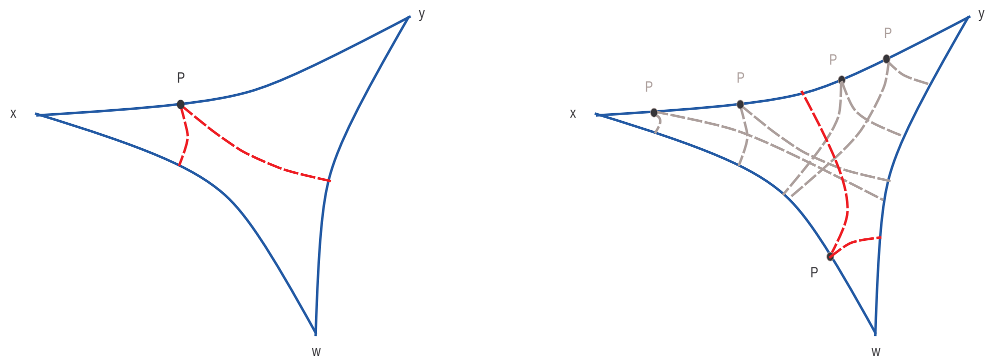

Figure 1.4: First steps in order to compute the hyperbolicity constant of X.

For a general graph or a general geodesic metric space deciding whether or not a space is hyperbolic is usually very difficult. We have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to (see Figure 1.4). And then we have to take the supremum over all the possible choices for P and then over all the possible choices for T (see Figures 1.4 and 1.5).

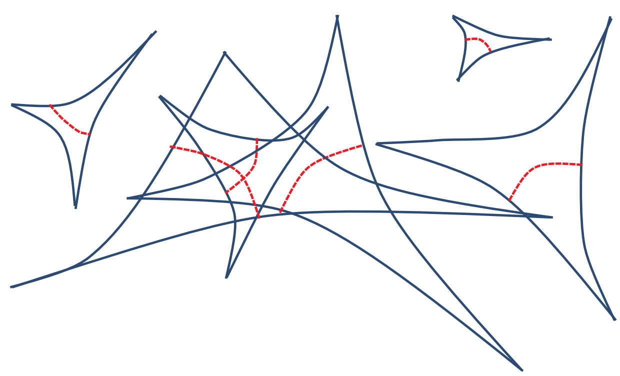

Figure 1.5: Calculating the supremum over all geodesic triangles.

Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant and other parameters of graphs. Another natural problem is to study the invariance of the hyperbolicity of graphs under appropriate transformations.

Since to obtain a characterization of hyperbolic graphs is a very ambitious goal, it seems reasonable to study this problem for particular classes of graphs (see Chapters 2, 3, 4 and 5). We are interested in to characterize the hyperbolicity of several graph products. In fact, we obtain this characterization for strong and lexicographic products and the Cartesian sum; for direct product of graphs, we provide a necessary condition, and we prove that this condition is also sufficient in many cases.

1.2 Equivalent definitions of hyperbolicity

Gromov product definition

Definition 1.2.1. *Given a metric space X, we define the* Gromov product *of $x, y \in X$ with base point $w \in X$ by*

$$(x|y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big). \tag{1.1}$$

We say that X is δ-hyperbolic product if there is a constant $\delta \geq 0$ such that

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta \tag{1.2}$$

for every $x, y, z, w \in X$ (see, e.g., [51]).

It is well known that (1.2) is equivalent to our definition of Gromov hyperbolicity for geodesic metric spaces (Definition 1.1.1). Furthermore, we have the following quantitative result about this equivalence.

Theorem 1.2.2. *[51, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-hyperbolic product.*
(2) *If X is δ-hyperbolic product, then it is 3δ-hyperbolic.*

Fine definition

First, we recall the definition of fine triangles.

Definition 1.2.3. *Given a geodesic triangle $T = \{x, y, z\}$ in a geodesic metric space X, let T_E be a Euclidean triangle with sides of the same length than T. Since there is no possible confusion, we will use the same notation for the corresponding points in T and T_E. The maximum inscribed circle in T_E meets the side $[xy]$ (respectively $[yz]$, $[zx]$) in a point z' (respectively x', y') such that $d(x, z') = d(x, y')$, $d(y, x') = d(y, z')$ and $d(z, x') = d(z, y')$. We call the points x', y', z', the* internal points *of $\{x, y, z\}$. There is a unique isometry f_{xyz} of $\{x, y, z\}$ onto a tripod (a star graph with one vertex w of degree 3, and three vertices*

*x'', y'', z'' of degree one, such that $d(x'', w) = d(x, z') = d(x, y')$, $d(y'', w) = d(y, x') = d(y, z')$ and $d(z'', w) = d(z, x') = d(z, y')$), see Figure 1.6. The triangle $\{x, y, z\}$ is δ-*fine *if $f_{xyz}(p) = f_{xyz}(q)$ implies that $d(p, q) \le \delta$. The space X is δ-*fine *if every geodesic triangle in X is δ-fine.*

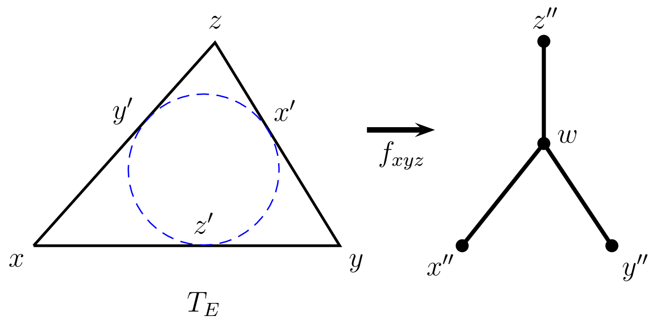

Figure 1.6: Isometry f_{xyz} of the triangle $T_E = \{x, y, z\}$ onto a tripod.

We also allow *degenerated tripods*, i.e., path graphs P_1, P_2 with one or two vertices, respectively. These situations correspond with triangles with several vertices repeated; in these cases the inscribed circle in T_E is a point.

It is known that this definition of fine is also equivalent to our definition of Gromov hyperbolicity. Furthermore, we have the following quantitative result.

Theorem 1.2.4. *[51, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-fine.*
(2) *If X is δ-fine, then it is δ-hyperbolic.*

Insize definition

Definition 1.2.5. *Given a geodesic metric space X, let $T = \{x, y, z\}$ be a geodesic triangle in X and let x', y', z' be the internal points on T in Definition 1.2.3. Let us define the* insize *of the geodesic triangle T as*

$$insize(T) := \operatorname{diam}\{x', y', z'\}. \tag{1.3}$$

*The space X is δ-*insize *if every geodesic triangle in X has insize at most δ.*

This definition of insize is also equivalent to our definition of Gromov hyperbolicity. Besides, we have the following quantitative result.

Theorem 1.2.6. *[51, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-insize.*
(2) *If X is δ-insize, then it is 2δ-hyperbolic.*

Minsize definition

Definition 1.2.7. *Given a geodesic metric space X, let $T = \{x, y, z\}$ be a geodesic triangle in X and let $x_0 \in [yz]$, $y_0 \in [zx]$, $z_0 \in [xy]$. We define the* minsize *of the geodesic triangle T to be*

$$minsize(T) := \min_{x_0,y_0,z_0 \in T} \operatorname{diam}\{x_0, y_0, z_0\}. \tag{1.4}$$

*The space X is δ-*minsize *if every geodesic triangle in X has minsize at most δ.*

It is known that this definition of minsize is also equivalent to Definition in a quantitative way.

Theorem 1.2.8. *[51, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-minsize.*
(2) *If X is δ-minsize, then it is 8δ-hyperbolic.*

1.3 Background on hyperbolic graphs

Let us return to our framework: graphs as geodesic metric spaces. In this section we present some previous results about hyperbolic graphs. These results are used throughout the thesis or are benchmark results on the subject.

Definition 1.3.1. *The diameter of the vertices of the graph G, denoted by* $\operatorname{diam} V(G)$, *is defined as*

$$\operatorname{diam} V(G) := \sup\{d_G(u, v) : u, v \in V(G)\},$$

and the diameter of the graph G, denoted by $\operatorname{diam} G$, *is defined as*

$$\operatorname{diam} G := \sup\{d_G(x, y) : x, y \in G\}.$$

Definition 1.3.2. *We say that a subgraph Γ of G is* isometric *if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$.*

We will need the following results (see [103, Lemma 5] and [105, Lemma 2.1]).

Lemma 1.3.3. *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \le \delta(G)$.*

Lemma 1.3.4. *Let us consider a geodesic metric space X. If every geodesic triangle in X that is a simple closed curve is δ-thin, then X is δ-hyperbolic.*

This lemma has the following direct consequence. As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices in a graph, except for the last one, which is equal to the first vertex.

Corollary 1.3.5. *In any graph G,*

$$\delta(G) = \sup\{\delta(T) : T \text{ is a geodesic triangle that is a cycle}\}.$$

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \geq 1$, $\beta \geq 0$ if, for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function f is ε-*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that f is an ε-full (α, β)-quasi-isometric embedding.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \longrightarrow Y$. One can check that to be quasi-isometric is an equivalence relation. An (α, β)-*quasi-geodesic* in X is an (α, β)-quasi-isometric embedding between an interval of $\mathbb{R}$ and X.

A fundamental property of hyperbolic spaces is the following (see, e.g., [51, p.88]):

Theorem 1.3.6 (Invariance of hyperbolicity)**.** *Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic.*

Besides, if f is ε-full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

Furthermore, if X (respectively, Y) is δ-hyperbolic, then Y (respectively, X) is δ'-hyperbolic, where δ' is a constant which just depends on δ, α, β and ε (respectively, δ, α and β).

The following result (see [103, Theorem 8]) will be useful.

Lemma 1.3.7. *In any graph G the inequality $\delta(G) \leq (\operatorname{diam} G)/2$ holds, and it is sharp.*

If G and H are isomorphic, we write $G \simeq H$. It is clear that if $G \simeq H$, then $\delta(G) = \delta(H)$.

The following results appear in [13, Theorems 8 and 10]. They allow to reduce the study of the hyperbolicity of non-simple graphs to the study of the hyperbolicity of simple graphs. Theorems 8 and 10 in [13] are, in fact, stronger, but these versions below are good enough for this work.

Given a non-simple graph G, we define $A(G)$ as the graph G without its loops, and $B(G)$ as the graph G without its multiple edges, obtained by replacing each multiple edge by a single edge.

Theorem 1.3.8. *If G is a graph with some loop, then G is hyperbolic if and only if $A(G)$ is hyperbolic. Besides,*

$$\delta(G) = \max\left\{\delta(A(G))\ ,\ \frac{1}{4}\right\}.$$

Theorem 1.3.9. *If G is a graph with some multiple edge, then G is hyperbolic if and only if $B(G)$ is hyperbolic. Besides,*

$$\delta(G) = \max\left\{\delta(B(G)), \frac{1}{2}\right\} = \max\left\{\delta(A(B(G))), \frac{1}{2}\right\}.$$

In particular, if $A(B(G))$ is not a tree, then $\delta(G) = \delta(B(G)) = \delta(A(B(G)))$.

Therefore, in what follows, by graph we mean simple graph.

We will also need the following result (see [103, Theorem 11]).

Theorem 1.3.10. *The following graphs have the following hyperbolicity constants:*

- *The path graphs verify $\delta(P_n) = 0$ for every $n \geq 1$.*
- *The cycle graphs verify $\delta(C_n) = n/4$ for every $n \geq 3$.*
- *The complete graphs verify $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$, $\delta(K_n) = 1$ for every $n \geq 4$.*
- *The complete bipartite graphs verify $\delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0$, $\delta(K_{m,n}) = 1$ for every $m, n \geq 2$.*
- *The Petersen graph P verifies $\delta(P) = 3/2$.*
- *The wheel graph with n vertices W_n verifies $\delta(W_4) = \delta(W_5) = 1$, $\delta(W_n) = 3/2$ for every $7 \leq n \leq 10$, and $\delta(W_n) = 5/4$ for $n = 6$ and for every $n \geq 11$.*

We will use the following results which allow to reduce the study of the hyperbolicity of graphs to a countable set of geodesic triangles.

If $[v_1, v_2] \in E(G)$, then we say that the point $x \in [v_1, v_2]$ with $d_G(x, v_1) = d_G(x, v_2) = 1/2$ is the *midpoint* of $[v_1, v_2]$. Given a graph G, we define $J(G)$ as the set of points of the graph G which are either vertices or midpoints of the edges. Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

The following three results, which appear in [11].

Theorem 1.3.11. *[11, Theorem 2.5] For every graph G we have $\delta_1(G) = \delta(G)$.*

The next result will narrow the posible values for the hyperbolicity constant δ.

Theorem 1.3.12. *[11, Theorem 2.6] For every hyperbolic graph G, $\delta(G)$ is a multiple of $1/4$.*

Theorem 1.3.13. *[11, Theorem 2.7] For any hyperbolic graph G, there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

Finally, we define some families of graphs which will be useful. Denote by C_n the cycle graph with $n \geq 3$ vertices and by $V(C_n) := \{v_1^{(n)}, \ldots, v_n^{(n)}\}$ the set of their vertices such that $[v_n^{(n)}, v_1^{(n)}] \in E(C_n)$ and $[v_i^{(n)}, v_{i+1}^{(n)}] \in E(C_n)$ for $1 \leq i \leq n-1$. Let $\mathcal{C}_6^{(1)}$ be the set of graphs obtained from C_6 by adding a (proper or not) subset of the set of edges $\{[v_2^{(6)}, v_6^{(6)}], [v_4^{(6)}, v_6^{(6)}]\}$. Let us define the set of graphs

$$\mathcal{F}_6 := \{\text{graphs containing, as induced subgraph, an isomorphic graph}$$

$$\text{to some element of } \mathcal{C}_6^{(1)}\}.$$

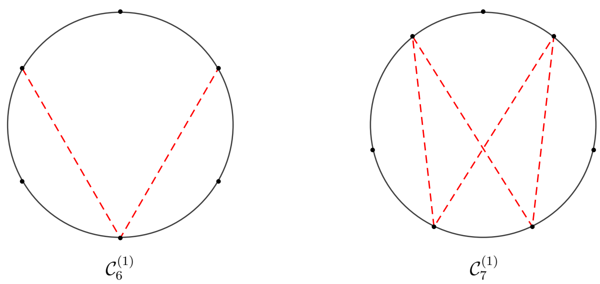

Figure 1.7: Generators of $\mathcal{C}_6^{(1)}$ and $\mathcal{C}_7^{(1)}$.

Let $\mathcal{C}_7^{(1)}$ be the set of graphs obtained from C_7 by adding a (proper or not) subset of the set of edges $\{[v_2^{(7)}, v_6^{(7)}], [v_2^{(7)}, v_7^{(7)}], [v_4^{(7)}, v_6^{(7)}], [v_4^{(7)}, v_7^{(7)}]\}$. Define

$$\mathcal{F}_7 := \{\text{graphs containing, as induced subgraph, an isomorphic graph}$$

$$\text{to some element of } \mathcal{C}_7^{(1)}\}.$$

Let $\mathcal{C}_8^{(1)}$ be the set of graphs obtained from C_8 by adding a (proper or not) subset of the set $\{[v_2^{(8)}, v_6^{(8)}], [v_2^{(8)}, v_8^{(8)}], [v_4^{(8)}, v_6^{(8)}], [v_4^{(8)}, v_8^{(8)}]\}$. Also, let $\mathcal{C}_8^{(2)}$ be the set of graphs obtained from C_8 by adding a (proper or not) subset of $\{[v_2^{(8)}, v_8^{(8)}], [v_4^{(8)}, v_6^{(8)}], [v_4^{(8)}, v_7^{(8)}], [v_4^{(8)}, v_8^{(8)}]\}$. Define

$$\mathcal{F}_8 := \{\text{graphs containing, as induced subgraph, an isomorphic graph}$$

$$\text{to some element of } \mathcal{C}_8^{(1)} \cup \mathcal{C}_8^{(2)}\}.$$

Let $\mathcal{C}_9^{(1)}$ be the set of graphs obtained from C_9 by adding a (proper or not) subset of the set of edges $\{[v_2^{(9)}, v_6^{(9)}], [v_2^{(9)}, v_9^{(9)}], [v_4^{(9)}, v_6^{(9)}], [v_4^{(9)}, v_9^{(9)}]\}$. Define

$$\mathcal{F}_9 := \{\text{graphs containing, as induced subgraph, an isomorphic graph}$$

to some element of $\mathcal{C}_9^{(1)}\}$.

Finally, we define the set $\mathcal{F}$ by

$$\mathcal{F} := \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8 \cup \mathcal{F}_9.$$

Note that $\mathcal{F}_6$, $\mathcal{F}_7$, $\mathcal{F}_8$ and $\mathcal{F}_9$ are not disjoint sets of graphs.

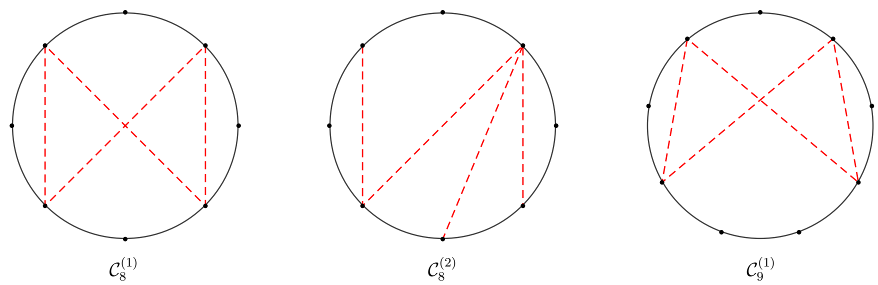

Figure 1.8: Generators of $\mathcal{C}_8^{(1)}$, $\mathcal{C}_8^{(2)}$ and $\mathcal{C}_9^{(1)}$.

Chapter 2

Gromov hyperbolicity in strong product graphs

The strong product graph operation has been extensively investigated in relation to a wide range of subjects [1, 20, 73, 115]. A fundamental principle for network design is extendability. That is to say, the possibility of building larger versions of a network preserving certain desirable properties. For designing large-scale interconnection networks, the strong product is a useful method to obtain large graphs from smaller ones whose invariants can be easily calculated [20, 73, 115].

2.1 The distance in strong product graphs

In order to estimate the hyperbolicity constant of the strong product of two graphs G_1 and G_2, we must obtain lower and upper bound on the distances between any two arbitrary points in $G_1 \boxtimes G_2$. The lemmas of this section provide these estimations. We will use the strong product definition given by Sabidussi in [106].

Definition 2.1.1. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ two graphs. The strong product $G_1 \boxtimes G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \boxtimes G_2$ are adjacent if either $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$, or $[u_1, u_2] \in E(G_1)$ and $v_1 = v_2$, or $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.*

Note that the strong product of two graphs is commutative.

Next, we will bound the distances between any two different pair of points in the strong product graph. For this aim we must distinguish some cases depending on the situation of the considered points. Let $p \in G_1$ and $q \in G_2$ be two points of G_1 and G_2 respectively. The pair (p, q) is an *inner point* in $G_1 \boxtimes G_2$, if $p \in G_1 \setminus V(G_1)$ and $q \in V(G_2)$ or $p \in V(G_1)$ and $q \in G_2 \setminus V(G_2)$ or $p \in G_1 \setminus V(G_1)$ and $q \in G_2 \setminus V(G_2)$ (i.e., $(p, q) \in G_1 \boxtimes G_2 \setminus V(G_1 \boxtimes G_2)$). Notice that the first and second cases of the inner points in $G_1 \boxtimes G_2$ are contained in the

Cartesian product graph $G_1 \square G_2 \subset G_1 \boxtimes G_2$; so the first and second cases are the inner points of the Cartesian edges properly. In order to represent the inner points of the non Cartesian edges in $G_1 \boxtimes G_2$ we will consider the following assumptions. Let $[A_1, A_2] \in E(G_1)$ and $[B_1, B_2] \in E(G_2)$ be edges in G_1 and G_2, respectively. Let $p \in [A_1, A_2]$ and $q \in [B_1, B_2]$ be inner points of theses fixed edges; we have $(p, q) \in G_1 \boxtimes G_2 \setminus G_1 \square G_2$ if $L([pA_1]) = L([qB_1])$ or $L([pA_1]) = L([qB_2])$.

Notice that there are different points on $G_1 \boxtimes G_2$ with the same representation: the midpoints of $[(A_1, B_1), (A_2, B_2)]$ and $[(A_1, B_2), (A_2, B_1)]$. Then, this notation is ambiguous, but it is convenient.

The following lemmas provide bounds on the distance between any two pair of points in the strong product graph $(p_1, q_1), (p_2, q_2) \in G_1 \boxtimes G_2$.

The first one is a well known property about distances between vertices in the strong product of graphs proved in [64].

Lemma 2.1.2 (Lemma 5.1 in [64])**.** *Let G_1, G_2 be any graphs. If $p_1, p_2 \in V(G_1)$ and $q_1, q_2 \in V(G_2)$, then*

$$d_{G_1 \boxtimes G_2}((p_1, q_1), (p_2, q_2)) = \max\{d_{G_1}(p_1, p_2), d_{G_2}(q_1, q_2)\}.$$

Next, a lower bound on the distance between any two points in the strong product graph.

Proposition 2.1.3. *Let G_1, G_2 be any graphs. For every $(p_1, q_1), (p_2, q_2) \in G_1 \boxtimes G_2$ we have*

$$d_{G_1 \boxtimes G_2}((p_1, q_1), (p_2, q_2)) \geq \max\{d_{G_1}(p_1, p_2), d_{G_2}(q_1, q_2)\}. \tag{2.1}$$

Proof. By symmetry, it suffices to prove $d_{G_1 \boxtimes G_2}((p_1, q_1), (p_2, q_2)) \geq d_{G_1}(p_1, p_2)$. Seeking for a contradiction, assume that $d_{G_1 \boxtimes G_2}((p_1, q_1), (p_2, q_2)) < d_{G_1}(p_1, p_2)$.

Hence, there exist a geodesic Γ joining (p_1, q_1) and (p_2, q_2) in $G_1 \boxtimes G_2$ with $L(\Gamma) < d_{G_1}(p_1, p_2)$. Denote by $(A_1, B_1), \ldots, (A_k, B_k)$ the vertices of $G_1 \boxtimes G_2$ in Γ; without loss of generality we can assume that Γ meets $(A_1, B_1), \ldots, (A_k, B_k)$ in this order. Then, we have

$$\Gamma := [(p_1, q_1)(A_1, B_1)] \bigcup \left\{ \bigcup_{j=1}^{k-1} [(A_j, B_j), (A_{j+1}, B_{j+1})] \right\} \bigcup [(A_k, B_k)(p_2, q_2)].$$

By Definition 2.1.1, we obtain that

$$\gamma := [p_1 A_1] \bigcup \left\{ \bigcup_{j=1}^{k-1} [A_j A_{j+1}] \right\} \bigcup [A_k p_2]$$

is a path joining p_1 and p_2 such that $L(\gamma) \leq L(\Gamma) < d_{G_1}(p_1, p_2)$. This is the contradiction we were looking for. $\square$

The following result provides an upper bound for the distance between a vertex and an inner point, as well as between two inner points in $G_1 \boxtimes G_2$.

Proposition 2.1.4. *Let G_1, G_2 be any graphs.*

(i) If $(u,v) \in V(G_1 \boxtimes G_2)$ and $(p,q) \in G_1 \boxtimes G_2 \setminus V(G_1 \boxtimes G_2)$, then

$$d_{G_1\boxtimes G_2}((u,v),(p,q)) \le \max\{d_{G_1}(u,p), d_{G_2}(v,q)\} + 1. \tag{2.2}$$

(ii) If $(p_1,q_1),(p_2,q_2) \in G_1 \boxtimes G_2 \setminus V(G_1 \boxtimes G_2)$, then

$$d_{G_1\boxtimes G_2}((p_1,q_1),(p_2,q_2)) \le \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} + 2. \tag{2.3}$$

Proof. In order to prove (i), let us consider $[(u_1,v_1),(u_2,v_2)] \in E(G_1 \boxtimes G_2)$ such that $(p,q) \in [(u_1,v_1),(u_2,v_2)]$. Let γ be a geodesic in $G_1 \boxtimes G_2$ joining (u,v) and (p,q). Without loss of generality we can assume that $(u_1,v_1) \in \gamma$. Define $\varepsilon := d_{G_1\boxtimes G_2}((u_1,v_1),(p,q))$. By Lemma 5.1.8, we have

$$\begin{aligned} d_{G_1\boxtimes G_2}((u,v),(p,q)) &= \max\{d_{G_1}(u,u_1), d_{G_2}(v,v_1)\} + \varepsilon \\ &\le \max\{d_{G_1}(u,p) + d_{G_1}(p,u_1), d_{G_2}(v,q) + d_{G_2}(q,v_1)\} + \varepsilon \\ &\le \max\{d_{G_1}(u,p), d_{G_2}(v,q)\} + 2\varepsilon. \end{aligned}$$

If $\varepsilon \le 1/2$, then we have (2.2). If $\varepsilon > 1/2$, then we have $\max\{d_{G_1}(u,u_2), d_{G_2}(v,v_2)\} = \max\{d_{G_1}(u,u_1), d_{G_2}(v,v_1)\} + 1$; thus, $d_{G_1\boxtimes G_2}\big((u,v),(p,q)\big) = \max\{d_{G_1}(u,p), d_{G_2}(v,q)\}$.

In order to proof (ii), notice that if $(p_1,q_1),(p_2,q_2)$ belong to the same edge of $G_1 \boxtimes G_2$, then we have the result since $d_{G_1\boxtimes G_2}((p_1,q_1),(p_2,q_2)) < 1$. Assume now that $(p_1,q_1),(p_2,q_2)$ belong to different edges of $G_1 \boxtimes G_2$. Let us consider $(u_1,v_1),(u_2,v_2),(u_3,v_3),(u_4,v_4) \in V(G_1 \boxtimes G_2)$ such that $(p_1,q_1) \in [(u_1,v_1),(u_2,v_2)]$ and $(p_2,q_2) \in [(u_3,v_3),(u_4,v_4)]$. Let γ^* be a geodesic in $G_1 \boxtimes G_2$ joining (p_1,q_1) and (p_2,q_2). Without loss of generality we can assume that $(u_2,v_2),(u_3,v_3) \in \gamma^*$. Define $\varepsilon_1 := d_{G_1\boxtimes G_2}((u_2,v_2),(p_1,q_1))$ and $\varepsilon_2 := d_{G_1\boxtimes G_2}((u_3,v_3),(p_2,q_2))$. Then, we have

$$\begin{aligned} d_{G_1\boxtimes G_2}((p_1,q_1),(p_2,q_2)) &= \varepsilon_1 + \max\{d_{G_1}(u_2,u_3), d_{G_2}(v_2,v_3)\} + \varepsilon_2 \\ &\le 2\varepsilon_1 + \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} + 2\varepsilon_2. \end{aligned}$$

Notice that if $\varepsilon_1, \varepsilon_2 \le 1/2$, then (2.3) holds directly. If $\varepsilon_1 > 1/2$ (the case $\varepsilon_2 > 1/2$ is analogous), then $\max\{d_{G_1}(u_1,u_3), d_{G_2}(v_1,v_3)\} = \max\{d_{G_1}(u_2,u_3), d_{G_2}(v_2,v_3)\} + 1$; thus, $d_{G_1\boxtimes G_2}\big((p_1,q_1),(u_3,v_3)\big) = \max\{d_{G_1}(p_1,u_3), d_{G_2}(q_1,v_3)\}$. Hence, we have

$$\begin{aligned} d_{G_1\boxtimes G_2}\big((p_1,q_1),(p_2,q_2)\big) &= \max\{d_{G_1}(p_1,u_3), d_{G_2}(q_1,v_3)\} + \varepsilon_2 \\ &\le \max\{d_{G_1}(p_1,p_2) + d_{G_1}(p_2,u_3), d_{G_2}(q_1,q_2) + d_{G_2}(q_2,v_3)\} + \varepsilon_2 \\ &\le \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} + 2\varepsilon_2. \end{aligned}$$

This finishes the proof. □

The previous lemmas let us announce the following general result on the distances in the strong product of two graphs.

Theorem 2.1.5. *For all graphs G_1, G_2 we have:*

(a) $d_{G_1\boxtimes G_2}((p_1,q_1),(p_2,q_2)) = \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\}$*, for every* $(p_1,q_1),(p_2,q_2) \in V(G_1 \boxtimes G_2)$,

(b) $\max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} \le d_{G_1\boxtimes G_2}((p_1,q_1),(p_2,q_2)) \le \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} +1$*, for every* $(p_1,q_1) \in V(G_1 \boxtimes G_2)$ *and* $(p_2,q_2) \in G_1 \boxtimes G_2$,

(c) $\max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} \le d_{G_1\boxtimes G_2}((p_1,q_1),(p_2,q_2)) \le \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} +2$*, for every* $(p_1,q_1),(p_2,q_2) \in G_1 \boxtimes G_2$.

Let us consider the projection $P_k : G_1 \boxtimes G_2 \longrightarrow G_k$ for $k \in \{1,2\}$.

Corollary 2.1.6. *Let $\{i,j\}$ be a permutation of $\{1,2\}$. Then, for every x,y in $G_1 \boxtimes G_2$,*

$$d_{G_i}(P_i(x),P_i(y)) \le d_{G_1\boxtimes G_2}(x,y) \le d_{G_i}(P_i(x),P_i(y)) + \operatorname{diam} G_j + 2. \tag{2.4}$$

These results provide information about the geodesics in $G_1 \boxtimes G_2$. Notice that, if γ is a geodesic joining x and y in $G_1 \boxtimes G_2$, then it is possible that $P_j(\gamma)$ does not contain a geodesic joining $P_j(x)$ and $P_j(y)$ in G_j, as the following example shows.

Example 2.1.7. *Consider a cycle graph G_1 with vertices $\{v_1,\dots,v_n\}$ such that $v_i \sim v_{i+1}$ for every $i \in \{1,\dots,n-1\}$ and a path graph G_2 with vertices $\{w_1,\dots,w_n\}$ such that $w_i \sim w_{i+1}$ for every $i \in \{1,\dots,n-1\}$. By Lemma 5.1.8, we have that $\gamma := \cup_{i=1}^{n-1}[(v_i,w_i),(v_{i+1},w_{i+1})]$ is a geodesic joining (v_1,w_1) and (v_n,w_n) in $G_1 \boxtimes G_2$, but $P_1(\gamma) = \cup_{i=1}^{n-1}[v_i,v_{i+1}]$ does not contain the geodesic joining v_1 and v_n in G_1 (the edge $[v_1,v_n]$).*

In this work by trivial graph we mean a graph having just a single vertex, and we denote it by E_1.

The following result allows to compute the diameter of the strong product of two graphs.

Theorem 2.1.8. *Let G_1, G_2 be any graphs. Then we have*

$$\operatorname{diam} G_1 \boxtimes G_2 = \begin{cases} \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}, & \text{if } G_1 \text{ or } G_2 \text{ is an isomorphic graph to } E_1, \\ \max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1, & \text{otherwise.} \end{cases}$$

Proof. Since for any graph G, $E_1 \boxtimes G$ is isomorphic to G we have the first equality. By Lemma 5.1.8, we have $\operatorname{diam} V(G_1 \boxtimes G_2) = \max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\}$; hence,

$$\max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} \le \operatorname{diam} G_1 \boxtimes G_2 \le \max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1.$$

Without loss of generality we can assume that $\operatorname{diam} V(G_1) \le \operatorname{diam} V(G_2)$. If $\operatorname{diam} V(G_2) = \infty$, then the inequality holds. Hence, we can assume that G_1 and G_2 are bounded. Let B_1, B_2 be vertices of G_2 such that $d_{G_2}(B_1, B_2) = \operatorname{diam} V(G_2)$, and let A_1, A_2 be two adjacent vertices of G_1. Let M_1 (respectively, M_2) be the midpoint of $[(A_1, B_1), (A_2, B_1)]$ (respectively, $[(A_1, B_2), (A_2, B_2)]$). One can check that $d_{G_1 \boxtimes G_2}(M_1, M_2) = \operatorname{diam} V(G_2) + 1$.

This finish the proof. □

Note that, in particular, $\operatorname{diam} G_1 \boxtimes G_2 = \operatorname{diam} V(G_1 \boxtimes G_2) + 1$ if G_1 and G_2 are not isomorphic to E_1.

We can deduce several results from Theorem 2.1.8. The first one says that $\max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}$ is a good approximation of the diameter of $G_1 \boxtimes G_2$.

Corollary 2.1.9. *For all graphs G_1, G_2 we have*

$$\max\{\operatorname{diam} G_1, \operatorname{diam} G_2\} \le \operatorname{diam} G_1 \boxtimes G_2 \le \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\} + 1.$$

Proof. If v is a vertex of G_1 (respectively, G_2), then, by Proposition 2.1.3, we have that $\{v\} \boxtimes G_2$ (respectively, $G_1 \boxtimes \{v\}$) is an isometric subgraph of $G_1 \boxtimes G_2$. Hence, we obtain the first inequality. The second one is a consequence of Theorem 2.1.8 and the inequality $\operatorname{diam} V(G) \le \operatorname{diam} G$. □

Furthermore, we characterize the graphs with $\operatorname{diam} G_1 \boxtimes G_2 = \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}$.

Corollary 2.1.10. *The equality $\operatorname{diam} G_1 \boxtimes G_2 = \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}$ holds if and only if G_1 or G_2 is isomorphic to E_1, or $\operatorname{diam} G = \operatorname{diam} V(G) + 1$ for $G \in \{G_1, G_2\}$ with $\operatorname{diam} G = \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}$.*

2.2 Bounds for the hyperbolicity constant

Some bounds for the hyperbolicity constant of the strong product of two graphs are studied in this section. These bounds allow to prove Theorem 2.2.11, which characterizes the hyperbolic strong product graphs.

Thanks to the Lemma 1.3.7 and Theorem 2.1.8 we obtain the following consequence.

Corollary 2.2.1. *For all graphs G_1, G_2, we have*

$$\delta(G_1 \boxtimes G_2) \le \frac{\max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1}{2},$$

and the inequality is sharp.

Theorems 2.3.6, 2.3.8 and 2.3.9 are families of examples for which the equality in the previous corollary is attained.

Taking into account that $E_1 \boxtimes G$ is an isomorphic graph to G, we have the following result.

Corollary 2.2.2. *For every graph G we have*

$$\delta(G \boxtimes E_1) = \delta(E_1 \boxtimes G) = \delta(G).$$

All the previous results allow us to present the following theorem which provides some lower bounds for $\delta(G_1 \boxtimes G_2)$.

Theorem 2.2.3. *For all graphs G_1, G_2 we have:*

(a) $\delta(G_1 \boxtimes G_2) \geq \max\{\delta(G_1), \delta(G_2)\}$,

(b) $\delta(G_1 \boxtimes G_2) \geq \frac{1}{2} \min\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\}$,

(c) $\delta(G_1 \boxtimes G_2) \geq \frac{1}{2} \big(\operatorname{diam} V(G_1) + 1\big)$, *if* $0 < \operatorname{diam} V(G_1) < \operatorname{diam} V(G_2)$,

(d) $\delta(G_1 \boxtimes G_2) \geq \frac{1}{4} \min\{\operatorname{diam} V(G_1) + 2\delta(G_2), \operatorname{diam} V(G_2) + 2\delta(G_1)\}$.

Proof. Part (a) is immediate due to $G_1 \boxtimes \{v\}$ and $\{u\} \boxtimes G_2$ are isometric subgraphs of $G_1 \boxtimes G_2$ for every $(u, v) \in V(G_1 \boxtimes G_2)$. Then Lemma 1.3.3 gives that $\delta(G_1 \boxtimes G_2) \geq \delta(G_1 \boxtimes \{v\}) = \delta(G_1)$ and $\delta(G_1 \boxtimes G_2) \geq \delta(\{u\} \boxtimes G_2) = \delta(G_2)$. Hence, we obtain $\delta(G_1 \boxtimes G_2) \geq \max\{\delta(G_1), \delta(G_2)\}$.

Let $D := \min\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\}$.

Let us prove (b). If $D = 0$, then (b) holds; so, we just consider $D > 0$. If $D < \infty$, let us consider a geodesic square $K := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ in $G_1 \square G_2 \subset G_1 \boxtimes G_2$ with sides of length D; then $T := \{\gamma_1, \gamma_2, \gamma\}$ is a geodesic triangle in $G_1 \boxtimes G_2$, where γ is a diagonal geodesic joining the endpoints of $\gamma_1 \cup \gamma_2$. It is clear that the midpoint p of γ satisfies $d_{G_1 \boxtimes G_2}(p, \gamma_1 \cup \gamma_2) = D/2$; therefore $\delta(T) \geq D/2$ and, consequently, $\delta(G_1 \boxtimes G_2) \geq D/2$. If $D = \infty$, we can repeat the same argument for any integer N instead of D, and we obtain $\delta(G_1 \boxtimes G_2) \geq N/2$, for every N: hence, $\delta(G_1 \boxtimes G_2) = \infty = D/2$.

In order to prove (c), note that $D < \infty$. Let us consider a geodesic rectangle $R := \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ in $G_1 \square G_2 \subset G_1 \boxtimes G_2$ with $L(\sigma_1) = L(\sigma_3) = \operatorname{diam} V(G_1)$ and $L(\sigma_2) = L(\sigma_4) = \operatorname{diam} V(G_1) + 1$. Denote by γ a geodesic in $G_1 \boxtimes G_2$ joining the endpoints of $\sigma_1 \cup \sigma_2$ which contains the edge in σ_4 incident to $\sigma_1 \cap \sigma_4$; we may choose γ such that it contains a diagonal of a geodesic square in $G_1 \boxtimes G_2$. Then $B := \{\sigma_1, \sigma_2, \gamma\}$ is a geodesic triangle in $G_1 \boxtimes G_2$. If p is the midpoint of γ, then

$$d_{G_1 \boxtimes G_2}(p, \sigma_1 \cup \sigma_2) = \frac{\operatorname{diam} V(G_1) + 1}{2}.$$

Consequently, $\delta(G_1 \boxtimes G_2) \geq \delta(B) \geq (\operatorname{diam} V(G_1) + 1)/2$.

Finally, (d). Let $E := \max\{\delta(G_1), \delta(G_2)\}$. Then from parts (a) and (b), we have

$$\begin{aligned}\delta(G_1 \boxtimes G_2) \geq \max\left\{\frac{D}{2}, E\right\} &\geq \frac{1}{2}\left(\frac{D}{2} + E\right)\\ &= \frac{1}{4}\min\{\operatorname{diam} V(G_1) + 2E, \operatorname{diam} V(G_2) + 2E\}\\ &\geq \frac{1}{4}\min\{\operatorname{diam} V(G_1) + 2\delta(G_2), \operatorname{diam} V(G_2) + 2\delta(G_1)\}.\end{aligned}$$

□

Theorems 2.3.8 and 2.3.9 provide a family of examples for which the equality in Theorem 2.2.3 (a) is attained.

Corollary 2.2.1 and Theorem 2.2.3 provide lower and upper bounds for $\delta(G_1 \boxtimes G_2)$ just in terms of distances in G_1 and G_2.

Corollary 2.2.4. *For all graphs G_1, G_2, we have*

$$\frac{1}{2}\min\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} \leq \delta(G_1 \boxtimes G_2) \leq \frac{1}{2}\big(\max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1\big).$$

From Theorem 2.2.3 we have obtained several interesting consequences. The following one is a qualitative result about the hyperbolicity of $G_1 \boxtimes G_2$.

Theorem 2.2.5. *If G_1 and G_2 are infinite graphs, then $G_1 \boxtimes G_2$ is not hyperbolic.*

Theorem 2.2.6. *Let G_1, G_2 be graphs with at least two vertices. Let m and M be the minimum and the maximum between $\operatorname{diam} V(G_1)$ and $\operatorname{diam} V(G_2)$, respectively. Then we have*

$$\delta(G_1 \boxtimes G_2) \geq \min\left\{m + \frac{1}{2}, \frac{M}{2}\right\}. \tag{2.5}$$

Proof. First of all, we prove

$$\delta(G_1 \boxtimes G_2) \geq \min\left\{m, \frac{M}{2}\right\}. \tag{2.6}$$

In order to prove this inequality, assume first that $2m \leq M$. If $m < \infty$, then let us consider a geodesic rectangle $R := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ in $G_1 \square G_2 \subset G_1 \boxtimes G_2$ with $L(\gamma_1) = L(\gamma_3) = 2m$ and $L(\gamma_2) = L(\gamma_4) = m$, and consider a geodesic γ joining the endpoints of γ_1 and containing the midpoint of γ_3, then $B := \{\gamma_1, \gamma\}$ is a geodesic bigon in $G_1 \boxtimes G_2$. If p is the midpoint of γ_3; then $d_{G_1 \boxtimes G_2}(p, \gamma_1) = m$; therefore $\delta(B) \geq m$, and consequently $\delta(G_1 \boxtimes G_2) \geq m$. If $m = \infty$, then we can repeat the same argument for any integer N instead of m, and we obtain $\delta(G_1 \boxtimes G_2) \geq N$, for every N; hence, $\delta(G_1 \boxtimes G_2) = \infty = m$.

If $2m > M$, then $M < \infty$ and we can repeat the previous argument with $\lfloor M/2 \rfloor$ instead of m, and we obtain the result when M is even. If M is odd, let us consider a geodesic rectangle $R := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ in $G_1 \Box G_2 \subset G_1 \boxtimes G_2$ with $L(\gamma_1) = L(\gamma_3) = 2\lfloor M/2 \rfloor + 1 = M$ and $L(\gamma_2) = L(\gamma_4) = \lfloor M/2 \rfloor$; let p_1, p_2 be points on γ_3 such that $d_{G_1 \boxtimes G_2}(p_1, \gamma_4) = \lfloor M/2 \rfloor$ and $d_{G_1 \boxtimes G_2}(p_2, \gamma_2) = \lfloor M/2 \rfloor$; consider a geodesic γ joining the endpoints of γ_1 and containing p_1 and p_2; then $B := \{\gamma_1, \gamma\}$ is a geodesic bigon in $G_1 \boxtimes G_2$. Denote by p the midpoint of $[p_1 p_2] \subset \gamma_3$; so, $d_{G_1 \boxtimes G_2}(p, \gamma_1) = M/2$; therefore, $\delta(G_1 \boxtimes G_2) \geq \delta(B) \geq M/2$.

Since we have proved (2.6), in order to obtain (2.5), we can assume that $0 < 2m < M$; then we have $m < \infty$. If we replace $\lfloor M/2 \rfloor$ by m in the previous argument, we obtain $\delta(G_1 \boxtimes G_2) \geq m + 1/2$. □

Corollary 2.3.7 and Theorems 2.3.8 and 2.3.9 show that the inequality in Theorem 2.2.6 is sharp.

Theorem 2.2.7. *Let G_1, G_2 be any graphs. Let m and M be the minimum and the maximum between* $\operatorname{diam} V(G_1)$ *and* $\operatorname{diam} V(G_2)$*, respectively. If $2m \geq M$, then*

$$\frac{M}{2} \leq \delta(G_1 \boxtimes G_2) \leq \frac{M+1}{2}. \tag{2.7}$$

Furthermore, if $2m > M > 0$, then

$$\delta(G_1 \boxtimes G_2) = \frac{M+1}{2}. \tag{2.8}$$

Proof. If $M = 0$, then $\delta(G_1 \boxtimes G_2) = 0$ and (2.7) holds. If $M > 0$, then, by Corollary 2.2.1 and Theorem 2.2.6, the inequalities in (2.7) hold directly.

In order to prove (2.8), without loss of generality we can assume that $\operatorname{diam} V(G_1) = m$ and $\operatorname{diam} V(G_2) = M$. Assume first that M is an even number. Since $m > M/2$, let us consider $A_0, A_1, \dots, A_{M/2+1} \in V(G_1)$ and $B_0, B_1, \dots, B_M \in V(G_2)$ with $\gamma_1 := A_0 A_1 \dots A_{M/2+1}$ is a geodesic in G_1 and $\gamma_2 := B_0 B_1 \dots B_M$ is a geodesic in G_2. Denote by X (respectively, Y) the midpoint of $[(A_0, B_0), (A_1, B_0)]$ (respectively, $[(A_0, B_M), (A_1, B_M)]$). Let us consider

$$\Gamma^* := [X(A_0, B_0)] \bigcup \left\{ \bigcup_{i=1}^{M} [(A_0, B_{i-1}), (A_0, B_i)] \right\} \bigcup [(A_0, B_M) Y]$$

and

$$\begin{aligned} \Gamma' := & [X(A_1, B_0)] \bigcup \left\{ \bigcup_{i=1}^{M/2} [(A_i, B_{i-1}), (A_{i+1}, B_i)] \right\} \bigcup \\ & \bigcup \left\{ \bigcup_{j=M/2+1}^{M} [(A_{M+2-j}, B_{j-1}), (A_{M+1-j}, B_j)] \right\} \bigcup [(A_1, B_M) Y]. \end{aligned}$$

Then $B := \{\Gamma^*, \Gamma'\}$ is a geodesic bigon in $G_1 \boxtimes G_2$. If p is the midpoint of Γ', then $d_{G_1 \boxtimes G_2}(p, \Gamma^*) = (M+1)/2$; therefore, $\delta(G_1 \boxtimes G_2) \geq \delta(B) \geq (M+1)/2$. Then, Corollary 2.2.1 gives the equality.

Assume now that M is an odd number. Since $m \geq (M+1)/2$, let us consider $A_0, A_1, \dots, A_{(M+1)/2} \in V(G_1)$ and $B_0, B_1, \dots, B_M \in V(G_2)$ with $\gamma_1 := A_0 A_1 \dots A_{(M+1)/2}$ is a geodesic in G_1 and $\gamma_2 := B_0 B_1 \dots B_M$ is a geodesic in G_2. Denote by X (respectively, Y) the midpoint of $[(A_0, B_0), (A_1, B_0)]$ (respectively, $[(A_0, B_M), (A_1, B_M)]$). Let us consider

$$\Gamma^* := [X(A_0, B_0)] \bigcup \left\{ \bigcup_{i=1}^{M} [(A_0, B_{i-1}), (A_0, B_i)] \right\} \bigcup [(A_0, B_M)Y]$$

and

$$\begin{aligned} \Gamma' := & [X(A_1, B_0)] \bigcup \left\{ \bigcup_{i=1}^{(M-1)/2} [(A_i, B_{i-1}), (A_{i+1}, B_i)] \right\} \bigcup \\ & \bigcup [(A_{(M+1)/2}, B_{(M-1)/2}), (A_{(M+1)/2}, B_{(M+1)/2})] \bigcup \\ & \bigcup \left\{ \bigcup_{j=(M+1)/2}^{M} [(A_{M+1-j}, B_{j-1}), (A_{M-j}, B_j)] \right\} \bigcup [(A_1, B_M)Y]. \end{aligned}$$

Then $B := \{\Gamma^*, \Gamma'\}$ is a geodesic bigon in $G_1 \boxtimes G_2$. If p is the midpoint of Γ', then $d_{G_1 \boxtimes G_2}(p, \Gamma^*) = (M+1)/2$; therefore, $\delta(G_1 \boxtimes G_2) \geq \delta(B) \geq (M+1)/2$. Finally, Corollary 2.2.1 gives the equality. $\square$

Theorems 2.3.8 and 2.3.9 show that the first inequality in Theorem 2.2.7 is attained.

Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call *ε-neighborhood* of Y in X, denoted by $V_\varepsilon(Y)$ to the set $\{x \in X : d_X(x, Y) \leq \varepsilon\}$.

The next result will be useful in order to prove the upper bound for $\delta(G_1 \boxtimes G_2)$ in Theorem 2.2.9 below.

Theorem 2.2.8 (Theorem 2.9 in [99]). *Let X be a δ-hyperbolic geodesic metric space, $u, v \in X$, b a non-negative constant, h a curve joining u and v with $L(h) \leq d(u,v) + b$, and $g = [uv]$. Then,*

$$h \subseteq V_{8\delta + b/2}(g), \qquad g \subseteq V_{16\delta + b}(h).$$

Theorem 2.2.9. *Let G_1, G_2 be any graphs. Then, we have*

$$\delta(G_1 \boxtimes G_2) \leq \frac{5}{2} \operatorname{diam} G_1 + 25\delta(G_2) + 5. \tag{2.9}$$

Proof. It suffices to prove (2.9) if G_1 is bounded and G_2 is hyperbolic, since otherwise the inequality $\delta(G_1 \boxtimes G_2) \le \infty$ holds. Let us consider any fixed geodesic triangle $T = \{x, y, z\}$ in $G_1 \boxtimes G_2$ and $\alpha \in T$. In order to bound $\delta(T)$, without loss of generality we can assume that $\alpha \in [xy]$. Consider the projection $P_2 : G_1 \boxtimes G_2 \longrightarrow G_2$ and any geodesic $\gamma := [uv]$ in $G_1 \boxtimes G_2$. By Corollary 2.1.6, we obtain

$$L\big(P_2(\gamma)\big) \le L(\gamma) = d_{G_1 \boxtimes G_2}(u, v) \le d_{G_2}\big(P_2(v), P_2(v)\big) + b, \qquad \text{with } b = \operatorname{diam} G_1 + 2.$$

Then, by Theorem 2.2.8, there is $\alpha' \in [P_2(x)P_2(y)]$ such that

$$d_{G_2}(P_2(\alpha), \alpha') \le 8\delta(G_2) + \frac{b}{2}. \tag{2.10}$$

Since G_2 is hyperbolic, there is $\beta' \in [P_2(y)P_2(z)] \cup [P_2(z)P_2(x)]$ such that

$$d_{G_2}(\alpha', \beta') \le \delta(G_2). \tag{2.11}$$

By Theorem 2.2.8, there is $\beta'' \in P_2([yz] \cup [zx])$ such that

$$d_{G_2}(\beta', \beta'') \le 16\delta(G_2) + b. \tag{2.12}$$

Consequently, by (2.10), (2.11) and (2.12) we obtain

$$d_{G_2}(P_2(\alpha), P_2([yz] \cup [zx])) \le d_{G_2}(P_2(\alpha), \beta'') \le 25\delta(G_2) + \frac{3b}{2}. \tag{2.13}$$

Finally, by Corollary 2.1.6 and (2.13) we obtain

$$d_{G_1 \boxtimes G_2}(\alpha, [yz] \cup [zx]) \le d_{G_2}(P_2(\alpha), P_2([yz] \cup [zx])) + b \le 25\delta(G_2) + \frac{5b}{2}.$$

This finishes the proof. □

Theorems 2.2.3 and 2.2.9 provide lower and upper bounds of $\delta(G_1 \boxtimes G_2)$ in terms of linear combinations of hyperbolicity constants and diameters of its generator graphs, as the following result shows.

Corollary 2.2.10. *For all graphs G_1, G_2, we have*

$$\frac{1}{4} \min\{2\delta(G_1) + \operatorname{diam} V(G_2), 2\delta(G_2) + \operatorname{diam} V(G_1)\} \le \delta(G_1 \boxtimes G_2)$$

$$\le \frac{5}{2} \min\left\{\operatorname{diam} G_1 + 10\delta(G_2), \operatorname{diam} G_2 + 10\delta(G_1)\right\} + 5.$$

Corollary 2.2.10 allows to obtain the main result of this work: the characterization of the hyperbolic graphs $G_1 \boxtimes G_2$.

Theorem 2.2.11. *For all graphs G_1, G_2 we have that $G_1 \boxtimes G_2$ is hyperbolic if and only if G_1 is hyperbolic and G_2 is bounded or G_2 is hyperbolic and G_1 is bounded.*

Many parameters γ of graphs satisfy the inequality $\gamma(G_1 \boxtimes G_2) \geq \gamma(G_1) + \gamma(G_2)$. Therefore, one could think that the inequality $\delta(G_1 \boxtimes G_2) \geq \delta(G_1) + \delta(G_2)$ holds for all graphs G_1, G_2. However, this is false, as the following example shows:

Example 2.2.12. *$\delta(P \boxtimes C_4) < \delta(P) + \delta(C_4)$, where P is the Petersen graph.*

We have that $\operatorname{diam} V(P) = 2$, $\operatorname{diam} V(C_4) = 2$. Besides, Theorem 1.3.10 gives that $\delta(P) = 3/2$ and $\delta(C_4) = 1$. By Theorem 2.2.7, we obtain $\delta(P \boxtimes C_4) = 3/2 < 3/2 + 1 = \delta(P) + \delta(C_4)$.

The inequality $\delta(G_1 \boxtimes G_2) \leq \delta(G_1) + \delta(G_2)$ is also false, since $\delta(P_2 \boxtimes P_2) = \delta(K_4) = 1 > 2\delta(P_2) = 0$.

2.3 Computation of the hyperbolicity constant for some product graphs

This last section present the value of the hyperbolicity constant for many product of graphs.

Remark 2.3.1. *By Theorems 1.3.12 and 1.3.13, in order to compute the hyperbolicity constant of a graph G it suffices to consider $d_G(p, [xz] \cup [yz])$ where $T = \{x, y, z\}$ is a geodesic triangle that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ satisfies $d_G(p, V(G)) \in \{0, 1/4, 1/2\}$.*

The following results characterize the hyperbolicity constant of the strong product of trees and certain graphs. These results are interesting by themselves and, furthermore, they will be useful in order to prove the last theorems of this Chapter.

Theorem 2.3.2. *Let T be any tree and G any graph with $0 < \operatorname{diam} V(G) < \operatorname{diam} T/2$. Then, we have*

$$\delta(G \boxtimes T) = \operatorname{diam} V(G) + \frac{1}{2}.$$

Proof. On the one hand, Theorem 2.2.6 gives $\delta(G \boxtimes T) \geq \operatorname{diam} V(G) + 1/2$. On the other hand, by Theorem 1.3.13 it suffices to consider geodesic triangles $\triangle = \{x, y, z\}$ in $G \boxtimes T$ which are cycles with $x, y, z \in J(G \boxtimes T)$. Let (v, w) be a vertex in $[xy]$. If $d_{G\boxtimes T}((v, w), \{x, y\}) \leq \operatorname{diam} V(G)$, then $d_{G\boxtimes T}((v, w), [yz] \cup [zx]) \leq \operatorname{diam} V(G)$. Assume that $d_{G\boxtimes T}((v, w), \{x, y\}) > \operatorname{diam} V(G)$. Let V_x (respectively, V_y) be the closest vertex to x (respectively, y) in $[xy]$. Note that $d_{G\boxtimes T}(V_x, V_y) = d_{G\boxtimes T}(V_x, (v, w)) + d_{G\boxtimes T}((v, w), V_y) \geq 2 \operatorname{diam} V(G)$. Consider the projection P_T on T. By Lemma 5.1.8 we have $d_{G\boxtimes T}(V_x, V_y) = d_T(P_T(V_x), P_T(V_y))$. Due to $d_T(P_T(V_x), P_T(V_y)) \leq d_T(P_T(V_x), w) + d_T(w, P_T(V_y))$, we have $d_{G\boxtimes T}(V_x, (v, w)) = d_T(P_T(V_x), w)$ and $d_{G\boxtimes T}((v, w), V_y) = d_T(w, P_T(V_y))$. Then, $w \in [P_T(x)P_T(y)] = P_T([xy])$. Since T is a tree, $w \in P_T\big([yz] \cup [zx]\big)$. Then, $([yz] \cup [zx]) \cap (G \boxtimes \{w\}) \neq \emptyset$ and $d_{G\boxtimes T}((v, w), [yz] \cup$

$[zx]) \le \operatorname{diam} V(G)$. So, we have $d_{G\boxtimes T}((v,w),[yz]\cup[zx]) \le \operatorname{diam} V(G)$ for every vertex (v,w) in $[xy]$. Since $x,y \in J(G\boxtimes T)$, $d_{G\boxtimes T}(p,[yz]\cup[zx]) \le \operatorname{diam} V(G)+1/2$ for every $p\in[xy]$. Hence, $\delta(\triangle) \le \operatorname{diam} V(G)+1/2$, and we obtain $\delta(G\boxtimes T)\le \operatorname{diam} V(G)+1/2$. □

Theorem 2.3.3. *Let T be any tree and G any graph with $0 < \operatorname{diam} V(G) = \operatorname{diam} T/2$. Then, we have*

$$\delta(G\boxtimes T) = \operatorname{diam} V(G) + \frac{1}{4}.$$

Proof. By Theorem 2.2.7, we have that $\operatorname{diam} V(G) \le \delta(G\boxtimes T) \le \operatorname{diam} V(G)+1/2$.

Now we show a geodesic bigon B in $G\boxtimes T$ with $\delta(B) = \operatorname{diam} V(G)+1/4$. Define by $n := \operatorname{diam} V(G)$ and consider $v_1,\dots,v_{n+1}\in V(G)$ with $v_i \sim v_{i+1}$ for $i=1,\dots,n$ and $d_G(v_1,v_{n+1}) = n$. Also, consider $w_1,\dots,w_{2n+1}\in V(T)$ with $w_i\sim w_{i+1}$ for $i = 1,\dots,2n$ and $d_T(w_1,w_{2n+1}) = \operatorname{diam} T = 2n$. Denote by a (respectively, b) the midpoint of $[(v_1,w_1),(v_2,w_1)]$ (respectively, $[(v_1,w_{2n+1}),(v_2,w_{2n+1})]$). Let us consider

$$\gamma^* := [a(v_1,w_1)]\bigcup\left\{\bigcup_{i=1}^{2n}[(v_1,w_i),(v_1,w_{i+1})]\right\}\bigcup[(v_1,w_{2n+1})b]$$

and

$$\begin{aligned}\gamma' :=&[a(v_2,w_1)]\bigcup\left\{\bigcup_{i=1}^{n-1}[(v_{i+1},w_i),(v_{i+2},w_{i+1})]\right\}\bigcup[(v_{n+1},w_n),(v_{n+1},w_{n+1})]\bigcup\\ &\bigcup[(v_{n+1},w_{n+1}),(v_{n+1},w_{n+2})]\bigcup\left\{\bigcup_{j=1}^{n-1}[(v_{n+2-j},w_{n+1+j}),(v_{n+1-j},w_{n+2+j})]\right\}\bigcup\\ &\bigcup[(v_2,w_{2n+1})b].\end{aligned}$$

Consider the geodesic bigon $B := \{\gamma^*,\gamma'\}$ in $G\boxtimes T$. Let p be the midpoint of γ' and let p_0 be a point in γ' with $d_{G\boxtimes T}(p_0,p) = 1/4$; then $d_{G\boxtimes T}(p_0,\gamma^*) = n+1/4$ and $\delta(G\boxtimes T)\ge\delta(B)\ge n+1/4$.

Hence, by Theorem 1.3.12 we have $\delta(G\boxtimes T)\in\{n+1/4\ ,\ n+1/2\}$. Seeking for a contradiction assume that $\delta(G\boxtimes T) = n+1/2$. Then there are a geodesic triangle $\triangle = \{x,y,z\}$ in $G\boxtimes T$ and $p\in[xy]$ with $d_{G\boxtimes T}(p,[yz]\cup[zx]) = n+1/2$. By Theorem 1.3.13 we can assume that $\triangle$ is a cycle with $x,y,z\in J(G\boxtimes T)$. By Theorem 2.1.8, $\operatorname{diam}(G\boxtimes T) = 2n+1$ and we conclude that $L([xy]) = 2n+1$ and p is the midpoint of $[xy]$. Since $\operatorname{diam} V(G\boxtimes T) = 2n$, we have that x,y are midpoints of edges in $G\boxtimes T$, and so, p is a vertex of $G\boxtimes T$. We can write $[xy]\cap V(G\boxtimes T) = \{(a_1,b_1),(a_2,b_2),\dots,(a_{2n+1},b_{2n+1})\}$ with $a_1,\dots,a_{2n+1}\in V(G)$, $(a_i,b_i)\sim(a_{i+1},b_{i+1})$ for $i=1,\dots,2n$ and $d_T(b_1,b_{2n+1}) = 2n$. Thus, $p = (a_{n+1},b_{n+1})$ and $p\in V(G\boxtimes\{b_{n+1}\})$. Since T is a tree we have that $([yz]\cup[zx])\cap(G\boxtimes\{b_{n+1}\})\ne\emptyset$; in particular, $d_{G\boxtimes T}(p,[yz]\cup[zx])\le\operatorname{diam} V(G)$. This is the contradiction we were looking for, and then $\delta(G\boxtimes T) = \operatorname{diam} V(G)+1/4$. □

The following lemma will be useful.

Lemma 2.3.4. *Let C_m be a cycle graph and G any graph with* $\operatorname{diam} V(G) < \operatorname{diam} V(C_m)$. *Let $\gamma = [xy]$ be a geodesic in $G \boxtimes C_m$ such that $x, y \in J(G \boxtimes C_m)$. Then, $L(P_{C_m}(\gamma)) \le m/2$ where P_{C_m} is the projection on C_m.*

Proof. If $\operatorname{diam} V(G) = 0$, then the result is direct. Assume now that $\operatorname{diam} V(G) > 0$.

If $L(\gamma) \le m/2$, then we have the result since $L(P_{C_m}(\gamma)) \le L(\gamma)$. Assume that $L(\gamma) > m/2$. Seeking for a contradiction, assume that $L(P_{C_m}(\gamma)) > m/2$.

Assume that m is even (the case m odd is similar). Since $x, y \in J(G \boxtimes C_m)$ and $L(P_{C_m}(\gamma)) > m/2$, there are $x', y' \in \gamma \cap J(G \boxtimes C_m)$ such that $d_{C_m}(P_{C_m}(x'), P_{C_m}(y')) = (m+1)/2$. Without loss of generality we can assume that $x' \in V(G \boxtimes C_m)$ and $y' \notin V(G \boxtimes C_m)$. Let $A, A_1, A_2 \in V(G)$ and $B, B_1, B_2 \in V(C_m)$ such that $x' = (A, B)$ and $y' \in [(A_1, B_1), (A_2, B_2)]$. Since $d_{C_m}(P_{C_m}(x'), P_{C_m}(y')) = (m+1)/2$, without loss of generality we can assume that $d_{C_m}(B, B_1) + 1 = d_{C_m}(B, B_2) = m/2$. Since $\operatorname{diam} V(C_m) > \operatorname{diam} V(G)$, by Lemma 5.1.8 we have $d_{G\boxtimes C_m}((A, B), (A_1, B_1)) = m/2-1$; thus, $d_{G\boxtimes C_m}(x', y') \le (m-1)/2$. This is the contradiction we were looking for. □

The following theorem provides the exact value of the hyperbolicity constant of the strong product of a cycle C_m and any graph G with $\operatorname{diam} V(G) \le \operatorname{diam} V(C_m)/2$. This result is interesting by itself and, furthermore, it will be useful in order to prove the last theorems of this Chapter.

Theorem 2.3.5. *Let C_m be a cycle graph and G any graph with* $\operatorname{diam} V(G) \le \operatorname{diam} V(C_m)/2$. *Then, we have*

$$\delta(G \boxtimes C_m) = \begin{cases} \lfloor m/2 \rfloor /2 + 1/4, & \text{if } \operatorname{diam} V(G) = \operatorname{diam} V(C_m)/2, \\ m/4, & \text{if } \operatorname{diam} V(G) < \operatorname{diam} V(C_m)/2. \end{cases} \tag{2.14}$$

Proof. If $\operatorname{diam} V(G) = 0$, then the equality is trivial. Assume now that $\operatorname{diam} V(G) > 0$. Let $V(C_m) = \{w_1, \dots, w_m\}$ where $w_i \sim w_{i+1}$ for $i = 1, \dots, m-1$. Let P_{C_m} be the projection on C_m.

First, we prove that $\delta(G \boxtimes C_m) < (\lfloor m/2 \rfloor + 1)/2$. Seeking for a contradiction, assume that there are a geodesic triangle $T = \{x, y, z\}$ in $G \boxtimes C_m$ and a point $p \in \gamma := [xy]$ with $d_{G\boxtimes C_m}(p, [yz] \cup [zx]) = (\lfloor m/2 \rfloor + 1)/2 = \operatorname{diam}(G \boxtimes C_m)/2$. Then $L(\gamma) = \operatorname{diam}(G \boxtimes C_m)$ and $d_{G\boxtimes C_m}(p, [yz] \cup [zx]) = \operatorname{diam}(G \boxtimes C_m)/2$, and we conclude that p is the midpoint of γ. By Theorem 1.3.13, we can assume that T is a cycle with $x, y, z \in J(G \boxtimes C_m)$. Since $\operatorname{diam} V(G \boxtimes C_m) = \operatorname{diam}(G \boxtimes C_m) - 1$, by Theorem 2.1.8 we have that x, y are midpoints of edges in $G\boxtimes C_m$. Let V_x (respectively, V_y) be the closest vertex to x (respectively, y) in γ. Let V_x' (respectively, V_y') be the closest vertex to x (respectively, y) in $[xz]$ (respectively, $[yz]$). By Lemma 5.1.8, we have $d_{G\boxtimes C_m}(V_x, V_y) = d_{C_m}(P_{C_m}(V_x), P_{C_m}(V_y)) = \lfloor m/2 \rfloor$. Therefore, since $\operatorname{diam} V(G) \le \operatorname{diam} V(C_m)/2$ we have $d_{C_m}(P_{C_m}(V_x), P_{C_m}(p)) = d_{C_m}(P_{C_m}(p), P_{C_m}(V_y)) =$

$\lfloor m/2 \rfloor/2$. By Lemma 2.3.4 we have $L(P_{C_m}(\gamma)) \le m/2$, since $2\big(\lfloor m/2 \rfloor/2 + 1/2\big) > m/2$ we have either $P_{C_m}(V_x) = P_{C_m}(x) = P_{C_m}(V'_x)$ or $P_{C_m}(V_y) = P_{C_m}(y) = P_{C_m}(V'_y)$. So, we have

$$d_{G\boxtimes C_m}(p, [xz] \cup [yz]) \le d_{G\boxtimes C_m}(p, \{V'_x, V'_y\}) \le \lfloor m/2 \rfloor/2 \le m/4.$$

This is the contradiction we were looking for, and we have $\delta(G \boxtimes C_m) < (\lfloor m/2 \rfloor + 1)/2$. So, by Theorem 1.3.12 we have $\delta(G \boxtimes C_m) \le \lfloor m/2 \rfloor/2 + 1/4$.

Assume now that $\lfloor m/2 \rfloor = 2 \operatorname{diam} V(G)$. If m is odd (i.e., $m = 4k+1$), then Theorem 2.2.3 (a) gives $\delta(G \boxtimes C_m) \ge m/4 = \lfloor m/2 \rfloor/2 + 1/4$. So, (2.14) holds. Assume that m in even (i.e., $m = 4k$). Now we show a geodesic bigon B in $G \boxtimes C_m$ with $\delta(B) = \lfloor m/2 \rfloor/2 + 1/4 = k + 1/4$. Note that $k = \operatorname{diam} V(G)$ and consider $v_1, \dots, v_{k+1} \in V(G)$ with $v_i \sim v_{i+1}$ for $i = 1, \dots, k$ and $d_G(v_1, v_{k+1}) = k$. Denote by a (respectively, b) the midpoint of $[(v_1, w_1), (v_2, w_1)]$ (respectively, $[(v_1, w_{2k+1}), (v_2, w_{2k+1})]$). Let us consider

$$\gamma^* := [a(v_1, w_1)] \bigcup \left\{ \bigcup_{i=1}^{2k} [(v_1, w_i), (v_1, w_{i+1})] \right\} \bigcup [(v_1, w_{2k+1}) b]$$

and

$$\begin{aligned} \gamma' :=& [a(v_2, w_1)] \bigcup \left\{ \bigcup_{i=1}^{k-1} [(v_{i+1}, w_i), (v_{i+2}, w_{i+1})] \right\} \bigcup [(v_{k+1}, w_k), (v_{k+1}, w_{k+1})] \bigcup \\ & \bigcup [(v_{k+1}, w_{k+1}), (v_{k+1}, w_{k+2})] \bigcup \left\{ \bigcup_{j=1}^{k-1} [(v_{k+2-j}, w_{k+1+j}), (v_{k+1-j}, w_{k+2+j})] \right\} \bigcup \\ & \bigcup [(v_2, w_{2k+1}) b]. \end{aligned}$$

Then $B := \{\gamma^*, \gamma'\}$ is a geodesic bigon in $G \boxtimes C_m$ with $\delta(B) = k + 1/4 = \lfloor m/2 \rfloor/2 + 1/4$.

Finally, assume that $\lfloor m/2 \rfloor > 2 \operatorname{diam} V(G)$. By Theorem 2.2.3 (a) it suffices to prove $\delta(G \boxtimes C_m) \le m/4$. If m is odd, then $\lfloor m/2 \rfloor/2 + 1/4 = m/4$ and (2.14) holds.

Assume that m is even, then $\operatorname{diam} V(G) \le m/4 - 1/2$. Fix any geodesic triangle $T = \{x, y, z\}$ in $G \boxtimes C_m$ and $p \in [xy]$. By Remark 2.3.1, we can assume that T is a cycle, $x, y, z \in J(G \boxtimes C_m)$ and p satisfies $d_G(p, V(G)) \in \{0, 1/4, 1/2\}$. If $d_{G\boxtimes C_m}(p, \{x, y\}) \le m/4$, then $d_{G\boxtimes C_m}(p, [yz] \cup [zx]) \le m/4$. Assume that $d_{G\boxtimes C_m}(p, \{x, y\}) > m/4$; since $x, y \in J(G \boxtimes C_m)$ and $d_G(p, V(G)) \in \{0, 1/4, 1/2\}$, we have $d_{G\boxtimes C_m}(p, \{x, y\}) \ge m/4 + 1/4$. We have $L([xy]) > m/2$. Let V_x (respectively, V_y) be the closest vertex to x (respectively, y) in $[xy]$; then $d_{G\boxtimes C_m}(p, \{V_x, V_y\}) \ge m/4 - 1/4$. Let V'_x (respectively, V'_y) be the closest vertex to x (respectively, y) in $[xz]$ (respectively, $[yz]$). Since m is even and $x, y \in J(G \boxtimes C_m)$ we have $d_{G\boxtimes C_m}(V_x, V_y) \ge m/2$ and we conclude $d_{G\boxtimes C_m}(V_x, V_y) = m/2$. By Lemma 5.1.8 we have $d_{G\boxtimes C_m}(V_x, V_y) = d_{C_m}(P_{C_m}(V_x), P_{C_m}(V_y)) = m/2$; by Lemma 2.3.4 we conclude $L\big(P_{C_m}([xy])\big) = m/2$. Since $m/2 = \lfloor m/2 \rfloor > \operatorname{diam} V(G)$, we have $P_{C_m}(V_x) = P_{C_m}(x) =$

$P_{C_m}(V_x')$ and $P_{C_m}(V_y) = P_{C_m}(y) = P_{C_m}(V_y')$. Since $d_{G\boxtimes C_m}(p,\{V_x,V_y\}) \le d_{G\boxtimes C_m}(V_x,V_y)/2 = m/4$, without loss of generality we can assume that $d_{G\boxtimes C_m}(p,\{V_x,V_y\}) = d_{G\boxtimes C_m}(p,V_x) \le m/4$. Let V_p be the closest vertex to p in $[xp]$. Since $d_{G\boxtimes C_m}(p,V_x) \ge m/4-1/4 > m/4-1/2 \ge \operatorname{diam} V(G)$, we have $\operatorname{diam} V(G) \ge d_{G\boxtimes C_m}(V_p,V_x) = d_{C_m}(P_{C_m}(V_p),P_{C_m}(V_x)) = d_{C_m}(P_{C_m}(V_p),P_{C_m}(V_x'))$ and we conclude $d_{G\boxtimes C_m}(V_p,V_x) = d_{G\boxtimes C_m}(V_p,V_x')$ and $d_{G\boxtimes C_m}(p,[xz]\cup[yz]) \le d_{G\boxtimes C_m}(p,V_x') \le d_{G\boxtimes C_m}(p,V_x) \le m/4$. Then $\delta(G\boxtimes C_m) \le m/4$. □

As a consequence of Theorems 2.2.7, 2.3.2, 2.3.3 and 2.3.5 we obtain the precise values of the hyperbolicity constants of the following families of graphs.

Theorem 2.3.6. *Let T_1, T_2 be two trees with $\operatorname{diam} T_1 \le \operatorname{diam} T_2$. Then*

$$\delta(T_1\boxtimes T_2) = \begin{cases} 0, & \text{if } \operatorname{diam} T_1 = 0,\\ \operatorname{diam} T_1 + 1/2, & \text{if } 0<\operatorname{diam} T_1 < (\operatorname{diam} T_2)/2,\\ \operatorname{diam} T_1 + 1/4, & \text{if } 0<\operatorname{diam} T_1 = (\operatorname{diam} T_2)/2,\\ (\operatorname{diam} T_2 + 1)/2, & \text{if } \operatorname{diam} T_1 > (\operatorname{diam} T_2)/2.\end{cases}$$

Corollary 2.3.7. *Let P_n, P_m be two path graphs with $2 \le n \le m$. Then*

$$\delta(P_n\boxtimes P_m) = \begin{cases} m/2, & \text{if } m-1 < 2(n-1),\\ n-3/4, & \text{if } m-1 = 2(n-1),\\ n-1/2, & \text{if } m-1 > 2(n-1).\end{cases}$$

Theorem 2.3.8. *Let C_n, C_m be two cycle graphs with $3 \le n \le m$. Then*

$$\delta(C_n\boxtimes C_m) = \begin{cases} \lfloor m/2\rfloor/2 + 1/2, & \text{if } \lfloor m/2\rfloor < 2\lfloor n/2\rfloor,\\ \lfloor m/2\rfloor/2 + 1/4, & \text{if } \lfloor m/2\rfloor = 2\lfloor n/2\rfloor,\\ m/4, & \text{if } \lfloor m/2\rfloor > 2\lfloor n/2\rfloor.\end{cases}$$

Theorem 2.3.9. *For every $m \ge 2, n \ge 3$,*

$$\delta(C_n\boxtimes P_m) = \begin{cases} \lfloor n/2\rfloor + 1/2, & \text{if } \lfloor n/2\rfloor < (m-1)/2,\\ \lfloor n/2\rfloor + 1/4, & \text{if } \lfloor n/2\rfloor = (m-1)/2,\\ m/2, & \text{if } (m-1)/2 < \lfloor n/2\rfloor \le (m-1),\\ \big(\lfloor n/2\rfloor + 1\big)/2, & \text{if } m-1 < \lfloor n/2\rfloor < 2(m-1),\\ \lfloor n/2\rfloor/2 + 1/4, & \text{if } \lfloor n/2\rfloor = 2(m-1),\\ n/4, & \text{if } \lfloor n/2\rfloor > 2(m-1).\end{cases}$$

Chapter 3

Gromov hyperbolicity in lexicographic product graphs

The lexicographic product of graphs has been extensively investigated in relation to a wide range of subjects (see, *e.g.*, [76, 98, 107, 120, 121] and the references therein).

3.1 Distances in lexicographic products

In order to estimate the hyperbolicity constant of the lexicographic product of two graphs G_1 and G_2, we must obtain bounds on the distances between any two arbitrary points in $G_1 \circ G_2$. Besides, we study the geodesics in $G_1 \circ G_2$, relating them with the geodesics in G_1. The lemmas of this section provide these results.

We will use the lexicographic product definition given in [64].

Definition 3.1.1. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The lexicographic product $G_1 \circ G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \circ G_2$ are adjacent if either $[u_1, u_2] \in E(G_1)$, or $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$.*

Note that the lexicographic product of two graphs is not always commutative (see Figure 3.1). We use the notation (x, y) for the points of the graph $G_1 \circ G_2$ with $x \in V(G_1)$ or $y \in V(G_2)$. Otherwise, this notation can be ambiguous. We consider that every edge of $G_1 \circ G_2$ has length 1.

Remark 3.1.2. *The Cartesian and the strong product of two graphs are subgraphs of the lexicographic product of two graphs,* i.e., $G_1 \Box G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \circ G_2$.

Remark 3.1.3. *Let G be any graph. Then $G \circ E_1 \simeq G$ and $E_1 \circ G \simeq G$.*

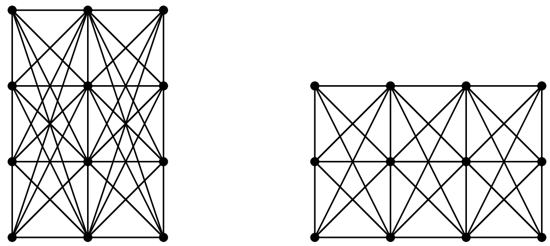

Figure 3.1: Non commutative lexicographic product of two graphs ($P_3 \circ P_4 \not\simeq P_4 \circ P_3$).

In what follows we denote by π the projection $\pi : G_1 \circ G_2 \to G_1$. The following result allows to compute the distance between any two vertices of $G_1 \circ G_2$.

Lemma 3.1.4. *Let G_1 be a non-trivial graph and G_2 any graph and (u, v), (u', v') two vertices in $G_1 \circ G_2$. Then*

$$d_{G_1\circ G_2}((u,v),(u',v')) = \begin{cases} \min\{2, d_{G_2}(v,v')\}, & \text{if } u = u', \\ d_{G_1}(u,u'), & \text{if } u \neq u'. \end{cases}$$

Proof. Assume first that $u = u'$, thus $(u, v), (u, v') \in V\big(\{u\} \circ G_2\big)$. If $d_{G_2}(v, v') \leq 2$ then $d_{G_1\circ G_2}\big((u, v), (u, v')\big) = d_{G_2}(v, v')$ since a path in $G_1 \circ G_2$ joining (u, v) and (u, v') which is not contained in $\{u\} \circ G_2$ has a vertex out of $\{u\} \circ G_2$, and so, its length is at least 2. If $d_{G_2}(v, v') > 2$ then

$$d_{G_1\circ G_2}((u,v),(u,v')) = d_{G_1\circ G_2}((u,v),\{w\}\circ G_2) + d_{G_1\circ G_2}(\{w\}\circ G_2,(u,v')) = 2,$$

where $[u, w] \in E(G_1)$.

Assume now that $u \neq u'$. If $\gamma := [uu']$ is a geodesic in G_1 joining the points u and u' with $L(\gamma) = k$, then there exist vertices $A_1, \ldots, A_{k-1}$ in $\gamma \setminus \{u, u'\}$. Without loss of generality we can assume that γ meets $A_1, \ldots, A_{k-1}$ in this order. If we fix $v_0 \in V(G_2)$, then

$$d_{G_1\circ G_2}((u,v),(u',v')) \leq d_{G_1\circ G_2}((u,v),(A_1,v_0)) + \ldots + d_{G_1\circ G_2}((A_{k-1},v_0),(u',v')) = k.$$

If $d_{G_1\circ G_2}((u, v), (u', v')) < k$, then there exists a geodesic Γ in $G_1 \circ G_2$ joining (u, v) and (u', v') with $L(\Gamma) = r < k$. Denote by $B_1, \ldots, B_{r-1}$ the vertices in $\Gamma \setminus \{(u, v), (u', v')\}$. Without loss of generality we can assume that Γ meets $B_1, \ldots, B_{r-1}$ in this order. Then we have

$$\Gamma := [(u,v),B_1] \bigcup \left\{ \bigcup_{j=1}^{r-2} [B_j, B_{j+1}] \right\} \bigcup [B_{r-1},(u',v')].$$

By Definition 3.1.1,

$$\gamma_1 := [u,\pi(B_1)] \bigcup \left\{ \bigcup_{j=1}^{r-2} [\pi(B_j),\pi(B_{j+1})] \right\} \bigcup [\pi(B_{r-1}),u']$$

is a path joining u and u' in G_1 such that $L(\gamma_1) \le L(\Gamma) < L(\gamma)$. This is a contradiction, thus

$$d_{G_1 \circ G_2}((u,v),(u',v')) = d_{G_1}(u,u').$$

□

Lemma 3.1.5. *Let G_1 be a non-trivial graph and G_2 any graph. Then $G_1 \circ G_2 \subseteq \mathcal{V}_{3/2}(G_1 \circ \{v\})$ for every $v \in V(G_2)$.*

Proof. Let p be any point of $G_1 \circ G_2$. If $p \in V(G_1 \circ G_2)$, then consider any $u_0 \in V(G_1)$ such that $[\pi(p), u_0] \in E(G_1)$. Definition 3.1.1 gives $d_{G_1 \circ G_2}(p, G_1 \circ \{v\}) \le d_{G_1 \circ G_2}(p, (u_0, v)) = 1$ for every $v \in V(G_2)$ since G_1 is non-trivial. Assume that $p \notin V(G_1 \circ G_2)$. Let $A \in V(G_1 \circ G_2)$ with $d_{G_1 \circ G_2}(p, A) \le 1/2$. Hence, we have

$$d_{G_1 \circ G_2}(p, G_1 \circ \{v\}) \le d_{G_1 \circ G_2}(p, A) + d_{G_1 \circ G_2}(A, G_1 \circ \{v\}) \le 3/2.$$

□

Lemma 3.1.6. *Let y_1, y_2 be any points in G_2 with $d_{G_2}(y_1, y_2) \le 5/2$ and x_0 a fixed vertex in G_1. Then $\gamma := \{x_0\} \times [y_1 y_2]$ is a geodesic in $G_1 \circ G_2$ joining the points (x_0, y_1) and (x_0, y_2).*

Proof. If G_1 is the trivial graph, then $G_1 \circ G_2 \simeq G_2$ and we have the result. Assume that G_1 is a non-trivial graph. Seeking for a contradiction assume that γ is not a geodesic in $G_1 \circ G_2$. Therefore, there is a geodesic Γ in $G_1 \circ G_2$ joining (x_0, y_1) and (x_0, y_2) which is not contained in $\{x_0\} \circ G_2$. Hence, Γ has a vertex V outside of $\{x_0\} \circ G_2$; thus, we have $2 \le L(\Gamma) < L(\gamma) \le 5/2$. We have

$$\Gamma = [(x_0, y_1)(x_0, B_1)] \cup [(x_0, B_1), V] \cup [V, (x_0, B_2)] \cup [(x_0, B_2)(x_0, y_2)],$$

where B_i is a closest vertex to y_i in G_2, for $i = 1, 2$. Since $\gamma \cup \Gamma$ contains a cycle C with $(x_0, B_1), (x_0, B_2) \in C$ and $L(\gamma) + L(\Gamma) < 5$ we have $L(C) \le 4$ and $d_{G_2}(B_1, B_2) \le 2$, and so, we obtain

$$\begin{aligned} d_{G_2}(y_1, y_2) &\le d_{G_2}(y_1, B_1) + d_{G_2}(B_1, B_2) + d_{G_2}(B_2, y_2) \\ &\le d_{G_2}(y_1, B_1) + 2 + d_{G_2}(B_2, y_2) = L(\Gamma) < L(\gamma) = d_{G_2}(y_1, y_2). \end{aligned}$$

This is the contradiction we were looking for, and so, γ is a geodesic in $G_1 \circ G_2$. □

Corollary 3.1.7. *Let G_1 be a non-trivial graph and G_2 any graph, y_1, y_2 any points in G_2 with $d_{G_2}(y_1, y_2) > 3$ and x_0 a fixed vertex in G_1. Then $\{x_0\} \times [y_1 y_2]$ is not a geodesic in $G_1 \circ G_2$.*

Proof. Let B_i be the closest vertex to y_i in G_2, for $i = 1, 2$. Since G_1 is a non-trivial graph there is a vertex $u_0 \in V(G_1)$ such that $[x_0, u_0] \in E(G_1)$. For any fixed $v_0 \in V(G_2)$ we have

$$\Gamma := [(x_0, y_1)(x_0, B_1)] \cup [(x_0, B_1), (u_0, v_0)] \cup [(u_0, v_0), (x_0, B_2)] \cup [(x_0, B_2)(x_0, y_2)]$$

is a path in $G_1 \circ G_2$ joining (x_0, y_1) and (x_0, y_2). Besides, since $d_{G_2}(y_1, B_1) \le 1/2$ and $d_{G_2}(y_2, B_2) \le 1/2$ we have $L(\Gamma) \le 3 < d_{G_2}(y_1, y_2) = L(\{x_0\} \times [y_1 y_2])$. □

Remark 3.1.8. *Let y_1, y_2 be two midpoints in any graph G_2 with $d_{G_2}(y_1, y_2) = 3$ and x_0 a fixed vertex in any graph G_1. Then $\{x_0\} \times [y_1y_2]$ is a geodesic in $G_1 \circ G_2$ joining (x_0, y_1) and (x_0, y_2).*

Lemma 3.1.9. *Let G_1 be a non-trivial graph and G_2 be any graph. If γ is a geodesic in $G_1 \circ G_2$ joining x and y with $L(\gamma) > 3$, then $\pi(\gamma)$ contains at least three vertices in G_1.*

Furthermore, if σ is a path in $G_1 \circ G_2$ joining x and y, then $\pi(\sigma)$ contains at least three vertices in G_1.

Proof. Since $L(\gamma) > 3$ then γ contains at least three vertices in $G_1 \circ G_2$. Let V_1 and V_2 be the closest vertices to x and y in γ, respectively. Seeking for a contradiction assume that $\pi(\gamma)$ contains either one or two vertices in G_1. Since G_1 is a non-trivial graph and $\pi(\gamma)$ contains at most two vertices, Lemma 3.1.4 gives that $d_{G_1\circ G_2}(V_1, V_2) = 2$ and $\pi(V_1) = \pi(V_2)$. Furthermore, since $L(\gamma) > 3$ we have either $d_{G_1\circ G_2}(x, V_1) > 1/2$ or $d_{G_1\circ G_2}(y, V_2) > 1/2$. Without loss of generality we can assume that $d_{G_1\circ G_2}(x, V_1) > 1/2$. Let W be the vertex in $G_1 \circ G_2$ with x in the edge $[V_1, W]$. Then $d_{G_1\circ G_2}(x, W) < 1/2 < d_{G_1\circ G_2}(x, V_1)$. Consider now a path $\gamma_1 := [xW] \cup [WV_2] \cup [V_2y]$ joining x and y in $G_1 \circ G_2$. Hence, $L(\gamma_1) < L(\gamma)$ since $d_{G_1\circ G_2}(W, V_2) \le 2$. This is the contradiction we were looking for, and then $\pi(\gamma)$ contains at least three vertices in G_1. Finally, since $L(\sigma) \ge L(\gamma)$ and $\pi(\gamma)$ contains at least three vertices, the proof is straightforward. □

Lemma 3.1.10. *Let G_1 be a non-trivial graph and G_2 be any graph. Consider a geodesic γ in $G_1 \circ G_2$ joining x and y. If $L(\gamma) > 3$, then $\pi(\gamma)$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$. Besides, if $L(\gamma) = 3$ then $\pi(\gamma)$ contains a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.*

Proof. Assume first that $L(\gamma) > 3$. By Lemma 3.1.9, $\pi(\gamma)$ contains at least three vertices in G_1. Denote by $V_1, \dots, V_r$ the vertices of $G_1 \circ G_2$ in γ with $r \ge 3$, and $v_1, \dots, v_r$ their projections in G_1 (there are at least three different vertices). Without loss of generality we can assume that γ meet $V_1, \dots, V_r$ in this order. Let V_1', V_r' be two vertices in $G_1 \circ G_2$ such that $x \in [V_1', V_1]$ and $y \in [V_r', V_r]$, and denote by v_1', v_r' their projections in G_1, respectively. Since $d_{G_1\circ G_2}(V_1, V_r) \ge 2$ and $d_{G_1\circ G_2}(x, y) \ge 3$, Lemma 3.1.4 gives $d_{G_1}\big(\{v_1, v_1'\}, \{v_r, v_r'\}\big) \ge 2$.

Seeking for a contradiction assume that there is a geodesic Γ in G_1 joining $\pi(x)$ and $\pi(y)$ with length less than $L(\pi(\gamma))$. Let us consider $v_i^* := \{v_i, v_i'\} \cap \Gamma$ and $V_i^* \in \{V_i, V_i'\}$ with $\pi(V_i^*) = v_i^*$ for $i \in \{1, r\}$. Now, we have three cases.

1. $\pi(x) \ne v_1$ and $\pi(y) \ne v_r$. Then $\pi(x) \in [v_1', v_1]$ and $\pi(y) \in [v_r', v_r]$. Let $\gamma_1 := [xV_1^*] \cup [V_1^*V_r^*] \cup [V_r^*y] \subset G_1 \circ G_2$. Since $d_{G_1}(v_1^*, v_r^*) \ge 2$, Lemma 3.1.4 gives $d_{G_1\circ G_2}(V_1^*, V_r^*) = d_{G_1}(v_1^*, v_r^*)$, and so $L(\gamma_1) = L(\Gamma) < L(\pi(\gamma)) \le L(\gamma)$. This is the contradiction we were looking for, and so, $\pi(\gamma)$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.

2. $\pi(x) = v_1$ and $\pi(y) \ne v_r$ or $\pi(x) \ne v_1$ and $\pi(y) = v_r$. By symmetry, we can assume $\pi(x) = v_1$ and $\pi(y) \ne v_r$. Then $\pi(y) \in [v_r', v_r]$ and $d_{G_1\circ G_2}(x, V_1) \le 1/2$. Let $\gamma_1 := [xV_1] \cup [V_1V_r^*] \cup [V_r^*y] \subset G_1 \circ G_2$. Since $d_{G_1}(v_1, v_r^*) \ge 2$, Lemma 3.1.4 gives

$d_{G_1 \circ G_2}(V_1, V_r^*) = d_{G_1}(v_1, v_r^*)$, and so $L(\gamma_1) = L(\Gamma) + L([xV_1]) < L(\pi(\gamma)) + L([xV_1]) \le L(\gamma)$. This is the contradiction we were looking for, and so, $\pi(\gamma)$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.

3. $\pi(x) = v_1$ and $\pi(y) = v_r$. Then $\pi(\gamma) = \pi([V_1 V_r])$. Since $d_{G_1}(v_1, v_r) \ge 2$, Lemma 3.1.4 gives $d_{G_1 \circ G_2}(V_1, V_r) = d_{G_1}(v_1, v_r)$. Then $L\big(\pi(\gamma)\big) = d_{G_1}(v_1, v_r)$, and $\pi(\gamma)$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.

Assume now that $L(\gamma) = 3$. Then $\pi(\gamma)$ contains either one, two, three or four vertices in G_1.

If $\pi(\gamma)$ contains a single vertex in G_1, then γ is contained in $\{v\} \circ G_2$ for some $v \in V(G_1)$. Thus, $\pi(\gamma) = v$ is a geodesic in G_1 joining $\pi(x)$ with $\pi(y)$.

If $\pi(\gamma)$ contains exactly two vertices in G_1, then x, y are midpoints of edges and $\pi(x) = \pi(y)$.

If $\pi(\gamma)$ contains three or four vertices in G_1, then $\pi(\gamma)$ contains a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$, and the argument used in the proof of the case $L(\gamma) > 3$ gives that $\pi(\gamma)$ is a geodesic. □

Remark 3.1.11. *Let γ be a geodesic in $G_1 \circ G_2$ joining x and y. If $L(\gamma) = 3$ and $\pi(\gamma)$ is not a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$, then x, y are midpoints of edges, $\pi(x) = \pi(y) \in V(G_1)$ and* $\operatorname{diam}(\pi(\gamma)) = 1$.

Corollary 3.1.12. *Let γ be a geodesic in $G_1 \circ G_2$ joining x and y. If $\pi(\gamma)$ is not a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$, then* $\operatorname{diam}\big(\pi(\gamma)\big) < 3$.

Notice that, if γ is a geodesic in $G_1 \circ G_2$ joining the points x and y, then it is possible that $\pi(\gamma)$ does not contain a geodesic in G_1 joining the points $\pi(x)$ and $\pi(y)$, as the following example shows.

Example 3.1.13. *Consider G_1 as the* cycle graph *C_3 with vertices $\{v_1, v_2, v_3\}$ and G_2 as the* path graph *P_3 with vertices $\{w_1, w_2, w_3\}$ and $E(G_2) = \{[w_1, w_2], [w_2, w_3]\}$. Let x and y be the midpoints of edges $[(v_1, w_1), (v_2, w_1)]$ and $[(v_1, w_3), (v_3, w_3)]$, respectively. We have that $\gamma := [x(v_2, w_1)] \cup [(v_2, w_1), (v_3, w_3)] \cup [(v_3, w_3)y]$ is a geodesic in $G_1 \circ G_2$ joining x and y, but $\pi(\gamma) = [\pi(x)v_2] \cup [v_2, v_3] \cup [v_3\pi(y)]$ does not contain the geodesic in G_1 joining $\pi(x)$ and $\pi(y)$ (note that this geodesic is $[\pi(x)v_1] \cup [v_1\pi(y)]$).*

3.2 Hiperbolicity in lexicographic products

Some bounds for the hyperbolicity constant of the lexicographic product of two graphs are studied in this section. These bounds allow to prove Theorem 3.2.14, which characterizes the hyperbolic lexicographic products of two graphs.

The next theorem shows an important qualitative result: if G_1 is not hyperbolic then $G_1 \circ G_2$ is not hyperbolic.

Theorem 3.2.1. *Let G_1 and G_2 two graphs, then $\delta(G_1) \le \delta(G_1 \circ G_2)$.*

Proof. Since $G_1 \circ \{y\}$ is an isometric subgraph of $G_1 \circ G_2$ for every $y \in V(G_2)$, Lemma 1.3.3 gives the result. □

Example 4.2.3 shows that the equality in Theorem 3.2.1 is attained: $\delta(C_n) = \delta(C_n \circ P_2)$ for $n \ge 5$.

Note that the strong product graph $G \boxtimes P_2$ is isomorphic to $G \circ P_2$ for any graph G. We recall that $\delta(P_n) = 0$ since the path graph P_n is a tree; besides, it is well known that the hyperbolicity constant of the cycle graph C_n is $n/4$, see Theorem 1.3.10. The following results which appear in [24] give the hyperbolicity constant of some lexicographic product graphs.

Example 3.2.2. *Let P_n be the path graph with $n \ge 2$. Then*

$$\delta(P_n \circ P_2) = \begin{cases} 1, & \text{if } n = 2, \\ 5/4, & \text{if } n = 3, \\ 3/2, & \text{if } n \ge 4. \end{cases}$$

Example 3.2.3. *Let C_n be the cycle graph with $n \ge 3$. Then*

$$\delta(C_n \circ P_2) = \begin{cases} 1, & \text{if } n = 3, \\ 5/4, & \text{if } n = 4, \\ n/4, & \text{if } n \ge 5. \end{cases}$$

Example 3.2.4. *Let K_m, K_n be the complete graphs with m, n vertices, respectively, and $m, n \ge 2$. Then $K_m \circ K_n$ is isomorphic to K_{mn} and $\delta(K_m \circ K_n) = 1$.*

Proposition 3.2.5. *Let G_1 be a non-trivial graph and G_2 any graph. Consider isometric subgraphs Γ_1, Γ_2 of G_1, G_2, respectively, with Γ_1 non-trivial. Then $\Gamma_1 \circ \Gamma_2$ is an isometric subgraph to $G_1 \circ G_2$.*

Note that taking Γ_1 as a trivial graph, $\Gamma_1 \circ \Gamma_2$ is not an isometric subgraph to $G_1 \circ G_2$ if $\operatorname{diam} V(\Gamma_2) \ge 3$.

Proof. Since $\Gamma_1 \circ \Gamma_2$ is a subgraph of $G_1 \circ G_2$, we have $d_{\Gamma_1 \circ \Gamma_2}(x, y) \ge d_{G_1 \circ G_2}(x, y)$ for every $x, y \in \Gamma_1 \circ \Gamma_2$. Let x, y be any points of $\Gamma_1 \circ \Gamma_2$. If $x, y \in V(\Gamma_1 \circ \Gamma_2)$ then by Lemma 3.1.4 we have $d_{G_1 \circ G_2}(x, y) = d_{\Gamma_1 \circ \Gamma_2}(x, y)$ and we obtain the result. Without loss of generality we can assume that $x, y \notin V(\Gamma_1 \circ \Gamma_2)$. Let $A_1, A_2, B_1, B_2 \in V(\Gamma_1 \circ \Gamma_2)$ with $x \in [A_1, A_2]$, $y \in [B_1, B_2]$. Consider a geodesic γ in $G_1 \circ G_2$ joining x and y with $\gamma := [xA_i] \cup [A_i B_j] \cup [B_j y]$ for some $i, j \in \{1, 2\}$. Then

$$d_{\Gamma_1 \circ \Gamma_2}(x, y) \le d_{\Gamma_1 \circ \Gamma_2}(x, A_i) + d_{\Gamma_1 \circ \Gamma_2}(A_i, B_j) + d_{\Gamma_1 \circ \Gamma_2}(B_j, y) = d_{G_1 \circ G_2}(x, y).$$

Thus, $d_{G_1 \circ G_2}(x, y) = d_{\Gamma_1 \circ \Gamma_2}(x, y)$. □

Theorem 3.2.6. *Let G_1 be a non-trivial graph and G_2 any graph. Then*

$$\delta(G_1 \circ G_2) = \max\{\delta(\Gamma_1 \circ \Gamma_2) : \Gamma_i \textit{ is isometric to } G_i \textit{ for } i = 1,2 \textit{ and } \Gamma_1 \textit{ non-trivial}\}.$$

Proof. By Lemma 1.3.3 and Proposition 3.2.5 we have $\delta(G_1 \circ G_2) \geq \delta(\Gamma_1 \circ \Gamma_2)$ for any Γ_1, Γ_2. Besides, since any graph is an isometric subgraph of itself we obtain the equality by taking $\Gamma_1 = G_1$ and $\Gamma_2 = G_2$. □

Theorem 3.2.7. *If G_1 and G_2 are non-trivial graphs, then $\delta(G_1 \circ G_2) \geq 1$.*

Proof. Since G_i is a non-trivial graph there is a subgraph P_2^i in G_i isomorphic to an edge, for $i = 1,2$. Hence, by Example 3.2.2 and Theorem 3.2.6 we have $\delta(G_1 \circ G_2) \geq \delta(P_2^1 \circ P_2^2) = 1$. □

Theorem 3.2.8. *Let G_2 be any non-trivial graph and G_1 any graph. If* $\operatorname{diam} V(G_1) = 2$*, then $\delta(G_1 \circ G_2) \geq 5/4$. If* $\operatorname{diam} V(G_1) \geq 3$*, then $\delta(G_1 \circ G_2) \geq 3/2$.*

Proof. Assume that $\operatorname{diam} V(G_1) = 2$. Since G_2 is a non-trivial graph there is a subgraph P_2 in G_2 isomorphic to an edge. Besides, since $\operatorname{diam} V(G_1) = 2$ then there is an isometric subgraph in G_1 isomorphic to a path P_3 with 3 vertices. Example 3.2.2 and Theorem 3.2.6 give $5/4 = \delta(P_3 \circ P_2) \leq \delta(G_1 \circ G_2)$.

If $\operatorname{diam} V(G_1) \geq 3$, then a similar argument replacing P_3 by P_4 gives $\delta(G_1 \circ G_2) \geq 3/2$. □

Theorem 3.2.9. *If G_1 is any non-trivial graph and G_2 is any graph with* $\operatorname{diam} G_2 > 2$*, then* $\delta(G_1 \circ G_2) \geq 5/4$.

Proof. Since $\operatorname{diam} G_2 \geq 5/2$ we have that there exist a midpoint $x \in J(G_2) \setminus V(G_2)$ and a vertex $y \in V(G_2)$ such that $d_{G_2}(x,y) = 5/2$. Hence, by Lemma 4.1.11 we have that $\gamma_1 := \{v_0\} \times [xy]$ is a geodesic in $G_1 \circ G_2$ joining the points (v_0, x) and (v_0, y) for some $v_0 \in V(G_1)$. Without loss of generality we can assume that $(v_0, x) \in [A_1, A_2]$ such that $A_1 \in \gamma_1$. Denote it by $\gamma_2 := [(v_0, x)A_2] \cup [A_2 W] \cup [W(v_0, y)]$ where $W \in V(\{v_1\} \circ G_2)$ with $[v_0, v_1] \in E(G_1)$. Therefore, $L(\gamma_2) = 5/2$ and γ_2 is a geodesic in $G_1 \circ G_2$ joining the points (v_0, x) and (v_0, y). Now we have a geodesic bigon $B := \{\gamma_1, \gamma_2\}$ in $G_1 \circ G_2$. If p is the midpoint of γ_1, then $d_{G_1 \circ G_2}(p, \gamma_2) = 5/4$ and we conclude that $\delta(G_1 \circ G_2) \geq \delta(B) = 5/4$. □

Theorem 3.2.10. *Let G_1 be any non-trivial graph and G_2 any graph. Then we have $\delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$.*

Proof. If G_1 is not hyperbolic, then $\delta(G_1) = \infty$, and so, Theorem 3.2.1 gives the result (with equality). Assume now that G_1 is hyperbolic. By Theorem 1.3.13 it suffices to consider geodesic triangles $T = \{x, y, z\}$ in $G_1 \circ G_2$ that are cycles with $x, y, z \in J(G_1 \circ G_2)$. Let $\gamma_1 := [xy]$, $\gamma_2 := [yz]$ and $\gamma_3 := [zx]$. It suffices to prove that $d_{G_1 \circ G_2}(p, \gamma_2 \cup \gamma_3) \leq \delta(G_1) + 3/2$ for every $p \in \gamma_1$. If $d_{G_1 \circ G_2}(p, \{x, y\}) \leq 3/2$, then $d_{G_1 \circ G_2}(p, \gamma_2 \cup \gamma_3) \leq d_{G_1 \circ G_2}(p, \{x, y\}) \leq 3/2$.

Assume that $d_{G_1 \circ G_2}(p, \{x, y\}) > 3/2$; then $L(\gamma_1) > 3$. Let $V_p := (v, w)$ be a closest vertex to p in γ_1. Consider the canonical projection $\pi : G_1 \circ G_2 \longrightarrow G_1 \circ \{w\}$. By Lemma 3.1.10, $\pi(\gamma_1)$ is a geodesic in $G_1 \circ \{w\}$ joining the points $\pi(x)$ and $\pi(y)$.

If $\pi(\gamma_2)$ and $\pi(\gamma_3)$ are geodesics in $G_1 \circ \{w\}$, then there is a point $\alpha \in \pi(\gamma_2) \cup \pi(\gamma_3)$ such that $d_{G_1\circ\{w\}}(V_p, \alpha) \le \delta(G_1)$. Assume that $\alpha \in V\big(\pi(\gamma_2) \cup \pi(\gamma_3)\big)$. Since $L(\gamma_1) > 3$ and $\gamma_2 \cup \gamma_3$ joins x and y, by Lemma 3.1.9, $\pi(\gamma_2) \cup \pi(\gamma_3)$ contains at least three vertices; hence, there exists a vertex $(v_\alpha, w) \in V(\pi(\gamma_2) \cup \pi(\gamma_3))$ such that $[\alpha, (v_\alpha, w)] \in E(G_1 \circ \{w\})$. Let V_α be a vertex in $\big(\{v_\alpha\} \circ G_2\big) \cap \big(\gamma_2 \cup \gamma_3\big)$. Thus, $[\alpha, V_\alpha] \in E(G_1 \circ G_2)$ and

$$d_{G_1\circ G_2}(p, \gamma_2 \cup \gamma_3) \le d_{G_1\circ G_2}(p, V_p) + d_{G_1\circ\{w\}}(V_p, \alpha) + d_{G_1\circ G_2}(\alpha, V_\alpha) \le \delta(G_1) + 3/2.$$

If $\alpha \notin V(\pi(\gamma_2) \cup \pi(\gamma_3))$, then $\alpha \in \{\pi(x), \pi(y)\}$ and α is a midpoint in $G_1 \circ \{w\}$. Without loss of generality we can assume that $\alpha = \pi(x)$ and, consequently, x is a midpoint in $G_1 \circ G_2$. Let V_x be the closest vertex to x in $\gamma_2 \cup \gamma_3$ and v_x the closest vertex to $\pi(x)$ in $\pi(\gamma_1)$. Hence, $[V_x, v_x] \in E(G_1 \circ G_2)$, $d_{G_1\circ\{w\}}(V_p, v_x) \le \delta(G_1) - 1/2$ and

$$d_{G_1\circ G_2}(p, \gamma_2 \cup \gamma_3) \le d_{G_1\circ G_2}(p, V_p) + d_{G_1\circ\{w\}}(V_p, v_x) + d_{G_1\circ G_2}(v_x, V_x) \le \delta(G_1) + 1.$$

If $\pi(\gamma_2)$ and $\pi(\gamma_3)$ are not geodesics in $G_1 \circ \{w\}$, then there is a point $\alpha \in [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]$ such that $d_{G_1\circ\{w\}}(V_p, \alpha) \le \delta(G_1)$. Notice that, if α is not a vertex in $G_1 \circ \{w\}$ then we repeat the previous argument and obtain the result. Assume now that $\alpha \in V([\pi(x)\pi(z)] \cup [\pi(z)\pi(y)])$; by symmetry, we can assume that $\alpha \in V([\pi(x)\pi(z)])$. If $\alpha \in \pi(\gamma_2) \cup \pi(\gamma_3)$, then the previous argument gives $d_{G_1\circ G_2}(p, \gamma_2 \cup \gamma_3) \le \delta(G_1) + 3/2$. Assume now that $\alpha \notin \pi(\gamma_2) \cup \pi(\gamma_3)$. Seeking for a contradiction assume that there is not a vertex $(v_\alpha, w) \in V(\pi(\gamma_2) \cup \pi(\gamma_3))$ such that $[\alpha, (v_\alpha, w)] \in E(G_1 \circ \{w\})$. Then $d_{G_1\circ\{w\}}(\alpha, V(\pi(\gamma_2) \cup \pi(\gamma_3))) \ge 2$; hence, $d_{G_1\circ\{w\}}(\alpha, \pi(x)) \ge 3/2$ and $d_{G_1\circ\{w\}}(\alpha, \pi(z)) \ge 3/2$. However, by Corollary 3.1.12 we have $d_{G_1\circ\{w\}}(\pi(x), \pi(z)) = d_{G_1\circ\{w\}}(\pi(x), \alpha) + d_{G_1\circ\{w\}}(\alpha, \pi(z)) < 3$, which is a contradiction. Therefore, there exists a vertex $(v_\alpha, w) \in V(\pi(\gamma_2) \cup \pi(\gamma_3))$ such that $[\alpha, (v_\alpha, w)] \in E(G_1 \circ \{w\})$. Let V_α be a vertex in $\big(\{v_\alpha\} \circ G_2\big) \cap \big(\gamma_2 \cup \gamma_3\big)$. Then $[\alpha, V_\alpha] \in E(G_1 \circ G_2)$ and

$$d_{G_1\circ G_2}(p, \gamma_2 \cup \gamma_3) \le d_{G_1\circ G_2}(p, V_p) + d_{G_1\circ\{w\}}(V_p, \alpha) + d_{G_1\circ G_2}(\alpha, V_\alpha) \le \delta(G_1) + 3/2.$$

In both cases, $\pi(\gamma_2)$ is a geodesic in $G_1 \circ \{w\}$ but $\pi(\gamma_3)$ is not a geodesic in $G_1 \circ \{w\}$, and $\pi(\gamma_3)$ is a geodesic in $G_1 \circ \{w\}$ but $\pi(\gamma_2)$ is not a geodesic in $G_1 \circ \{w\}$, a similar argument gives the inequality. □

Remark 3.2.11. *Let G_1 be any hyperbolic graph which is not a tree and let G_2 be any graph. The argument in the proof of Theorem 3.2.10 gives that if $\delta(G_1 \circ G_2) = \delta(G_1) + 3/2$ then there is a geodesic triangle $T = \{x, y, z\}$ with $x, y, z \in J(G_1 \circ G_2)$ and a midpoint $p \in [xy]$ such that $d_{G_1\circ G_2}(p, [xz] \cup [zy]) = \delta(G_1) + 3/2$. Besides, $d_{G_1\circ\{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1)$ and the distance is attained in a vertex $\alpha \in [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]$.*

Example 3.2.2 and Theorem 3.2.20 show that the equality in Theorem 3.2.10 is attained. We obtain the following consequence of Theorem 3.2.1 and Theorem 3.2.10.

Theorem 3.2.12. *Let G_1 be any non-trivial graph and G_2 any graph. Then*

$$\delta(G_1) \le \delta(G_1 \circ G_2) \le \delta(G_1) + 3/2.$$

Theorems 3.2.8 and 3.2.10 have the following consequence.

Corollary 3.2.13. *If G_1 is any infinite tree and G_2 is any non-trivial graph, then $\delta(G_1 \circ G_2) = 3/2$.*

Theorem 3.2.14. *Let G_1 be any non-trivial graph and G_2 any graph. The lexicographic product $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Remark 3.2.15. *For any graph G and the trivial graph E_1, the lexicographic product graph $E_1 \circ G$ is hyperbolic if and only if G is hyperbolic, since $\delta(E_1 \circ G) = \delta(G)$. This trivial result completes the characterization of hyperbolic lexicographic products.*

The following results allow to characterize the graphs for which the bound in Theorem 3.2.10 is attained.

Theorem 3.2.16. *Let G_1 be any hyperbolic graph and let G_2 be any graph. If $\delta(G_1 \circ G_2) = \delta(G_1) + 3/2$, then G_1 is a tree, G_2 is a non-trivial graph and $\delta(G_1 \circ G_2) = 3/2$.*

Proof. Seeking for a contradiction assume that G_1 is not a tree (*i.e.*, $\delta(G_1) > 0$). By hypothesis $G_1 \circ G_2$ is hyperbolic, thus, Theorem 1.3.13 and Remark 3.2.11 give that there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle with $x, y, z \in J(G_1 \circ G_2)$ and a midpoint $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = \delta(G_1) + 3/2$. Let $V_p := (v, w)$ be a closest vertex to p in $[xy] \cap V(G_1 \circ G_2)$ as in the proof of Theorem 3.2.10, *i.e.*, $d_{G_1 \circ \{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1)$ with π the canonical projection on $G_1 \circ \{w\}$; besides, this equality is attained in a vertex $\alpha \in [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]$. Note that $\delta(G_1)$ is an integer number since it is the distance between two vertices. Since $\delta(G_1) > 0$, we have $\delta(G_1) \geq 1$. Let V'_p be the vertex in $T \cap V(G_1 \circ G_2)$ such that $[V_p, V'_p]$ is the edge in $G_1 \circ G_2$ with $p \in [V_p, V'_p]$. Since $d_{G_1 \circ G_2}(p, \{x, y\}) \geq d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = \delta(G_1) + 3/2$, there exist $a, b \in [xy] \cap V(G_1 \circ G_2)$ with $d_{G_1 \circ G_2}(a, p) = d_{G_1 \circ G_2}(b, p) = 3/2$ and $d_{G_1 \circ G_2}(a, b) = 3$. If $\pi(V_p) = \pi(V'_p)$, then $d_{G_1 \circ \{w\}}(\pi(a), \pi(b)) = 2$. This contradicts Lemma 3.1.4, and so, we have $\pi(V_p) \neq \pi(V'_p)$ and $\pi(V_p) \neq \pi(p) \neq \pi(V'_p)$. If $d_{G_1 \circ \{w\}}(\pi(p), [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = d_{G_1 \circ \{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1) \geq 1$, then since $\pi(V_p) \neq \pi(p)$ we obtain that $d_{G_1 \circ \{w\}}(\xi, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1) + 1/4$ where ξ is the midpoint of $[\pi(p)V_p]$. But this is a contradiction since $d_{G_1 \circ \{w\}}(\xi, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) \leq \delta(G_1)$. Then we have $d_{G_1 \circ \{w\}}(\pi(p), [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) < d_{G_1 \circ \{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1)$; hence, $d_{G_1 \circ \{w\}}(\pi(p), [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1) - 1/2$ and $d_{G_1 \circ \{w\}}(\pi(V'_p), [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1) - 1$. We can repeat the same argument in the proof of Theorem 3.2.10 for V'_p instead of V_p, and we obtain $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) \leq \delta(G_1) + 1/2$. This is the contradiction we were looking for and G_1 is a tree.

Hence, $\delta(G_1 \circ G_2) = 3/2$. If G_2 is a trivial graph, then $3/2 = \delta(G_1 \circ G_2) = \delta(G_1) = 0$, which is a contradiction. Therefore, G_2 is a non-trivial graph. □

Theorem 3.2.20 below is a converse of Theorem 3.2.16; furthermore, it provides the exact value of the hyperbolicity constant of the lexicographic product of many trees and graphs. We need some lemmas.

Lemma 3.2.17. *Let G_1 be any tree with $1 \leq \operatorname{diam} G_1 \leq 2$ and G_2 any graph. Then $\delta(G_1 \circ G_2) = 3/2$ if and only if there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle contained in $\{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$ with $x, y, z \in J(\{v_0\} \circ G_2)$ and a vertex $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = d_{G_1 \circ G_2}(p, x) = d_{G_1 \circ G_2}(p, y) = 3/2$.*

Proof. Assume first that $\delta(G_1 \circ G_2) = 3/2$. By Theorem 1.3.13 there exists a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle with $x, y, z \in J(G_1 \circ G_2)$ and a point $p \in [xy]$ such that $\delta(T) = d_{G_1 \circ G_2}(p, [yz] \cup [zx]) = 3/2$. Thus, $d_{G_1 \circ G_2}(p, \{x, y\}) \geq d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = 3/2$ and $L([xy]) \geq 3$.

Assume that $\operatorname{diam} G_1 = 2$ (the case $\operatorname{diam} G_1 = 1$ is similar and simpler). We show now that $\operatorname{diam} G_1 \circ G_2 = 3$. Note that $\operatorname{diam} G_1 \circ G_2 \geq L([xy]) \geq 3$. Let $A, B \in V(G_1 \circ G_2)$. If $\pi(A) = \pi(B)$, then by Lemma 3.1.4 we have $d_{G_1 \circ G_2}(A, B) \leq 2$. If $\pi(A) \neq \pi(B)$, then by Lemma 3.1.4 we have $d_{G_1 \circ G_2}(A, B) \leq 2$ since that $\operatorname{diam} G_1 = 2$. Therefore, $\operatorname{diam} V(G_1 \circ G_2) = 2$ and $\operatorname{diam} G_1 \circ G_2 \leq 3$. Consequently, $\operatorname{diam} G_1 \circ G_2 = 3$, $L([xy]) = 3$ and $d_{G_1 \circ G_2}(p, x) = d_{G_1 \circ G_2}(p, y) = 3/2$. Notice that x, y are midpoints of $G_1 \circ G_2$ and p a vertex of $G_1 \circ G_2$.

Assume now that $x \in \{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$ and $y \notin \{v_0\} \circ G_2$, where $x \in [A_1, A_2]$ and $y \in [B_1, B_2]$ with $A_1, B_1 \in [xy]$; then $d_{G_1 \circ G_2}(A_1, B_1) = 2$ since that $L([xy]) = 3$. Note that $A_1 \in \{v_0\} \times V(G_2)$ and $B_1 \in \{w_0\} \times V(G_2)$ with $d_{G_1}(v_0, w_0) = 2$. We have that $[xy] \cap ([yz] \cup [zx]) = \{x, y\}$ since T is a cycle. Hence, $A_2, B_2 \in V([yz] \cup [zx])$ and $d_{G_1 \circ G_2}(p, [yz] \cup [zx]) = d_{G_1 \circ G_2}(p, \{A_2, B_2\}) = 1$ since p is a vertex, and this is a contradiction. If $y \in \{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$ and $x \notin \{v_0\} \circ G_2$, then the same argument gives a contradiction. If $x, y \notin \cup_{v_0 \in V(G_1)} \{v_0\} \circ G_2$, then one can check that $d_{G_1 \circ G_2}(x, y) \leq 2$, which is a contradiction. Hence, we conclude that $x, y \in \{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$. We also have $p \in \{v_0\} \circ G_2$ and we conclude that $[xy]$ is contained in $\{v_0\} \circ G_2$. If $[yz] \cup [zx]$ is not contained in $\{v_0\} \circ G_2$, then there is a vertex $W \in [yz] \cup [zx]$ such that $W \in \{w_0\} \circ G_2$ and $d_{G_1}(v_0, w_0) = 1$. Hence, $d_{G_1 \circ G_2}(p, W) = 1$, which is a contradiction. Then T is contained in $\{v_0\} \circ G_2$.

It is easy to check that if there exists such a geodesic triangle T, then $\delta(G_1 \circ G_2) \geq \delta(T) \geq 3/2$. Theorem 3.2.10 allows to conclude $\delta(G_1 \circ G_2) = 3/2$. □

For any non-empty set $S \subset V(G)$, the induced subgraph of S will be denoted by $\langle S \rangle$.

Lemma 3.2.18. *Let G be any graph. Then $G \in \mathcal{F}$ if and only if there is a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$, $L([xy]), L([yz]), L([zx]) \leq 3$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ where $p \in [xy] \cap V(G)$.*

Proof. Assume first that there is a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$, $L([xy]), L([yz]), L([zx]) \leq 3$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ for some $p \in [xy]$. Since $d_G(p, \{x, y\}) \geq d_G(p, [yz] \cup [zx]) = 3/2$, we have $L([xy]) = 3$ and p is the midpoint of $[xy]$. Since $L([yz]) \leq 3$, $L([zx]) \leq 3$ and $L([yz]) + L([zx]) \geq L([xy])$, we have $6 \leq L(T) \leq 9$.

Assume now that $L(T) = 6$. Denote by $\{v_1, \ldots, v_6\}$ the vertices in T such that $T = \bigcup_{i=1}^{6}[v_i, v_{i+1}]$ with $v_7 := v_1$. Without loss of generality we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_G(x, y) = 3$, we have that $\langle\{v_1, \ldots, v_6\}\rangle$ contains neither $[v_1, v_4]$, $[v_1, v_5]$, $[v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_G(p, [yz] \cup [zx]) > 1$ we have that $\langle\{v_1, \ldots, v_6\}\rangle$ contains neither $[v_3, v_1]$, $[v_3, v_5]$ nor $[v_3, v_6]$. Note that $[v_2, v_6]$, $[v_4, v_6]$ may be contained in $\langle\{v_1, \ldots, v_6\}\rangle$. Therefore, $G \in \mathcal{F}_6$.

Assume that $L(T) = 7$ and $G \notin \mathcal{F}_6$. Denote by $\{v_1, \ldots, v_7\}$ the vertices in T such that $T = \bigcup_{i=1}^{7}[v_i, v_{i+1}]$ with $v_8 := v_1$. Without loss of generality we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_G(x, y) = 3$, we have that $\langle\{v_1, \ldots, v_7\}\rangle$ contains neither $[v_1, v_4]$, $[v_1, v_5]$, $[v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_G(p, [yz] \cup [zx]) > 1$ we have that $\langle\{v_1, \ldots, v_7\}\rangle$ contains neither $[v_3, v_1]$, $[v_3, v_5]$, $[v_3, v_6]$ nor $[v_3, v_7]$. Since $G \notin \mathcal{F}_6$, $[v_1, v_6]$ and $[v_5, v_7]$ are not contained in $\langle\{v_1, \ldots, v_7\}\rangle$. Note that $[v_2, v_6]$, $[v_2, v_7]$, $[v_4, v_6]$, $[v_4, v_7]$ may be contained in $\langle\{v_1, \ldots, v_7\}\rangle$. Hence, $G \in \mathcal{F}_7$.

Assume that $L(T) = 8$ and $G \notin \mathcal{F}_6 \cup \mathcal{F}_7$. Denote by $\{v_1, \ldots, v_8\}$ the vertices in T such that $T = \bigcup_{i=1}^{8}[v_i, v_{i+1}]$ with $v_9 := v_1$. Without loss of generality we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_G(x, y) = 3$, we have that $\langle\{v_1, \ldots, v_8\}\rangle$ contains neither $[v_1, v_4]$, $[v_1, v_5]$, $[v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_G(p, [yz] \cup [zx]) > 1$ we have that $\langle\{v_1, \ldots, v_8\}\rangle$ contains neither $[v_3, v_1]$, $[v_3, v_5]$, $[v_3, v_6]$, $[v_3, v_7]$ nor $[v_3, v_8]$. Since $G \notin \mathcal{F}_6 \cup \mathcal{F}_7$, $[v_1, v_6]$, $[v_1, v_7]$, $[v_5, v_7]$, $[v_5, v_8]$ and $[v_6, v_8]$ are not contained in $\langle\{v_1, \ldots, v_8\}\rangle$. Since T is a geodesic triangle we have that $z \in \{v_{6,7}, v_7, v_{7,8}\}$ with $v_{6,7}$ and $v_{7,8}$ the midpoints of $[v_6, v_7]$ and $[v_7, v_8]$, respectively. If $z = v_7$ then $\langle\{v_1, \ldots, v_8\}\rangle$ contains neither $[v_2, v_7]$ nor $[v_4, v_7]$. Note that $[v_2, v_6]$, $[v_2, v_8]$, $[v_4, v_6]$, $[v_4, v_8]$ may be contained in $\langle\{v_1, \ldots, v_8\}\rangle$. If $z = v_{6,7}$ then $\langle\{v_1, \ldots, v_8\}\rangle$ contains neither $[v_2, v_6]$ nor $[v_2, v_7]$. Note that $[v_2, v_8]$, $[v_4, v_6]$, $[v_4, v_7]$, $[v_4, v_8]$ may be contained in $\langle\{v_1, \ldots, v_8\}\rangle$. By symmetry, we obtain an equivalent result for $z = v_{7,8}$. Therefore, $G \in \mathcal{F}_8$.

Assume that $L(T) = 9$ and $G \notin \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$. Denote by $\{v_1, \ldots, v_9\}$ the vertices in T such that $T = \bigcup_{i=1}^{9}[v_i, v_{i+1}]$ with $v_{10} := v_1$. Without loss of generality we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_G(x, y) = 3$, we have that $\langle\{v_1, \ldots, v_9\}\rangle$ contains neither $[v_1, v_4]$, $[v_1, v_5]$, $[v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_G(p, [yz] \cup [zx]) > 1$ we have that $\langle\{v_1, \ldots, v_9\}\rangle$ contains neither $[v_3, v_1]$, $[v_3, v_5]$, $[v_3, v_6]$, $[v_3, v_7]$, $[v_3, v_8]$ nor $[v_3, v_9]$. Since T is a geodesic triangle we have that z is the midpoint of $[v_7, v_8]$. Since $d_G(y, z) = d_G(z, x) = 3$, we have that $\langle\{v_1, \ldots, v_9\}\rangle$ contains neither $[v_1, v_7]$, $[v_1, v_8]$, $[v_2, v_7]$, $[v_2, v_8]$, $[v_4, v_7]$, $[v_4, v_8]$, $[v_5, v_7]$ nor $[v_5, v_8]$. Since $G \notin \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$, $[v_1, v_6]$, $[v_5, v_9]$, $[v_6, v_8]$, $[v_6, v_9]$ and $[v_7, v_9]$ are not contained in $\langle\{v_1, \ldots, v_9\}\rangle$. Note that $[v_2, v_6]$, $[v_2, v_9]$, $[v_4, v_6]$, $[v_4, v_9]$ may be contained in $\langle\{v_1, \ldots, v_9\}\rangle$. Hence, $G \in \mathcal{F}_9$.

Therefore, in any case $G \in \mathcal{F}$.

The previous argument also shows that if $G \in \mathcal{F}$, then there is a geodesic triangle with the required properties. □

Corollary 3.2.19. *Let G be any graph. Then $G \in \mathcal{F}$ if and only if there is a geodesic triangle $T = \{x, y, z\}$ in G with $x, y, z \in J(G)$, $L([xy]), L([yz]), L([zx]) \leq 3$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ for some $p \in [xy] \cap V(G)$.*

Proof. Assume that there is a geodesic triangle $T = \{x, y, z\}$ in G with $x, y, z \in J(G)$, $L([xy]), L([yz]), L([zx]) \leq 3$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ for some $p \in [xy] \cap V(G)$. Since $L([xy]) \leq 3$ and $d_G(p, [yz] \cup [zx]) = 3/2$, we deduce that $L([xy]) = 3$ and $[xy] \cap ([yz] \cup [zx]) = \{x, y\}$. Let Γ be the set of curves joining x and y, and contained in $[yz] \cup [zx]$. If $\gamma \in \Gamma$ satisfies $L(\gamma) \leq L(g)$ for every $g \in \Gamma$, then $[xy] \cup \gamma$ is a cycle and $\gamma \cap [yz] \cap [zx]$ is a single point. If $z' := \gamma \cap [yz] \cap [zx]$, then $\gamma = [yz'] \cup [z'x]$, $z' \in J(G)$, $L([yz']) \leq L([yz]) \leq 3$, $L([z'x]) \leq L([zx]) \leq 3$, $T' = \{x, y, z'\}$ is a cycle and $\delta(T') = 3/2 = d_G(p, [yz'] \cup [z'x])$. Since we have constructed a geodesic triangle T' that is a cycle from T verifying the properties of T, Lemma 3.2.18 gives the result. □

Theorem 3.2.16 and the following result characterize the graphs for which the bound in Theorem 3.2.10 is attained.

Theorem 3.2.20. *Let G_1 be any tree and G_2 any non-trivial graph.*

(1) *If* $\operatorname{diam} G_1 \geq 3$, *then* $\delta(G_1 \circ G_2) = 3/2$.

(2) *If* $1 \leq \operatorname{diam} G_1 \leq 2$, *then* $\delta(G_1 \circ G_2) = 3/2$ *if and only if* $G_2 \in \mathcal{F}$.

(3) *If G_1 is trivial, then* $\delta(G_1 \circ G_2) = 3/2$ *if and only if* $\delta(G_2) = 3/2$.

Proof. If $\operatorname{diam} G_1 \geq 3$, then Theorems 3.2.8 and 3.2.10 give the result since that $\delta(G_1) = 0$.

In order to prove (2), by Lemma 3.2.17, we have that $\delta(G_1 \circ G_2) = 3/2$ if and only if there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle contained in $\{v\} \circ G_2$ for some $v \in V(G_1)$ with $x, y, z \in J(\{v\} \circ G_2)$ and a vertex $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = d_{G_1 \circ G_2}(p, x) = d_{G_1 \circ G_2}(p, y) = 3/2$. By Lemma 3.1.4, $\operatorname{diam} V(G_1 \circ G_2) = 2$, hence, $L([yz]), L([zx]) \leq 3$ and x, y are midpoints with $L([xy]) = 3$. Hence, by Lemma 3.2.18 we have that $\delta(G_1 \circ G_2) = 3/2$ if and only if $\{v\} \circ G_2 \in \mathcal{F}$ and so, Remark 3.1.3 gives that this is equivalent to $G_2 \in \mathcal{F}$.

Finally, if G_1 is trivial, then Remark 3.1.3 gives the result. □

The following result allows to compute, in a simple way, the hyperbolicity constant of the lexicographic product of any tree and any graph.

Theorem 3.2.21. *Let G_1 be any tree and G_2 any graph. Then*

$$\delta(G_1 \circ G_2) = \begin{cases} \delta(G_2), & \text{if } G_1 \simeq E_1, \\ 0, & \text{if } G_2 \simeq E_1, \\ 1, & \text{if } \operatorname{diam} G_1 = 1 \text{ and } 1 \leq \operatorname{diam} G_2 \leq 2, \\ 5/4, & \text{if } \operatorname{diam} G_1 = 1 \text{ and } \operatorname{diam} G_2 > 2 \text{ and } G_2 \notin \mathcal{F}, \\ 5/4, & \text{if } \operatorname{diam} G_1 = 2 \text{ and } \operatorname{diam} G_2 \geq 1 \text{ and } G_2 \notin \mathcal{F}, \\ 3/2, & \text{if } 1 \leq \operatorname{diam} G_1 \leq 2 \text{ and } G_2 \in \mathcal{F}, \\ 3/2, & \text{if } \operatorname{diam} G_1 \geq 3 \text{ and } \operatorname{diam} G_2 \geq 1. \end{cases}$$

Proof. If $G_1 \simeq E_1$ or $G_2 \simeq E_1$, then we have the result by Remark 3.1.3.

If $\operatorname{diam} G_1 = 1$ and $1 \leq \operatorname{diam} G_2 \leq 2$, then Theorems 1.3.12, 3.2.7, 3.2.10 and 3.2.20 give $\delta(G_1 \circ G_2) \in \{1, 5/4\}$ since $G_2 \notin \mathcal{F}$. Seeking for a contradiction we can assume that $\delta(G_1 \circ G_2) = 5/4$. Then by Theorem 1.3.13 there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle with $x, y, z \in J(G_1 \circ G_2)$ and a point $p \in [xy]$ such that $\delta(T) = d_{G_1 \circ G_2}(p, [yz] \cup [zx]) = 5/4$. Thus, $d_{G_1 \circ G_2}(p, \{x, y\}) \geq d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = 5/4$, $L([xy]) \geq 5/2$ and $x, y \in \{v\} \circ G_2$ for some $v \in V(G_1)$ since $\operatorname{diam} G_1 = 1$. This is a contradiction since $\operatorname{diam} G_2 \leq 2$ and we conclude that $\delta(G_1 \circ G_2) = 1$.

If $\operatorname{diam} G_1 = 1$ and $\operatorname{diam} G_2 > 2$ or $\operatorname{diam} G_1 = 2$ and $\operatorname{diam} G_2 \geq 1$, then Theorems 1.3.12, 3.2.8, 3.2.9 and 3.2.10 give $\delta(G_1 \circ G_2) \in \{5/4, 3/2\}$. Finally, since $G_2 \notin \mathcal{F}$, Theorem 3.2.20 gives $\delta(G_1 \circ G_2) \neq 3/2$ and we have $\delta(G_1 \circ G_2) = 5/4$.

If $1 \leq \operatorname{diam} G_1 \leq 2$ and $G_2 \in \mathcal{F}$ or $\operatorname{diam} G_1 \geq 3$ and $\operatorname{diam} G_2 \geq 1$, then we have the result by Theorem 3.2.20. ☐

Corollary 3.2.22. *Let G_1, G_2 be any trees. Then*

$$\delta(G_1 \circ G_2) = \begin{cases} 0, & \textit{if } \; G_1 \simeq E_1 \quad \textit{or} \quad G_2 \simeq E_1, \\ 1, & \textit{if } \; \operatorname{diam} G_1 = 1 \quad \textit{and} \quad 1 \leq \operatorname{diam} G_2 \leq 2, \\ 5/4, & \textit{if } \; \operatorname{diam} G_1 = 1 \quad \textit{and} \quad \operatorname{diam} G_2 \geq 3, \\ 5/4, & \textit{if } \; \operatorname{diam} G_1 = 2 \quad \textit{and} \quad \operatorname{diam} G_2 \geq 1, \\ 3/2, & \textit{if } \; \operatorname{diam} G_1 \geq 3 \quad \textit{and} \quad \operatorname{diam} G_2 \geq 1. \end{cases}$$

Corollary 3.2.23. *Let P_n, P_m be two path graphs. Then*

$$\delta(P_n \circ P_m) = \begin{cases} 0, & \textit{if } \; n = 1 \quad \textit{or} \quad m = 1, \\ 1, & \textit{if } \; n = 2 \quad \textit{and} \quad m = 2, 3, \\ 5/4, & \textit{if } \; n = 2 \quad \textit{and} \quad m \geq 4 \quad \textit{or} \quad n = 3 \quad \textit{and} \quad m \geq 2, \\ 3/2, & \textit{if } \; n \geq 4 \quad \textit{and} \quad m \geq 2. \end{cases}$$

Chapter 4

Gromov hyperbolicity in the Cartesian sum of graphs

The Cartesian sum of graphs has been extensively investigated in relation to a wide range of subjects (see, *e.g.*, [40, 80, 89, 108, 112] and the references therein). This notion of graph product was introduced by Ore [89]. The Cartesian sum is also known as the disjunctive product [108].

4.1 Distance in the Cartesian sum graphs

In order to estimate the hyperbolicity constant of the Cartesian sum of two graphs $G_1 \oplus G_2$, we will need bounds for the distance between two arbitrary points. We will use the definition given in [45].

Definition 4.1.1. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The Cartesian sum $G_1 \oplus G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \oplus G_2$ are adjacent if either $[u_1, u_2] \in E(G_1)$ or $[v_1, v_2] \in E(G_2)$.*

From the definition, it follows that the Cartesian sum of two graphs is commutative, i.e., $G_1 \oplus G_2 \simeq G_2 \oplus G_1$.

Hence, the conclusion of any result in this Chapter with some "non-symmetric" hypothesis also holds if we replace G_1 by G_2 and G_2 by G_1 (see, e.g., Lemmas 4.1.10, 4.1.11 and 4.1.12).

We use the notation (x, y) for the points of the graph $G_1 \oplus G_2$ if $x \in V(G_1)$ or $y \in V(G_2)$. Otherwise, this notation can be ambiguous.

Remark 4.1.2. *The Cartesian, strong and lexicographic products of two graphs are subgraphs of the Cartesian sum product of two graphs, i.e., $G_1 \square G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \circ G_2 \subseteq G_1 \oplus G_2$.*

Remark 4.1.3. *For any graph G we have $E_1 \oplus G \simeq G \oplus E_1 \simeq G$.*

Remark 4.1.4. *Let G be any graph and K_n the complete graph with n vertices. Then $G \oplus K_n \simeq K_n \oplus G \simeq K_n \circ G$. Note that $K_n \oplus K_m \simeq K_{nm}$.*

The following result allows to compute the distance between any two points in $G_1 \oplus G_2$. Furthermore, this result provides information about the geodesics in the Cartesian sum.

Proposition 4.1.5. *For every non-trivial graphs G_1, G_2 we have:*

(a) Let $x_1, x_2 \in V(G_1 \oplus G_2)$ where $x_1 = (u_1, v_1)$ and $x_2 = (u_2, v_2)$. Then

$$d_{G_1 \oplus G_2}(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = x_2, \\ 1, & \text{if } [u_1, u_2] \in E(G_1) \quad \text{or} \quad [v_1, v_2] \in E(G_2), \\ 2, & \text{if } [u_1, u_2] \notin E(G_1) \quad \text{and} \quad [v_1, v_2] \notin E(G_2). \end{cases}$$

(b) Let $x_1 \in V(G_1 \oplus G_2), x_2 \notin V(G_1 \oplus G_2)$ where $x_1 = (u_1, v_1)$, $x_2 \in [(A_1, B_1), (A_2, B_2)]$ with

$d_{G_1 \oplus G_2}((A_1, B_1), x_2) \leq 1/2$. Then

$$d_{G_1 \oplus G_2}(x_1, x_2) \leq \begin{cases} 3/2, & \text{if } [u_1, A_1] \in E(G_1) \quad \text{or} \quad [v_1, B_1] \in E(G_2), \\ 5/2, & \text{if } [u_1, A_1] \notin E(G_1) \quad \text{and} \quad [v_1, B_1] \notin E(G_2). \end{cases}$$

(c) $d_{G_1 \oplus G_2}(x_1, x_2) \leq 3$ for every $x_1, x_2 \in G_1 \oplus G_2$.

Proof. In order to prove (a), if $[u_1, u_2] \in E(G_1)$ or $[v_1, v_2] \in E(G_2)$, then Definition 4.1.1 gives $d_{G_1 \oplus G_2}(x_1, x_2) = 1$. If $[u_1, u_2] \notin E(G_1)$ and $[v_1, v_2] \notin E(G_2)$ then there exist $u_3 \in V(G_1) \setminus \{u_1, u_2\}, v_3 \in V(G_2) \setminus \{v_1, v_2\}$ with $[u_1, u_3] \in E(G_1)$ and $[v_2, v_3] \in E(G_2)$; thus $x_3 := (u_3, v_3) \in V(G_1 \oplus G_2)$ and $d_{G_1 \oplus G_2}(x_1, x_2) \leq d_{G_1 \oplus G_2}(x_1, x_3) + d_{G_1 \oplus G_2}(x_3, x_2) = 2$ by Definition 4.1.1; but $d_{G_1 \oplus G_2}(x_1, x_2) \geq 2$ since that $[u_1, u_2] \notin E(G_1)$ and $[v_1, v_2] \notin E(G_2)$. Hence, $d_{G_1 \oplus G_2}(x_1, x_2) = 2$.

In order to prove (b), assume first that $[u_1, A_1] \in E(G_1)$ or $[v_1, B_1] \in E(G_2)$. Then $d_{G_1 \oplus G_2}(x_1, x_2) \leq d_{G_1 \oplus G_2}(x_1, (A_1, B_1)) + d_{G_1 \oplus G_2}((A_1, B_1), x_2) \leq 3/2$ by Definition 4.1.1. Assume now that $[u_1, A_1] \notin E(G_1)$ and $[v_1, B_1] \notin E(G_2)$. Then,

$$d_{G_1 \oplus G_2}(x_1, x_2) \leq d_{G_1 \oplus G_2}(x_1, (A_1, B_1)) + d_{G_1 \oplus G_2}((A_1, B_1), x_2) \leq 2 + d_{G_1 \oplus G_2}((A_1, B_1), x_2) \leq 5/2.$$

In order to prove (c), let us consider $X_1, X_2 \in V(G_1 \oplus G_2)$ such that $d_{G_1 \oplus G_2}(x_1, X_1) \leq 1/2$ and $d_{G_1 \oplus G_2}(x_2, X_2) \leq 1/2$. Then,

$$d_{G_1 \oplus G_2}(x_1, x_2) \leq d_{G_1 \oplus G_2}(x_1, X_1) + d_{G_1 \oplus G_2}(X_1, X_2) + d_{G_1 \oplus G_2}(X_2, x_2) \leq 3$$

since $d_{G_1 \oplus G_2}(X_1, X_2) \leq 2$ by item (a). □

Proposition 4.1.5 gives the following result.

Proposition 4.1.6. *Let G_1, G_2 be two non-trivial graphs and let Γ_1, Γ_2 be isometric subgraphs of G_1 and G_2, respectively. If Γ_1 and Γ_2 are non-trivial graphs, then $\Gamma_1 \oplus \Gamma_2$ is an isometric subgraph of $G_1 \oplus G_2$.*

Note that taking Γ_1 as a trivial graph, $\Gamma_1 \oplus \Gamma_2 \simeq \Gamma_2$ is not an isometric subgraph of $G_1 \oplus G_2$ if $\operatorname{diam} V(\Gamma_2) \geq 3$.

Proof. Since $\Gamma_1 \oplus \Gamma_2$ is a subgraph of $G_1 \oplus G_2$, we have $d_{\Gamma_1 \oplus \Gamma_2}(x, y) \geq d_{G_1 \circ G_2}(x, y)$ for every $x, y \in \Gamma_1 \oplus \Gamma_2$. Let x, y be any points of $\Gamma_1 \oplus \Gamma_2$. If $x, y \in V(\Gamma_1 \oplus \Gamma_2)$ then Proposition 4.1.5 gives $d_{G_1 \oplus G_2}(x, y) = d_{\Gamma_1 \oplus \Gamma_2}(x, y)$ and we obtain the result. Otherwise, let $A_1, A_2, B_1, B_2 \in V(\Gamma_1 \oplus \Gamma_2)$ with $x \in [A_1, A_2]$, $y \in [B_1, B_2]$ (it is possible to have x or y in $V(\Gamma_1 \oplus \Gamma_2)$). Consider a geodesic γ in $G_1 \oplus G_2$ joining x and y with $\gamma := [xA_i] \cup [A_i B_j] \cup [B_j y]$ for some $i, j \in \{1, 2\}$. Then

$$d_{\Gamma_1 \oplus \Gamma_2}(x, y) \leq d_{\Gamma_1 \oplus \Gamma_2}(x, A_i) + d_{\Gamma_1 \oplus \Gamma_2}(A_i, B_j) + d_{\Gamma_1 \oplus \Gamma_2}(B_j, y) = d_{G_1 \oplus G_2}(x, y).$$

Thus, $d_{G_1 \oplus G_2}(x, y) = d_{\Gamma_1 \oplus \Gamma_2}(x, y)$. □

The following result allows to compute the diameter of the set of vertices in the Cartesian sum of two graphs.

Proposition 4.1.7. *For every non-trivial graphs G_1, G_2 we have $1 \leq \operatorname{diam} V(G_1 \oplus G_2) \leq 2$. Furthermore, $\operatorname{diam} V(G_1 \oplus G_2) = 1$ if and only if G_1 and G_2 are complete graphs.*

Proof. Since $G_1 \oplus G_2$ is a non-trivial graph, $\operatorname{diam} V(G_1 \oplus G_2) \geq 1$. Besides, if $u, v \in V(G_1 \oplus G_2)$, then by Proposition 4.1.5 we have that $d_{G_1 \oplus G_2}(u, v) \leq 2$ and $\operatorname{diam} V(G_1 \oplus G_2) \leq 2$.

Finally, one can check that $G_1 \oplus G_2$ is a complete graph if and only if G_1 and G_2 are complete graphs. □

Since $\operatorname{diam} V(G) \leq \operatorname{diam} G \leq \operatorname{diam} V(G) + 1$ for every graph G, the previous proposition has the following consequence.

Corollary 4.1.8. *For every non-trivial graphs G_1, G_2 we have $1 \leq \operatorname{diam} G_1 \oplus G_2 \leq 3$.*

Proposition 4.1.5 gives the following result. Given a graph G, we say that $x \in G$ is a midpoint (of an edge) if $d_G(x, V(G)) = 1/2$.

Corollary 4.1.9. *Let G_1, G_2 be any non-trivial graphs. If $d_{G_1 \oplus G_2}(x, y) = 3$, then x, y are midpoints in $G_1 \oplus G_2$.*

Lemma 4.1.10. *Let G_1, G_2 be any non-trivial graphs. Then $G_1 \oplus G_2 \subseteq \mathcal{V}_{3/2}(G_1 \oplus \{v\})$ for every $v \in V(G_2)$.*

Proof. Let p be any point of $G_1 \oplus G_2$ and $v \in V(G_2)$. If $p \in V(G_1 \oplus G_2)$, then Definition 4.1.1 gives that there exists a vertex $u_0 \in V(G_1 \oplus \{v\})$ such that $[p, u_0] \in E(G_1 \oplus G_2)$ since G_1 is non-trivial. Assume that $p \notin V(G_1 \oplus G_2)$. Let $A \in V(G_1 \oplus G_2)$ with $d_{G_1 \oplus G_2}(p, A) \le 1/2$. Hence, we have

$$d_{G_1 \oplus G_2}(p, G_1 \oplus \{v\}) \le d_{G_1 \oplus G_2}(p, A) + d_{G_1 \oplus G_2}(A, G_1 \oplus \{v\}) \le 3/2.$$

□

Lemma 4.1.11. *Let G_1, G_2 be any graphs. Let y_1, y_2 be any points in G_2 with $d_{G_2}(y_1, y_2) \le 5/2$ and x_0 any fixed vertex in G_1. Then $\gamma := \{x_0\} \times [y_1 y_2]$ is a geodesic in $G_1 \oplus G_2$ joining the points (x_0, y_1) and (x_0, y_2).*

Proof. If G_1 is the trivial graph, then $G_1 \oplus G_2 \simeq G_2$ and we have the result. Assume that G_1 is a non-trivial graph. Seeking for a contradiction assume that γ is not a geodesic in $G_1 \oplus G_2$. Therefore, there is a geodesic Γ in $G_1 \oplus G_2$ joining (x_0, y_1) and (x_0, y_2) which is not contained in $\{x_0\} \oplus G_2$. Hence, Γ has a vertex A outside of $\{x_0\} \oplus G_2$; thus, we have $2 \le L(\Gamma) < L(\gamma) \le 5/2$. We have

$$\Gamma = [(x_0, y_1)(x_0, B_1)] \cup [(x_0, B_1), A] \cup [A, (x_0, B_2)] \cup [(x_0, B_2)(x_0, y_2)],$$

where B_i is a closest vertex to y_i in G_2, for $i = 1, 2$. Since $\gamma \cup \Gamma$ contains a cycle C with $(x_0, B_1), (x_0, B_2) \in C$ and $L(\gamma) + L(\Gamma) < 5$ we have $L(C) \le 4$ and $d_{G_2}(B_1, B_2) \le 2$. Then we obtain

$$\begin{aligned} d_{G_2}(y_1, y_2) &\le d_{G_2}(y_1, B_1) + d_{G_2}(B_1, B_2) + d_{G_2}(B_2, y_2) \\ &\le d_{G_2}(y_1, B_1) + 2 + d_{G_2}(B_2, y_2) = L(\Gamma) < L(\gamma) = d_{G_2}(y_1, y_2). \end{aligned}$$

This is the contradiction we were looking for, and so, γ is a geodesic in $G_1 \oplus G_2$. □

Lemma 4.1.12. *Let G_1, G_2 be any graphs. Let y_1, y_2 be two midpoints in G_2 with $d_{G_2}(y_1, y_2) = 3$ and x_0 any fixed vertex in G_1. Then $\{x_0\} \times [y_1 y_2]$ is a geodesic in $G_1 \oplus G_2$ joining (x_0, y_1) and (x_0, y_2).*

Proof. Seeking for a contradiction assume that $\{x_0\} \times [y_1 y_2]$ is not a geodesic in $G_1 \oplus G_2$. Let Γ be a geodesic in $G_1 \oplus G_2$ joining (x_0, y_1) and (x_0, y_2)(i.e., $L(\Gamma) < L(\{x_0\} \times [y_1 y_2]) = 3$). Then, Γ is not contained in $\{x_0\} \oplus G_2$ and there exists $v \in V(\Gamma)$ such that $v \notin V(\{x_0\} \oplus G_2)$. Hence, $\Gamma = [(x_0, y_1)v] \cup [v(x_0, y_2)]$ and we conclude $L(\Gamma) \ge 3$. This is the contradiction we were looking for. □

4.2 Hyperbolicity constant in the Cartesian sum graphs

In this section we obtain some bounds for the hyperbolicity constant of the Cartesian sum of graphs. These bounds allow to prove that the Cartesian sum is always hyperbolic with a small hyperbolicity constant, except if G_1 or G_2 is the trivial graph.

Theorem 4.2.1. *For every non-trivial graphs G_1, G_2, we have*

$$\delta(G_1 \oplus G_2) = \max\{\delta(\Gamma_1 \oplus \Gamma_2) : \Gamma_i \textit{ is an isometric subgraph of } G_i \textit{ and } \Gamma_i$$
$$\textit{is non-trivial for } i = 1, 2\}.$$

Proof. By Proposition 4.1.6 and Lemma 1.3.3 we have $\delta(G_1 \oplus G_2) \geq \delta(\Gamma_1 \oplus \Gamma_2)$ for any isometric subgraph Γ_i of G_i with Γ_i non-trivial for $i = 1, 2$. Besides, since any graph is an isometric subgraph of itself we obtain the equality by taking $\Gamma_1 = G_1$ and $\Gamma_2 = G_2$. □

The following result characterizes the hyperbolic Cartesian sums.

Theorem 4.2.2. *Let G_1 and G_2 be any graphs.*

(1) *If G_1 is a trivial graph, then the Cartesian sum $G_1 \oplus G_2$ is hyperbolic if and only if G_2 is hyperbolic. Furthermore,*

$$\delta(G_1 \oplus G_2) = \delta(G_2).$$

(2) *If G_2 is a trivial graph, then the Cartesian sum $G_1 \oplus G_2$ is hyperbolic if and only if G_1 is hyperbolic. Furthermore,*

$$\delta(G_1 \oplus G_2) = \delta(G_1).$$

(3) *For every non-trivial graphs G_1, G_2 the Cartesian sum $G_1 \oplus G_2$ is hyperbolic with*

$$1 \leq \delta(G_1 \oplus G_2) \leq 3/2.$$

Furthermore, the hyperbolicity constant $\delta(G_1 \oplus G_2)$ belongs to $\{1, 5/4, 3/2\}$.

Proof. Since $E_1 \oplus G \simeq G \oplus E_1 \simeq G$ for any graph G, the Cartesian sum of $E_1 \oplus G$ and $G \oplus E_1$ are hyperbolic if and only if G is hyperbolic.

Assume now that G_1 and G_2 are non-trivial graphs. Thus, there is a subgraph P_2^i in G_i isomorphic to an edge, for $i = 1, 2$. Hence, by Theorem 4.2.1 and Example 4.2.4 we have $\delta(G_1 \oplus G_2) \geq \delta(P_2^1 \oplus P_2^2) = 1$. Corollary 4.1.8 gives $\operatorname{diam} G_1 \oplus G_2 \leq \operatorname{diam} V(G_1 \oplus G_2) + 1 \leq 3$ and by Lemma 1.3.7 we have that $\delta(G_1 \oplus G_2) \leq 3/2$. The other statement is consequence of Theorem 1.3.12. □

Theorems 4.2.6 and 4.2.7 show that the inequalities in Theorem 4.2.2 are attained for many graphs.

The following results give the hyperbolicity constant of some Cartesian sum of graphs. The first and second examples are direct consequences of Remark 4.1.4, [27, Theorem 3.24 and Corollary 3.25].

Example 4.2.3. *Let C_n be the cycle graph with $n \geq 3$. Then*

$$\delta(C_n \oplus P_2) = \begin{cases} 1, & \text{if } n = 3, 4 \\ 5/4, & \text{if } n = 5 \quad \text{or} \quad n \geq 10, \\ 3/2, & \text{if } n = 6, 7, 8, 9. \end{cases}$$

Example 4.2.4. *Let G be any tree. Then*

$$\delta(G \oplus P_2) = \begin{cases} 0, & \text{if } G \simeq E_1, \\ 1, & \text{if } 1 \leq \operatorname{diam} G \leq 2, \\ 5/4, & \text{if } \operatorname{diam} G \geq 3. \end{cases}$$

Example 4.2.5. *Let K_m, K_n be the complete graphs with m, n vertices, respectively, and $mn \geq 4$. Then $K_m \oplus K_n$ is isomorphic to K_{mn} and $\delta(K_m \oplus K_n) = 1$.*

In what follows we denote by π_i the projection $\pi_i : V(G_1 \oplus G_2) \to V(G_i)$ for $i \in \{1, 2\}$.

Theorem 4.2.6. *Let G_1, G_2 be any graphs. Then $\delta(G_1 \oplus G_2) = 1$ if and only if we have either:*

(1) *G_1 is trivial and $\delta(G_2) = 1$,*

(2) *G_2 is trivial and $\delta(G_1) = 1$,*

(3) *$1 \leq \operatorname{diam} G_1 \leq 2$ and $1 \leq \operatorname{diam} G_2 \leq 2$.*

Proof. If G_1 (respectively, G_2) is trivial, then $G_1 \oplus G_2 \simeq G_2$ (respectively, $G_1 \oplus G_2 \simeq G_1$) and $\delta(G_1 \oplus G_2) = 1$ if and only if (1) holds (respectively, (2) holds).

Assume now that G_1 and G_2 are non-trivial graphs. Thus, $\operatorname{diam} G_1 \geq 1$ and $\operatorname{diam} G_2 \geq 1$. Seeking for a contradiction assume that $\delta(G_1 \oplus G_2) = 1$ and $\operatorname{diam} G_1 > 2$ or $\operatorname{diam} G_2 > 2$. By symmetry we can assume that $\operatorname{diam} G_1 > 2$. Since $\operatorname{diam} G_1 \geq 5/2$, there exist $x_0, x_1 \in J(G_1)$ such that $d_{G_1}(x_0, x_1) = 5/2$. Fix $y_0 \in V(G_2)$. Lemma 4.1.11 gives that $\gamma_1 := [x_0 x_1] \times \{y_0\}$ is a geodesic in $G_1 \oplus G_2$ joining the points (x_0, y_0) and (x_1, y_0). Now we show a geodesic bigon B in $G_1 \oplus G_2$ with $\delta(B) = 5/4$. Without loss of generality we can assume that there exist $A_1, A_2 \in V(G_1 \oplus \{y_0\})$ such that $(x_0, y_0), A_1 \in V(\gamma_1)$ and (x_1, y_0) is the midpoint of $[A_1, A_2]$. Since G_2 is non-trivial, there exists $y_1 \in V(G_2)$ such that $d_{G_2}(y_0, y_1) = 1$. Fix $A_3 \in V(G_1 \oplus \{y_1\})$ and define $B := \{\gamma_1, \gamma_2\}$ with

$$\gamma_2 := [(x_0, y_0), A_3] \cup [A_3, A_2] \cup [A_2(x_1, y_0)].$$

If p is the midpoint of γ_1, then $\delta(B) = d_{G_1 \oplus G_2}\big(p, \gamma_2\big) = 5/4$ and we have $1 = \delta(G_1 \oplus G_2) \geq \delta(B) = 5/4$, which is a contradiction. Therefore, (3) holds.

Finally, assume that (3) holds. We are going to prove that $\operatorname{diam} G_1 \oplus G_2 = 2$. Seeking for a contradiction assume that there exist $u \in V(G_1 \oplus G_2), [v, w] \in E(G_1 \oplus G_2)$ with $d_{G_1 \oplus G_2}(u, [v, w]) = 2$. We have three cases.

First case: $\pi_1(v) = \pi_1(w)$. Then $[\pi_2(v), \pi_2(w)] \in E(G_2)$ and Proposition 4.1.5 gives $d_{G_2}(\pi_2(u), [\pi_2(v), \pi_2(w)]) = 2$ since $d_{G_1 \oplus G_2}(u, [v, w]) = 2$, which is a contradiction since that $\operatorname{diam} G_2 \leq 2$.

Second case: $d_{G_1}(\pi_1(v), \pi_1(w)) = 1$. Proposition 4.1.5 gives $d_{G_1}(\pi_1(u), [\pi_1(v), \pi_1(w)]) = 2$ since
$d_{G_1 \oplus G_2}(u, [v, w]) = 2$, which contradicts $\operatorname{diam} G_1 \leq 2$.

Third case: $d_{G_1}(\pi_1(v), \pi_1(w)) = 2$. Thus, $d_{G_2}(\pi_2(v), \pi_2(w)) = 1$ and Proposition 4.1.5 gives
$d_{G_2}(\pi_2(u), [\pi_2(v), \pi_2(v)]) = 2$ since $d_{G_1 \oplus G_2}(u, [v, w]) = 2$, which is not possible since that $\operatorname{diam} G_2 \leq 2$.

Thus, we conclude that $\operatorname{diam} G_1 \oplus G_2 = 2$ and Lemma 1.3.7 and Theorem 4.2.2 give $\delta(G_1 \oplus G_2) = 1$. □

Note that if $1 \leq \operatorname{diam} G \leq 2$, then G is isomorphic to a complete graph K_2 or K_3, or it verifies $\operatorname{diam} G = 2$.

Theorem 4.2.7. *Let G_1, G_2 be any graphs. If $\operatorname{diam} V(G_i) \geq 3$ for $i \in \{1, 2\}$, then $\delta(G_1 \oplus G_2) = 3/2$.*

Proof. Since $\operatorname{diam} V(G_i) \geq 3$, there is an isometric subgraph in G_i isomorphic to a path graph P_4^i with 4 vertices for $i \in \{1, 2\}$; denote by $\{v_1^i, v_2^i, v_3^i, v_4^i\}$ the vertices of P_4^i with $[v_j^i, v_{j+1}^i] \in E(P_4^i)$ for $i \in \{1, 2\}$ and $1 \leq j \leq 3$. Now we show a geodesic bigon B in $P_4^1 \oplus P_4^2$ with $\delta(B) = 3/2$. Let x and y be the midpoints of $[(v_1^1, v_1^2), (v_2^1, v_1^2)]$ and $[(v_4^1, v_3^2), (v_4^1, v_4^2)]$, respectively. Hence, Proposition 4.1.5 gives $d_{P_4^1 \oplus P_4^2}(x, y) = 3$. Define $B := \{\gamma_1, \gamma_2\}$ with

$$\gamma_1 := [x(v_2^1, v_1^2)] \cup [(v_2^1, v_1^2), (v_1^1, v_4^2)] \cup [(v_1^1, v_4^2), (v_4^1, v_3^2)] \cup [(v_4^1, v_3^2)y]$$

and

$$\gamma_2 := [x(v_1^1, v_1^2)] \cup [(v_1^1, v_1^2), (v_3^1, v_2^2)] \cup [(v_3^1, v_2^2), (v_4^1, v_4^2)] \cup [(v_4^1, v_4^2)y].$$

If p is the midpoint of γ_1, then $d_{P_4^1 \oplus P_4^2}(p, \gamma_2) = 3/2$ and we have $\delta(P_4^1 \oplus P_4^2) \geq \delta(B) = 3/2$. Thus, Theorems 4.2.1 and 4.2.2 give $3/2 \leq \delta(P_4^1 \oplus P_4^2) \leq \delta(G_1 \oplus G_2) \leq 3/2$ and we conclude that $\delta(G_1 \oplus G_2) = 3/2$. □

We have the following direct consequence.

Corollary 4.2.8. *For every infinite graphs G_1, G_2 we have $\delta(G_1 \oplus G_2) = 3/2$.*

Lemma 4.2.9. *Let G_1, G_2 be any graphs. If $\operatorname{diam} V(G_1) \leq 2$, then $\delta(G_1 \oplus G_2) \geq \delta(G_1)$.*

Proof. By Theorem 1.3.13 there exist a geodesic triangle $T = \{x, y, z\}$ in G_1 that is a cycle with $x, y, z \in J(G_1)$ and $p \in [xy]$ with $d_{G_1}(p, [xz] \cup [zy]) = \delta(T) = \delta(G_1)$. Since $\operatorname{diam} V(G_1) \leq 2$ we have that $\operatorname{diam} G_1 \leq 3$ and Lemma 1.3.7 gives $\delta(G_1) \leq 3/2$. Hence, each one of the lengths $L([xy]), L([yz]), L([zx])$ is either 3 or at most 5/2. By Lemmas 4.1.11 and 4.1.12 we have that $T \times \{v\}$ is a geodesic triangle in $G_1 \oplus G_2$ for any fixed $v \in V(G_2)$.

Since $L([xy]) \le 3$, if $s \in [xz] \cup [zy]$ and $t \in [xy]$ with $d_{G_1}(t,s) = d_{G_1}(t, [xz] \cup [zy])$, then $d_{G_1}(t,s) \le 3/2$. Hence, Lemma 4.1.11 gives

$$d_{G_1}(t,s) = d_{G_1 \oplus G_2}((t,v),(s,v)) = d_{G_1 \oplus G_2}((t,v),([xz] \cup [zy] \times \{v\})).$$

A similar result holds for $[xz]$ and $[yz]$. Therefore, $\delta(G_1 \oplus G_2) \ge \delta(T \times \{v\}) = \delta(T) = \delta(G_1)$. □

Corollary 4.2.10. *Let G_1, G_2 be any graphs. If* $\operatorname{diam} V(G_i) \le 2$ *for $i = 1, 2$, then*

$$\delta(G_1 \oplus G_2) \ge \max\{\delta(G_1), \delta(G_2)\}.$$

Corollary 4.2.11. *Let G_1, G_2 be any graphs. If* $\operatorname{diam} V(G_1) = 2$ *and $\delta(G_1) = 3/2$, then $\delta(G_1 \oplus G_2) = 3/2$.*

Corollary 4.2.12. *Let G_1, G_2 be any graphs. If $\delta(G_1) > 1$, then $\delta(G_1 \oplus G_2) > 1$.*

Proof. Since $\delta(G_1) > 1$ we have that $\operatorname{diam} V(G_1) \ge 2$. If $\operatorname{diam} V(G_1) = 2$, then Lemma 4.2.9 gives the result. If $\operatorname{diam} V(G_1) \ge 3$, then Theorem 4.2.6 gives the result. □

If A is a subset of the graph G, we denote by $V(A)$ the set of vertices of G in A, i.e., $V(A) = V(G) \cap A$.

Theorem 4.2.13. *Let G_1, G_2 be any graphs with* $\operatorname{diam} V(G_2) = 2$.

(1) *If* $\operatorname{diam} V(G_1) = 1$*, then $\delta(G_1 \oplus G_2) = 3/2$ if and only if $\delta(G_2) = 3/2$.*

(2) *If* $\operatorname{diam} V(G_1) = 2$ *and we have $\delta(G_1) = 3/2$ or $\delta(G_2) = 3/2$, then $\delta(G_1 \oplus G_2) = 3/2$.*

Proof. If $\operatorname{diam} V(G_1) = 2$ and besides $\delta(G_1) = 3/2$ or $\delta(G_2) = 3/2$, then Corollary 4.2.11 gives $\delta(G_1 \oplus G_2) = 3/2$ since $\operatorname{diam} V(G_2) = 2$.

If $\operatorname{diam} V(G_1) = 1$ and $\delta(G_2) = 3/2$, then Corollary 4.2.11 gives $\delta(G_1 \oplus G_2) = 3/2$.

Assume now that $\operatorname{diam} V(G_1) = 1$ and $\delta(G_1 \oplus G_2) = 3/2$. Lemma 1.3.7 gives $\delta(G_2) \le 3/2$ since $\operatorname{diam} V(G_2) = 2$. We show now that $\delta(G_2) \ge 3/2$. By Theorem 1.3.13 there exist a geodesic triangle $T = \{x, y, z\}$ in $G_1 \oplus G_2$ that is a cycle with $x, y, z \in J(G_1 \oplus G_2)$ and $A_3 \in [xy]$ with $d_{G_1 \oplus G_2}(A_3, [xz] \cup [zy]) = \delta(T) = \delta(G_1 \oplus G_2) = 3/2$. Since $d_{G_1 \oplus G_2}(A_3, \{x, y\}) \ge d_{G_1 \oplus G_2}(A_3, [xz] \cup [zy]) = 3/2$ we have that $L([xy]) = 3$. Corollary 4.1.9 gives $x, y \in J(G_1 \oplus G_2) \setminus V(G_1 \oplus G_2)$ and $A_3 \in V(G_1 \oplus G_2)$ with $d_{G_1 \oplus G_2}(A_3, x) = d_{G_1 \oplus G_2}(A_3, y) = 3/2$. Without loss of generality we can assume that $x \in [A_1, A_2]$ and $y \in [A_4, A_5]$ with $A_2, A_4 \in [xy]$. Since $L([xy]) = 3$ and $d_{G_1 \oplus G_2}(A_3, [xz] \cup [zy]) = 3/2$ we have

$$d_{G_1 \oplus G_2}(\{A_1, A_2\}, \{A_4, A_5\}) = 2$$

and

$$d_{G_1 \oplus G_2}(A_3, V([xz] \cup [zy])) = 2.$$

Since $A_3 \in V(G_1 \oplus G_2)$, we have $A_3 \in V(\{v\} \oplus G_2)$ with $v \in V(G_1)$. Since $\operatorname{diam} V(G_1) = 1$, we have $d_{G_1 \oplus G_2}(A_3, A) = 1$ for every vertex $A \notin V(\{v\} \oplus G_2)$. Thus, $V([xz] \cup [zy]) \subset V(\{v\} \oplus G_2)$ since $d_{G_1 \oplus G_2}(A_3, V([xz] \cup [zy])) = 2$. If $A_2 \notin V(\{v\} \oplus G_2)$ (respectively, $A_4 \notin V(\{v\} \oplus G_2)$), then $d_{G_1 \oplus G_2}(A_2, A_5) = 1$ (respectively, $d_{G_1 \oplus G_2}(A_4, A_1) = 1$) by Proposition 4.1.5 and this is a contradiction since $d_{G_1 \oplus G_2}(\{A_1, A_2\}, \{A_4, A_5\}) = 2$. Thus, $A_2, A_4 \in V(\{v\} \oplus G_2)$. Finally, $V(T) \subset V(\{v\} \oplus G_2)$ and consequently $T \subset \{v\} \oplus G_2$. Since $d_{G_1 \oplus G_2}(x, y) = 3$, we have $\operatorname{diam} G_1 \oplus G_2 = 3$ and $\operatorname{diam} V(G_1 \oplus G_2) = 2$ and, consequently, T is a geodesic triangle in $\{v\} \oplus G_2$. Hence, $3/2 = \delta(T) \le \delta(\{v\} \oplus G_2) = \delta(G_2)$ and we can conclude that $\delta(G_2) = 3/2$. □

One can think that the converse of (2) in Theorem 4.2.13 holds. However, this is not true since $\delta(C_5 \oplus C_5) = 3/2$ (see Theorem 4.2.17) and $\delta(C_5) = 5/4$.

Theorem 4.2.14. *Let G_1, G_2 be any trees. Then*

$$\delta(G_1 \oplus G_2) = \begin{cases} 0, & \text{if } G_1 \simeq E_1 \quad \text{or} \quad G_2 \simeq E_1, \\ 1, & \text{if } 1 \le \operatorname{diam} G_1 \le 2 \quad \text{and} \quad 1 \le \operatorname{diam} G_2 \le 2, \\ 5/4, & \text{if } 1 \le \operatorname{diam} G_1 \le 2 \quad \text{and} \quad \operatorname{diam} G_2 \ge 3, \\ 3/2, & \text{if } \operatorname{diam} G_1 \ge 3 \quad \text{and} \quad \operatorname{diam} G_2 \ge 3. \end{cases}$$

Proof. If $G_1 \simeq E_1$ or $G_2 \simeq E_1$, then Remark 4.1.3 gives the result since $\delta(G) = 0$ for every tree G.

If $1 \le \operatorname{diam} G_1 \le 2$ and $1 \le \operatorname{diam} G_2 \le 2$, then Theorem 4.2.6 gives $\delta(G_1 \oplus G_2) = 1$.

If $\operatorname{diam} G_1 = 1$ and $\operatorname{diam} G_2 \ge 3$, then Example 4.2.4 gives the result.

If $\operatorname{diam} G_1 = 2$ and $\operatorname{diam} G_2 \ge 3$, then Theorems 1.3.12, 4.2.2 and 4.2.6 give $\delta(G_1 \oplus G_2) \in \{5/4, 3/2\}$. Seeking for a contradiction assume that $\delta(G_1 \oplus G_2) = 3/2$. By Theorem 1.3.13 there exist a geodesic triangle $T = \{x, y, z\}$ in $G_1 \oplus G_2$ that is a cycle with $x, y, z \in J(G_1 \oplus G_2)$ and $p \in [xy]$ with $d_{G_1 \oplus G_2}(p, [xz] \cup [zy]) = \delta(T) = 3/2$. Since $d_{G_1 \oplus G_2}(p, \{x, y\}) \ge d_{G_1 \oplus G_2}(p, [xz] \cup [zy]) = 3/2$ we have that $L([xy]) = 3$. Corollary 4.1.9 gives $x, y \in J(G_1 \oplus G_2) \setminus V(G_1 \oplus G_2)$ and $p \in V(G_1 \oplus G_2)$ with $d_{G_1 \oplus G_2}(p, x) = d_{G_1 \oplus G_2}(p, y) = 3/2$. Without loss of generality we can assume that $x \in [A_1, A_2]$ and $y \in [A_3, A_4]$ with $A_1, A_3 \in [xy]$. Since $L([xy]) = 3$ we have that $d_{G_1 \oplus G_2}(\{A_1, A_2\}, \{A_3, A_4\}) = 2$. Let W be the point in $V([xz] \cup [zy]) \setminus \{A_2, A_4\}$ such that $d_{G_1 \oplus G_2}(A_2, W) = 1$. Since $d_{G_1 \oplus G_2}(p, [xz] \cup [zy]) = 3/2$ we have $d_{G_1 \oplus G_2}(p, V([xz] \cup [zy])) = 2$ and, in particular, $d_{G_1 \oplus G_2}(p, \{A_2, A_4, W\}) = 2$. Since G_1 is a tree with $\operatorname{diam} G_1 = 2$, there exists a unique $v \in V(G_1)$ with $d_{G_1}(v, w) = 1$ for every $w \in V(G_1) \setminus \{v\}$; note that $d_{G_1 \oplus G_2}((v, u_1), (w, u_2)) = 1$ for every $w \in V(G_1) \setminus \{v\}$ and $u_1, u_2 \in V(G_2)$. Hence, if $A_i \in \{v\} \oplus G_2$ for some $i \in \{1, 2, 3, 4\}$, then $A_i \in \{v\} \oplus G_2$ for every $i \in \{1, 2, 3, 4\}$. Assume first that $p \in \{v\} \oplus G_2$. Therefore, $V([xz] \cup [zy]) \setminus \{v\} \oplus G_2 = \emptyset$ and $A_i \in \{v\} \oplus G_2$ for every $i \in \{1, 2, 3, 4\}$. Thus, $T \subseteq \{v\} \oplus G_2$, and this is a contradiction since $\delta(G_2) = 0$ and $d_{G_1 \oplus G_2}(p, [xz] \cup [zy]) = 3/2$. Assume that $p \in \{w\} \oplus G_2$, where $w \in V(G_1) \setminus \{v\}$. Since $d_{G_1 \oplus G_2}(p, A_1) = d_{G_1 \oplus G_2}(p, A_3) = 1$ and $d_{G_1 \oplus G_2}(p, \{A_2, A_4, W\}) = 2$ we have $d_{G_2}(\pi_2(p), \pi_2(A_1)) = d_{G_2}(\pi_2(p), \pi_2(A_3)) = 1$ and $d_{G_2}(\pi_2(p), \pi_2(A_2)) = d_{G_2}(\pi_2(p), \pi_2(A_4)) =$

$d_{G_2}(\pi_2(p), \pi_2(W)) = 2$, this is a contradiction since $d_{G_2}(\pi_2(A_2), \pi_2(W)) = 1$. Finally, we have $\delta(G_1 \oplus G_2) \neq 3/2$ and we conclude that $\delta(G_1 \oplus G_2) = 5/4$.

If $\operatorname{diam} G_1 \geq 3$ and $\operatorname{diam} G_2 \geq 3$, then Theorem 4.2.7 gives the result. □

Corollary 4.2.15. *Let P_n, P_m be two path graphs. Then*

$$\delta(P_n \oplus P_m) = \begin{cases} 0, & \text{if } n = 1 \quad \text{or} \quad m = 1, \\ 1, & \text{if } n = 2,3 \quad \text{and} \quad m = 2,3, \\ 5/4, & \text{if } n = 2,3 \quad \text{and} \quad m \geq 4, \\ 3/2, & \text{if } n \geq 4 \quad \text{and} \quad m \geq 4. \end{cases}$$

Proposition 4.2.16. *Let G be any graph with $\operatorname{diam} V(G) = 2$. Then $\delta(G) \leq 3/2$, and $\delta(G) = 3/2$ if and only if $G \in \mathcal{F}$.*

Proof. By Lemma 1.3.7 and $\operatorname{diam} G \leq \operatorname{diam} V(G) + 1 = 3$, we have $\delta(G) \leq 3/2$.

If $G \in \mathcal{F}$, then Lemma 3.2.18 gives $\delta(G) \geq 3/2$, and we conclude $\delta(G) = 3/2$.

Finally, assume that $\delta(G) = 3/2$. By Theorem 1.3.13 there exist a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ with $d_G(p, [xz] \cup [zy]) = \delta(T) = 3/2$. Since $d_G(p, \{x, y\}) \geq d_G(p, [xz] \cup [zy]) = 3/2$ and $\operatorname{diam} G \leq 3$ we have that $d_G(p, \{x, y\}) = 3/2, L([xy]) = 3, L([yz]), L([zx]) \leq 3$. Thus, $x, y \in J(G) \setminus V(G)$ and $p \in [xy] \cap V(G)$. By Lemma 3.2.18 we conclude that $G \in \mathcal{F}$. □

Theorem 4.2.17. *Let C_n, C_m be two cycle graphs. Then*

$$\delta(C_n \oplus C_m) = \begin{cases} 1, & \text{if } n = 3,4 \quad \text{and} \quad m = 3,4, \\ 5/4, & \text{if } n = 3,4 \quad \text{and} \quad m = 5 \quad \text{or} \quad m \geq 10, \\ 3/2, & \text{if } n = 3,4 \quad \text{and} \quad m = 6,7,8,9, \\ 3/2, & \text{if } n \geq 5 \quad \text{and} \quad m \geq 5. \end{cases}$$

Proof. If $n = 3,4$ and $m = 3,4$, then Theorem 4.2.6 gives $\delta(C_n \oplus C_m) = 1$.

If $n = 3,4$ and $m = 5$ or $m \geq 10$, then Theorems 4.2.2 and 4.2.6 give $\delta(C_n \oplus C_m) \in \{5/4, 3/2\}$. Seeking for a contradiction assume that $\delta(C_n \oplus C_m) = 3/2$. By Theorem 1.3.13 there exist a geodesic triangle $T = \{x, y, z\}$ in $C_n \oplus C_m$ that is a cycle with $x, y, z \in J(C_n \oplus C_m)$ and $p \in [xy]$ with $d_{C_n \oplus C_m}(p, [xz] \cup [zy]) = \delta(T) = 3/2$. Since $d_{C_n \oplus C_m}(p, \{x, y\}) \geq d_{C_n \oplus C_m}(p, [xz] \cup [zy]) = 3/2$ we have that $L([xy]) = 3$ and by Corollary 4.1.9 we have that x, y are midpoints in $C_n \oplus C_m$ and $p \in [xy] \cap V(C_n \oplus C_m)$. We have $x \in [A_1, A_2], y \in [A_3, A_4]$ with $A_2, A_3 \in [xy]$. Since $d_{C_n \oplus C_m}(x, y) = 3$ and $d_{C_n \oplus C_m}(p, [xz] \cup [zy]) = 3/2$ we have $d_{C_n \oplus C_m}(\{A_1, A_2\}, \{A_3, A_4\}) = 2$ and $d_{C_n \oplus C_m}(p, V([xz] \cup [zy])) = 2$. Let $v \in V(C_n)$ be the vertex with $p \in V(\{v\} \oplus C_m)$. If $n = 4$, then Proposition 4.1.5 gives $V([xz] \cup [zy]) \subset V(\{v\} \oplus C_m) \cup V(\{w\} \oplus C_m)$ with $w \in V(C_4)$ such that $d_{C_4}(v, w) = 2$. Since $d_{C_n \oplus C_m}(A_2, A_4) = 2$ and $d_{C_n \oplus C_m}(A_1, A_3) = 2$, we obtain $A_2, A_3 \in V(\{v\} \oplus C_m) \cup V(\{w\} \oplus C_m)$. Hence, we conclude $V(T) \subset V(\{v\} \oplus C_m) \cup V(\{w\} \oplus C_m)$. If $n = 3$, then a similar argument gives $V(T) \subset V(\{v\} \oplus C_m)$. Consequently, if $n =$

$3,4$ and $A,B \in V(T)$ with $[A,B] \in E(T)$, then Proposition 4.1.5 gives $[\pi_2(A),\pi_2(B)] \in E(C_m)$. By Proposition 4.1.5 we have $d_{C_m}(\{\pi_2(A_1),\pi_2(A_2)\},\{\pi_2(A_3),\pi_2(A_4)\}) = 2$ since $d_{C_n\oplus C_m}(\{A_1,A_2\},\{A_3,A_4\}) = 2$. Let $W \in V([xz]\cup[zy])$, Proposition 4.1.5 gives $\pi_2(W) \notin \{\pi_2(A_2),\pi_2(A_3)\}$ since $d_{C_n\oplus C_m}(p,V([xz]\cup[zy])) = 2$. Therefore, $\pi_2(W) \neq \pi_2(p)$ by continuity. Hence, there exist a geodesic triangle $T_1 = \{\pi_2(x),\pi_2(y),\pi_2(z)\} \subseteq \pi_2(T)$ in C_m with $\pi_2(x),\pi_2(y),\pi_2(z) \in J(C_m)$, $\pi_2(p) \in [\pi_2(x)\pi_2(y)]$, $L([\pi_2(x)\pi_2(y)]) = L([xy]) = 3$, $L([\pi_2(x)\pi_2(z)]) \le L([xz]) \le 3$, $L([\pi_2(z)\pi_2(y)]) \le L([zy]) \le 3$ and $d_{C_m}(\pi_2(p),[\pi_2(x)\pi_2(z)]\cup[\pi_2(z)\pi_2(y)]) = \delta(T_1) = 3/2$. Corollary 3.2.19 gives the contradiction we were looking for since $C_m \notin \mathcal{F}$. Thus, we conclude that $\delta(C_n\oplus C_m) = 5/4$.

If $n = 3,4$ and $m = 6,7,8,9$, then by Theorem 4.2.1 and Example 4.2.3 we have $\delta(C_n\oplus C_m) \ge \delta(P_2\oplus C_m) = \delta(C_m\oplus P_2) = 3/2$ since P_2 is an isometric subgraph of C_n, and Theorem 4.2.2 gives $\delta(C_n\oplus C_m) \le 3/2$. Thus, we conclude that $\delta(C_n\oplus C_m) = 3/2$.

Finally, we deal with the case $n \ge 5$ and $m \ge 5$. Consider C_n as the cycle graph with vertices $\{u_1,u_2,u_3,u_4,u_5,\dots,u_n\}$ and edges $[u_n,u_1]$ and $[u_j,u_{j+1}]$ for $1 \le j < n$ and C_m as the cycle graph with vertices $\{v_1,v_2,v_3,v_4,v_5,\dots,v_m\}$ and edges $[v_m,v_1]$ and $[v_j,v_{j+1}]$ for $1 \le j < m$. Let x and y be the midpoints of $[(u_3,v_1),(u_4,v_1)]$ and $[(u_1,v_3),(u_1,v_4)]$, respectively. Proposition 4.1.5 gives $d_{C_n\oplus C_m}(x,y) = 3$. Now we show a geodesic bigon B in $C_n\oplus C_m$ with $\delta(B) = 3/2$. Define $B := \{\gamma_1,\gamma_2\}$ with

$$\gamma_1 := [x(u_4,v_1)]\cup[(u_4,v_1),(u_2,v_2)]\cup[(u_2,v_2),(u_1,v_4)]\cup[(u_1,v_4)y]$$

and

$$\gamma_2 := [x(u_3,v_1)]\cup[(u_3,v_1),(u_4,v_4)]\cup[(u_4,v_4),(u_1,v_3)]\cup[(u_1,v_3)y].$$

If p is the midpoint of γ_2, then $d_{C_n\oplus C_m}\big(p,\gamma_1\big) = 3/2$ and we have $\delta(C_n\oplus C_m) \ge \delta(B) = d_{C_n\oplus C_m}\big(p,\gamma_1\big) = 3/2$. Thus, Theorem 4.2.2 gives $\delta(C_n\oplus C_m) \le 3/2$ and we conclude that $\delta(C_n\oplus C_m) = 3/2$. □

Remark 4.2.18. *Since $\delta(C_n\oplus C_m) = \delta(C_m\oplus C_n)$, Theorem 4.2.17 provides the precise value of $\delta(C_n\oplus C_m)$ for every $n,m \ge 3$.*

Theorem 4.2.19. *Let G_1, G_2 be any graphs.*

(1) *If $G_1 \in \mathcal{F}$ and G_2 is non-trivial, then $\delta(G_1\oplus G_2) = 3/2$.*

(2) *If $\delta(G_1\oplus G_2) = 3/2$, $\operatorname{diam} V(G_1) = 2$ and $\operatorname{diam} V(G_2) = 1$, then $G_1 \in \mathcal{F}$.*

Proof. Assume first that $G_1 \in \mathcal{F}$ and G_2 is non-trivial. Note that G_1 is a non-trivial graph since it belong to $\mathcal{F}$. By Lemma 3.2.18 there is a geodesic triangle $T = \{x,y,z\}$ in G_1 that is a cycle with $x,y,z \in J(G_1)$, $L([xy]), L([yz]), L([zx]) \le 3$ and $\delta(T) = 3/2 = d_{G_1}(p,[yz]\cup[zx])$ for some $p \in [xy]\cap V(G_1)$. Since $d_{G_1}(p,\{x,y\}) \ge d_{G_1}(p,[yz]\cup[zx]) = 3/2$ and $d_{G_1}(x,y) \le 3$, we obtain $d_{G_1}(x,y) = 3$. Then $x,y \in J(G_1)\setminus V(G_1)$, since $p \in V(G_1)$, and we have $d_{G_1\oplus G_2}((x,v),(y,v)) = d_{G_1}(x,y) = 3$ for any fixed $v \in V(G_2)$. Since $x,y \in J(G_1)\setminus V(G_1)$, $z \in J(G_1)$, $d_{G_1}(y,z) \le 3$ and $d_{G_1}(z,x) \le 3$, a similar argument gives $d_{G_1\oplus G_2}((y,v),(z,v)) =$

$d_{G_1}(y,z)$ and $d_{G_1 \oplus G_2}((z,v),(x,v)) = d_{G_1}(z,x)$. Hence, $T \times \{v\}$ is a geodesic triangle in $G_1 \oplus G_2$ and $3/2 = \delta(T) = \delta(T \times \{v\}) \le \delta(G_1 \oplus G_2)$, and we conclude $\delta(G_1 \oplus G_2) = 3/2$ by Theorem 4.2.2, since G_1 and G_2 are non-trivial.

Finally, if $\delta(G_1 \oplus G_2) = 3/2$, $\operatorname{diam} V(G_1) = 2$ and $\operatorname{diam} V(G_2) = 1$, then $\delta(G_1) = 3/2$ by Theorem 4.2.13 (1). Thus, Proposition 4.2.16 gives $G_1 \in \mathcal{F}$. □

One can think that the converse of (1) in Theorem 4.2.19 holds. However, this is not true, since the cycle graph C_5 does not belong to $\mathcal{F}$ and $\delta(C_5 \oplus C_5) = 3/2$ (see Theorem 4.2.17).

Finally, we have a characterization of the Cartesian sums with hyperbolicity constant 3/2 which does not involve properties of G_1 and G_2.

Theorem 4.2.20. *For any non-trivial graphs G_1, G_2, we have $\delta(G_1 \oplus G_2) = 3/2$ if and only if $G_1 \oplus G_2 \in \mathcal{F}$.*

Proof. By Proposition 4.1.7 we have $1 \le \operatorname{diam} V(G_1 \oplus G_2) \le 2$.

If $\operatorname{diam} V(G_1 \oplus G_2) = 1$, then $G_1 \oplus G_2$ is a complete graph. Hence, $\delta(G_1 \oplus G_2) = 1$ and $G_1 \oplus G_2 \notin \mathcal{F}$.

If $\operatorname{diam} V(G_1 \oplus G_2) = 2$, then Proposition 4.2.16 provides the equivalence. □

4.3 Hyperbolicity in the complement of the Cartesian sum graphs

In this section we obtain an upper bound for the hyperbolicity constant of the complement of the Cartesian sum of two graphs.

Given any graph G, we denote by $\overline{G}$ the complement of G, defined as the graph with $V(\overline{G}) = V(G)$ and $e \in E(\overline{G})$ if and only if $e \notin E(G)$.

The followingl result which will be useful.

Lemma 4.3.1. *[57, 79, 88] For any graphs G_1 and G_2,*

$$\overline{G_1 \oplus G_2} = \overline{G_1} \boxtimes \overline{G_2}.$$

The next lemma follows from Theorem 2.1.5.

Lemma 4.3.2. *Let G_1, G_2 be any graphs and let Γ_1, Γ_2 be isometric subgraphs of G_1 and G_2, respectively. We have that $\Gamma_1 \boxtimes \Gamma_2$ is an isometric subgraph of $G_1 \boxtimes G_2$.*

The proof of the following lemma is similar to the proof of Theorem 4.2.1, using Lemma 4.3.2 instead of Proposition 4.1.6.

Lemma 4.3.3. *For any graphs G_1, G_2, we have*

$$\delta(G_1 \boxtimes G_2) = \max\{\delta(\Gamma_1 \boxtimes \Gamma_2) : \Gamma_i \text{ is an isometric subgraph of } G_i, \text{ for } i = 1, 2\}.$$

Theorem 4.3.4. *Let G_1, G_2 be any graphs. If* $\operatorname{diam} V(G_i) \geq 3$ *for* $i \in \{1, 2\}$, *then*

$$\frac{3}{2} \leq \delta(\overline{G_1 \oplus G_2}) \leq 2.$$

Proof. It is well known that if $\operatorname{diam} V(G) \geq 3$ for any graph G, then $\overline{G}$ is connected and $\operatorname{diam} V(\overline{G}) \leq 3$. Thus, Corollary 2.2.1 and Lemma 4.3.1 give $\delta(\overline{G_1 \oplus G_2}) \leq 2$.

If $\operatorname{diam} V(\overline{G}) = 1$, then $\overline{G}$ is a complete graph and consequently G is a disconnected graph. Hence, $\operatorname{diam} V(\overline{G}_1) \geq 2$ and $\operatorname{diam} V(\overline{G}_2) \geq 2$. Consequently, there is an isometric subgraph in $\overline{G}_i$ isomorphic to a path graph P_3^i with 3 vertices, for $i = 1, 2$. Lemmas 4.3.1 and 4.3.3 give $\delta(\overline{G_1 \oplus G_2}) = \delta(\overline{G}_1 \boxtimes \overline{G}_2) \geq \delta(P_3^1 \boxtimes P_3^2)$. Thus, $\delta(\overline{G_1 \oplus G_2}) \geq 3/2$ since $\delta(P_3^1 \boxtimes P_3^2) = 3/2$ by [24, Corollary 33]. $\square$

Chapter 5

Hyperbolicity of direct products of graphs

The direct product is clearly commutative and associative. Weichsel observed that $G_1 \times G_2$ is connected if and only if G_1 and G_2 are connected and G_1 or G_2 is not a bipartite graph [118]. Many different properties of direct product of graphs have been studied (sometimes with various different names, such as cardinal product, tensor product, Kronecker product, categorical product, conjunction,...). The study includes structural results [8, 18, 56, 65, 66, 67], hamiltonian properties [6, 74], and above all the well-known Hedetniemi's conjecture on chromatic number of direct product of two graphs (see [64] and [122]). Open problems in the area suggest that a deeper structural understanding of this product would be welcome.

5.1 Hyperbolic direct products

In order to study the hyperbolicity constant of the direct product of two graphs $G_1 \times G_2$, we will need bounds for the distance between two arbitrary points. We will use the definition given in [57].

Definition 5.1.1. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The* direct product *$G_1 \times G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.*

From the definition, it follows that the direct product of two graphs is commutative, i.e., $G_1 \times G_2 \simeq G_2 \times G_1$. Hence, the conclusion of every result in this section with some "non-symmetric" hypothesis also holds if we change the roles of G_1 and G_2 (see, e.g., Theorems 5.1.9, 5.1.10, 5.1.20, 5.1.22 and 5.1.31 and Corollary 5.1.32).

In what follows we denote by π_i the projection $\pi_i : V(G_1 \times G_2) \to V(G_i)$ for $i \in \{1, 2\}$. Note that, in fact, this projection is well defined as a map $\pi_i : G_1 \times G_2 \to G_i$ for $i \in \{1, 2\}$.

We collect some previous results of [57], which will be useful. If G is a graph and $u, u' \in V(G)$, then by a u, u'-*walk* in G we mean a path joining u and u' where repeating vertices is allowed.

Proposition 5.1.2. *[57, Proposition 5.7] Suppose (u, v) and (u', v') are vertices of the direct product $G_1 \times G_2$, and n is an integer for which G_1 has a u, u'-walk of length n and G_2 has a v, v'-walk of length n. Then $G_1 \times G_2$ has a walk of length n from (u, v) to (u', v'). The smallest such n (if it exists) equals $d_{G_1\times G_2}((u, v), (u', v'))$. If no such n exists, then $d_{G_1\times G_2}((u, v), (u', v')) = \infty$.*

Proposition 5.1.3. *[57, Proposition 5.8] Suppose x and y are vertices of $G_1 \times G_2$. Then*

$$d_{G_1\times G_2}(x, y) = \min\big\{n \in \mathbb{N} \,|\, \text{each factor } G_i \text{ has a } \pi_i(x), \pi_i(y)\text{-walk of length } n \text{ for } i = 1, 2\big\},$$

where it is understood that $d_{G_1\times G_2}(x, y) = \infty$ if no such n exists.

Corollary 5.1.4. *We have for every $(u, v), (u', v') \in V(G_1 \times G_2)$*

$$d_{G_1\times G_2}((u, v), (u', v')) \geq \max\big\{d_{G_1}(u, u'),\, d_{G_2}(v, v')\big\}$$

and, consequently,

$$\operatorname{diam} V(G_1 \times G_2) \geq \max\big\{ \operatorname{diam} V(G_1),\, \operatorname{diam} V(G_2)\big\}.$$

Furthermore, if $d_{G_1}(u, u')$ and $d_{G_2}(v, v')$ have the same parity, then

$$d_{G_1\times G_2}((u, v), (u', v')) = \max\big\{d_{G_1}(u, u'),\, d_{G_2}(v, v')\big\}$$

and, consequently,

$$\operatorname{diam} V(G_1 \times G_2) = \max\big\{ \operatorname{diam} V(G_1),\, \operatorname{diam} V(G_2)\big\}.$$

The following theorem, first proved by Weichsel in 1962, characterizes connectedness in direct products of two factors. As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

Theorem 5.1.5. *[57, Theorem 5.9] Suppose G_1 and G_2 are connected non-trivial graphs. If at least one of G_1 or G_2 has an odd cycle, then $G_1 \times G_2$ is connected. If both G_1 and G_2 are bipartite, then $G_1 \times G_2$ has exactly two connected components.*

Corollary 5.1.6. *[57, Corollary 5.10] A direct product of connected non-trivial graphs is connected if and only if at most one of the factors is bipartite. In fact, the product has $2^{\max\{k,1\}-1}$ connected components, where k is the number of bipartite factors.*

Proposition 5.1.7. *Let G_1 and G_2 be two unbounded graphs. Then $G_1 \times G_2$ is not hyperbolic.*

Proof. Since G_1 and G_2 are unbounded graphs, for each positive integer n there exist two geodesic paths $P_1 := [w_1, w_2] \cup [w_2, w_3] \cup \cdots \cup [w_{n-1}, w_n]$ in G_1 and $P_2 := [v_1, v_2] \cup [v_2, v_3] \cup \cdots \cup [v_{n-1}, v_n]$ in G_2. If n is odd, then we can consider the geodesic triangle T in $G_1 \times G_2$ defined by the following geodesics:

$$\gamma_1 := [(w_1, v_2), (w_2, v_1)] \cup [(w_2, v_1), (w_3, v_2)] \cup [(w_3, v_2), (w_4, v_1)] \cup \cdots \cup [(w_{n-1}, v_1), (w_n, v_2)],$$
$$\gamma_2 := [(w_1, v_2), (w_2, v_3)] \cup [(w_2, v_3), (w_1, v_4)] \cup [(w_1, v_4), (w_2, v_5)] \cup \cdots \cup [(w_1, v_{n-1}), (w_2, v_n)],$$
$$\gamma_3 := [(w_2, v_n), (w_3, v_{n-1})] \cup [(w_3, v_{n-1}), (w_4, v_{n-2})] \cup [(w_4, v_{n-2}), (w_5, v_{n-3})] \cup \cdots$$
$$\cdots \cup [(w_{n-1}, v_3), (w_n, v_2)],$$

Corollary 5.1.4 gives that $\gamma_1, \gamma_2, \gamma_3$ are geodesics.

Let $m := \frac{n+1}{2}$ and consider the vertex (w_m, v_{m+1}) in γ_3. For every vertex (w_i, v_j) in γ_1, $j \in \{1, 2\}$, we have $d_{G_1 \times G_2}((w_m, v_{m+1}), (w_i, v_j)) \geq d_{G_2}(v_{m+1}, v_j) \geq m + 1 - 2 = \frac{n-1}{2}$ by Corollary 5.1.4. We have for every vertex (w_i, v_j) in γ_2, $i \in \{1, 2\}$, by Corollary 5.1.4, $d_{G_1 \times G_2}((w_m, v_{m+1}), (w_i, v_j)) \geq d_{G_1}(w_m, w_i) \geq m - 2 = \frac{n-3}{2}$. Hence, $d_{G_1 \times G_2}\big((w_m, v_{m+1}), \gamma_1 \cup \gamma_2\big) \geq \frac{n-3}{2}$ and $\delta(G_1 \times G_2) \geq \delta(T) \geq \frac{n-3}{2}$. Since n is arbitrarily large, $G_1 \times G_2$ is not hyperbolic. □

Lemma 5.1.8. *Consider two graphs G_1 and G_2. If $f : V(G_1) \longrightarrow V(G_2)$ is an (α, β)-quasi-isometric embedding, then there exists an $(\alpha, \alpha+\beta)$-quasi-isometric embedding $g : G_1 \longrightarrow G_2$ with $g = f$ on $V(G_1)$. Furthermore, if f is ε-full, then g is $(\varepsilon + \frac{1}{2})$-full.*

Proof. For each $x \in G_1$, let us choose a closest point $v_x \in V(G_1)$ from x, and define $g(x) := f(v_x)$. Note that $v_x = x$ if $x \in V(G_1)$ and so $g = f$ on $V(G_1)$. Given $x, y \in G_1$, we have

$$d_{G_2}(g(x), g(y)) = d_{G_2}(f(v_x), f(v_y)) \leq \alpha d_{G_1}(v_x, v_y) + \beta \leq \alpha\big(d_{G_1}(x, y) + 1\big) + \beta,$$
$$d_{G_2}(g(x), g(y)) = d_{G_2}(f(v_x), f(v_y)) \geq \alpha^{-1} d_{G_1}(v_x, v_y) - \beta \geq \alpha^{-1}\big(d_{G_1}(x, y) - 1\big) - \beta,$$

and g is an $(\alpha, \alpha + \beta)$-quasi-isometric embedding, since $\alpha \geq 1 \geq \alpha^{-1}$.

Furthermore, if f is ε-full, then g is $(\varepsilon + \frac{1}{2})$-full since $g(G_1) = f(V(G_1))$. □

Given a graph G, let $g_I(G)$ denote the *odd girth* of G, this is, the length of the shortest odd cycle in G.

Theorem 5.1.9. *Let G_1 be a graph and G_2 be a non-trivial bounded graph with some odd cycle. Then, $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Proof. Let $v_0 \in V(G_2)$ such that v_0 is contained in an odd cycle C with $L(C) = g_I(G_2)$. Consider the map $i : V(G_1) \to V(G_1 \times G_2)$ such that $i(w) := (w, v_0)$ for every $w \in V(G_1)$.

By Corollary 5.1.4, for any pair of vertices $w_1, w_2 \in V(G_1)$,

$$d_{G_1}(w_1, w_2) \leq d_{G_1 \times G_2}\big((w_1, v_0), (w_2, v_0)\big).$$

Also, Proposition 5.1.3 gives the following.

If a geodesic joining w_1 and w_2 has even length, then

$$d_{G_1\times G_2}\big((w_1,v_0),(w_2,v_0)\big) = d_{G_1}(w_1,w_2).$$

If a geodesic joining w_1 and w_2 has odd length, then C defines a v_0, v_0-walk with odd length and

$$d_{G_1\times G_2}\big((w_1,v_0),(w_2,v_0)\big) \le \max\{d_{G_1}(w_1,w_2), g_I(G_2)\} \le d_{G_1}(w_1,w_2) + g_I(G_2).$$

Thus, i is a $\big(1, g_I(G_2)\big)$ quasi-isometric embedding.

Consider any $(w,v) \in V(G_1\times G_2)$. Then, if the geodesic joining v and v_0 has even length,

$$d_{G_1\times G_2}\big((w,v),(w,v_0)\big) = d_{G_2}(v,v_0).$$

If a geodesic joining v and v_0 has odd length, $[vv_0]\cup C$ defines a v, v_0-walk with even length. Therefore,

$$d_{G_1\times G_2}\big((w,v),(w,v_0)\big) \le d_{G_2}(v,v_0) + g_I(G_2).$$

Thus, i is $\big(\operatorname{diam}(V(G_2)) + g_I(G_2)\big)$-full.

Hence, by Lemma 5.1.8, there is a $\big(\operatorname{diam}(V(G_2))+g_I(G_2)+\frac{1}{2}\big)$-full $\big(1, g_I(G_2)+1\big)$-quasi-isometry, $j : G_1 \to G_1 \times G_2$, and $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic by Theorem 1.3.6. □

Theorem 5.1.10. *Let G_1 be a graph without odd cycles and G_2 be a non-trivial bounded graph without odd cycles. Then, $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Proof. Fix some vertex $w_0 \in V(G_1)$ and some edge $[v_1,v_2] \in E(G_2)$.

By Theorem 5.1.5, there are exactly two components in $G_1 \times G_2$. Since there are no odd cycles, there is no $(w_0,v_1),(w_0,v_2)$-walk in $G_1 \times G_2$. Thus, let us denote by $(G_1\times G_2)^1$ the component containing the vertex (w_0,v_1) and by $(G_1\times G_2)^2$ the component containing the vertex (w_0,v_2).

Consider $i : V(G_1) \to V(G_1\times G_2)^1$ defined as $i(w) := (w,v_1)$ for every $w \in V(G_1)$ such that every w_0, w-walk has even length and $i(w) := (w,v_2)$ for every $w \in V(G_1)$ such that every w_0, w-walk has odd length.

By Proposition 5.1.3, $d_{G_1\times G_2}\big(i(w_1), i(w_2)\big) = d_{G_1}(w_1,w_2)$ for every $w_1, w_2 \in V(G_1)$ and i is a $(1,0)$-quasi-isometric embedding.

Let $(w,v) \in V(G_1\times G_2)^1$. Let v_j with $j \in \{1,2\}$ such that every v, v_j-walk has even length. Then, by Proposition 5.1.3, $d_{G_1\times G_2}\big((w,v),(w,v_j)\big) = d_{G_2}(v,v_j) \le \operatorname{diam}(G_2)$. Therefore, i is $\operatorname{diam}(G_2)$-full.

Hence, by Lemma 5.1.8, there is a $\big(\operatorname{diam}(G_2)+\frac{1}{2}\big)$-full $\big(1,1\big)$-quasi-isometry, $j : G_1 \to (G_1\times G_2)^1$, and $(G_1\times G_2)^1$ is hyperbolic if and only if G_1 is hyperbolic by Theorem 1.3.6.

The same argument proves that $(G_1\times G_2)^2$ is hyperbolic. □

Denote by P_2 the path graph with two vertices, i.e., a graph with two vertices and an edge.

Lemma 5.1.11. *Let G_1 be a graph with some odd cycle and G_2 a non-trivial bounded graph without odd cycles. Then $G_1 \times G_2$ and $G_1 \times P_2$ are quasi-isometric and $\delta(G_1 \times P_2) \leq \delta(G_1 \times G_2)$.*

Proof. By Theorem 5.1.5, we know that $G_1 \times G_2$ and $G_1 \times P_2$ are connected graphs.

Denote by v_1 and v_2 the vertices of P_2 and fix $[w_1, w_2] \in E(G_2)$. The map $f : V(G_1 \times P_2) \longrightarrow V(G_1 \times [w_1, w_2])$ defined as $f(u, v_j) := (u, w_j)$ for every $u \in V(G_1)$ and $j = 1, 2$, is an isomorphism of graphs; hence, it suffices to prove that $G_1 \times G_2$ and $G_1 \times [w_1, w_2]$ are quasi-isometric.

Consider the inclusion map $i : V(G_1 \times [w_1, w_2]) \longrightarrow V(G_1 \times G_2)$. Since $G_1 \times [w_1, w_2]$ is a subgraph of $G_1 \times G_2$, we have $d_{G_1\times G_2}(x, y) \leq d_{G_1\times[w_1,w_2]}(x, y)$ for every $x, y \in V(G_1 \times [w_1, w_2])$.

Since G_2 is a graph without odd cycles, every w_1, w_2-walk has odd length and every w_j, w_j-walk has even length for $j = 1, 2$. Thus Proposition 5.1.3 gives, for every $x = (u, w_1), y = (v, w_2) \in V(G_1 \times [w_1, w_2])$,

$$d_{G_1\times[w_1,w_2]}(x, y) = d_{G_1\times G_2}(x, y) = \min\big\{L(g) \mid g \text{ is a } u, v\text{-walk of odd length}\big\}.$$

Furthermore, for every $x = (u, w_j), y = (v, w_j) \in V(G_1 \times [w_1, w_2])$ and $j = 1, 2$,

$$d_{G_1\times[w_1,w_2]}(x, y) = d_{G_1\times G_2}(x, y) = \min\big\{L(g) \mid g \text{ is a } u, v\text{-walk of even length}\big\}.$$

Hence, $d_{G_1\times[w_1,w_2]}(x, y) = d_{G_1\times G_2}(x, y)$ for every $x, y \in V(G_1 \times [w_1, w_2])$, and the inclusion map i is an $(1, 0)$-quasi-isometric embedding. Therefore, $\delta(G_1 \times P_2) = \delta(G_1 \times [w_1, w_2]) \leq \delta(G_1 \times G_2)$.

Since G_2 is a graph without odd cycles, given any $w \in V(G_2)$, we have either that every w, w_1-walk has even length and every w, w_2-walk has odd length or that every w, w_2-walk has even length and every w, w_1-walk has odd length. Also, since G_1 is connected, for each $u \in V(G_1)$ there is some $u' \in V(G_1)$ such that $[u, u'] \in E(G_1)$. Therefore, by Proposition 5.1.3, for every $(u, w) \in V(G_1 \times G_2)$, if $\min\big\{d_{G_2}(w, w_1),\, d_{G_2}(w, w_2)\big\}$ is even, then

$$\begin{aligned} d_{G_1\times G_2}\big((u, w), V(G_1 \times [w_1, w_2])\big) &= d_{G_1\times G_2}\big((u, w), V(u \times [w_1, w_2])\big) \\ &= \min\big\{d_{G_2}(w, w_1),\, d_{G_2}(w, w_2)\big\}, \end{aligned}$$

and if $\min\big\{d_{G_2}(w, w_1),\, d_{G_2}(w, w_2)\big\}$ is odd, then

$$\begin{aligned} d_{G_1\times G_2}\big((u, w), V(G_1 \times [w_1, w_2])\big) &= d_{G_1\times G_2}\big((u, w), V(u' \times [w_1, w_2])\big) \\ &= \min\big\{d_{G_2}(w, w_1),\, d_{G_2}(w, w_2)\big\}. \end{aligned}$$

In both cases,

$$d_{G_1\times G_2}\big((u, w), V(G_1 \times [w_1, w_2])\big) \leq \operatorname{diam} V(G_2),$$

and i is $\big(\operatorname{diam} V(G_2)\big)$-full. By Lemma 5.1.8, there exists a $\big(\operatorname{diam} V(G_2) + \frac{1}{2}\big)$-full $(1, 1)$-quasi-isometry $g : G_1 \times [w_1, w_2] \longrightarrow G_1 \times G_2$. □

A u, v-walk g in G is a *shortcut* of a cycle C if $g \cap C = \{u, v\}$ and $L(g) < d_C(u, v)$ where d_C denotes the length metric on C.

A cycle C' is a *reduction* of the cycle C if both have odd length and C' is the union of a subarc η of C and a shortcut of C joining the endpoints of η. Note that $L(C') \leq L(C) - 2$. We say that a cycle is *minimal* if it has odd length and it does not have a reduction.

Lemma 5.1.12. *If C is a minimal cycle of G, then $L(C) \leq 4\delta(G)$.*

Proof. We prove first that C is an isometric subgraph of G. Seeking for a contradiction assume that C is not an isometric subgraph. Thus, there exists a shortcut g of C with endpoints u, v. There are two subarcs η_1, η_2 of C joining u and v; since C has odd length, we can assume that η_1 has even length and η_2 has odd length. If g has even length, then $C' := g \cup \eta_2$ is a reduction of C. If g has odd length, then $C'' := g \cup \eta_1$ is a reduction of C. Hence, C is not minimal, which is a contradiction, and so C is an isometric subgraph of G.

Let $x, y \in C$ with $d_C(x, y) = L(C)/2$ and σ_1, σ_2 the two subarcs of C joining x, y. Since C is an isometric subgraph, $T := \{\sigma_1, \sigma_2\}$ is a geodesic bigon. If p is the midpoint of σ_1, then Lemma 1.3.3 gives $L(C)/4 = d_G(p, \{x, y\}) = d_G(p, \sigma_2) \leq \delta(C) \leq \delta(G)$. □

Given any w_0, w_k-walk $g = [w_0, w_1] \cup [w_1, w_2] \cup \cdots \cup [w_{k-1}, w_k]$ in G_1 and $P_2 = [v_1, v_2]$, if $L(g)$ is either odd or even, then we define the $(w_0, v_1), (w_k, v_i)$-walk for $i \in 1, 2$,

$$\Gamma_1 g := [(w_0, v_1), (w_1, v_2)] \cup [(w_1, v_2), (w_2, v_1)] \cup [(w_2, v_1), (w_3, v_2)] \cup \cdots \cup [(w_{k-1}, v_1), (w_k, v_2)],$$
$$\Gamma_1 g := [(w_0, v_1), (w_1, v_2)] \cup [(w_1, v_2), (w_2, v_1)] \cup [(w_2, v_1), (w_3, v_2)] \cup \cdots \cup [(w_{k-1}, v_2), (w_k, v_1)],$$

respectively.

Remark 5.1.13. *By Proposition 5.1.3, if g is a geodesic path in G_1, then $\Gamma_1 g$ is a geodesic path in $G_1 \times P_2$.*

Let us define the map $R : V(G_1 \times P_2) \to V(G_1 \times P_2)$ as $R(w, v_1) = (w, v_2)$ and $R(w, v_2) = (w, v_1)$ for every $w \in V(G_1)$, and the path $\Gamma_2 g$ as $\Gamma_2 g = R(\Gamma_1 g)$.

Let us define the map $(\Gamma_1 g)' : g \to \Gamma_1 g$ which is an isometry on the edges and such that $(\Gamma_1 g)'(w_j) = (w_j, v_1)$ if j is even and $(\Gamma_1 g)'(w_j) = (w_j, v_2)$ if j is odd. Also, let $(\Gamma_2 g)' : g \to \Gamma_2 g$ be the map defined by $(\Gamma_2 g)' := R \circ (\Gamma_1 g)'$.

Given a graph G, denote by $\mathfrak{C}(G)$ the set of minimal cycles of G.

Lemma 5.1.14. *Let G_1 be a graph with some odd cycle and $P_2 = [v_1, v_2]$. Consider a geodesic $g = [w_0 w_k] = [w_0, w_1] \cup [w_1, w_2] \cup \cdots \cup [w_{k-1}, w_k]$ in G_1. Let us define $w_0' := (\Gamma_1 g)'(w_0) = (w_0, v_1)$ and $w_k' := (\Gamma_2 g)'(w_k)$, i.e., $w_k' := (w_k, v_1)$ or $w_k' := (w_k, v_2)$ if k is odd or even, respectively. Then $d_{G_1 \times P_2}(w_0', w_k') > \sqrt{d_{G_1}\big(w_j, \mathfrak{C}(G_1)\big)}$ for every $0 \leq j \leq k$.*

Proof. Fix $0 \leq j \leq k$. Define

$$\mathfrak{P} := \big\{\sigma \mid \sigma \text{ is a } w_0, w_k\text{-walk such that } L(\sigma) \text{ has a parity different from that of } k\big\}.$$

Proposition 5.1.3 gives

$$d_{G_1\times P_2}(w_0',w_k')=\min\big\{L(\sigma)\mid \sigma\in\mathfrak{P}\big\}.$$

Choose $\sigma_0\in\mathfrak{P}$ such that $L(\sigma_0)=d_{G_1\times P_2}(w_0',w_k')$. Since $L(g)+L(\sigma_0)$ is odd, we have $L(g)+L(\sigma_0)=2t+1$ for some positive integer t. Thus $d_{G_1\times P_2}(w_0',w_k')=L(\sigma_0)>\frac{1}{2}(2t+1)$.

If $g\cup\sigma_0$ is a cycle, then let us define $C_0:=g\cup\sigma_0$. Thus, $L(C_0)=2t+1$ and $d_{G_1}\big(w_j,C_0\big)=0$ for every $0\le j\le k$. Otherwise, we may assume that $g\cap\sigma_0=[w_0w_{i_1}]\cup[w_{i_2}w_k]$ for some $0\le i_1<i_2\le k$. If $\sigma_1=\sigma_0\setminus g$, then let us define $C_0:=[w_{i_1}w_{i_2}]\cup\sigma_1$ (where $[w_{i_1}w_{i_2}]\subset g$). Hence, C_0 is a cycle, $L(C_0)\le 2t-1$ and $d_{G_1}\big(w_j,C_0\big)<\frac{1}{2}(2t+1)$.

If C_0 is not minimal, then consider a reduction C_1 of C_0. Let us repeat the process until we obtain a minimal cycle C_s. Note that $L(C_1)\le L(C_0)-2$ and for every point $p_1\in C_0$, $d_{G_1}\big(p_1,C_1\big)<\frac{1}{2}L(C_0)$. Now, repeating the argument, for every $1<i\le s$, $L(C_i)\le L(C_{i-1})-2$ and for every point $p_i\in C_{i-1}$, $d_{G_1}\big(p_i,C_i\big)<\frac{1}{2}L(C_{i-1})$. Therefore,

$$\begin{aligned}d_{G_1}\big(w_j,\mathfrak{C}(G_1)\big)\le d_{G_1}\big(w_j,C_s\big)&\le d_{G_1}\big(w_j,C_0\big)+\frac{1}{2}L(C_0)+\frac{1}{2}L(C_1)+\cdots+\frac{1}{2}L(C_s)\\&<\frac{1}{2}(2t+1)+\frac{1}{2}(2t-1)+\cdots+\frac{5}{2}+\frac{3}{2}\,.\end{aligned}$$

Hence,

$$d_{G_1}\big(w_j,\mathfrak{C}(G_1)\big)<\frac{1}{2}\sum_{i=1}^{t}(2i+1)=\frac{1}{2}t^2+t<\Big(\frac{1}{2}(2t+1)\Big)^2<\Big(d_{G_1\times P_2}(w_0',w_k')\Big)^2.$$

□

Corollary 5.1.15. *Let G_1 be a hyperbolic graph with some odd cycle and $P_2=[v_1,v_2]$. Consider a geodesic $g=[w_0w_k]=[w_0,w_1]\cup[w_1,w_2]\cup\cdots\cup[w_{k-1},w_k]$ in G_1. Let us define $w_0':=(\Gamma_1 g)'(w_0)=(w_0,v_1)$ and $w_k':=(\Gamma_2 g)'(w_k)$. Then, we have for every $0\le j\le k$,*

$$\frac{1}{2}\Big(k+\sqrt{d_{G_1}\big(w_j,\mathfrak{C}(G_1)\big)}\,\Big)\le d_{G_1\times P_2}(w_0',w_k')\le k+2d_{G_1}\big(w_j,\mathfrak{C}(G_1)\big)+4\delta(G_1).$$

Proof. Corollary 5.1.4 and Lemma 5.1.14 give $d_{G_1\times P_2}(w_0',w_k')\ge k$ and $d_{G_1\times P_2}(w_0',w_k')\ge\sqrt{d_{G_1}\big(w_j,\mathfrak{C}(G_1)\big)}$, and these inequalities provide the lower bound of $d_{G_1\times P_2}(w_0',w_k')$.

Consider a geodesic γ joining w_j and $C\in\mathfrak{C}(G_1)$ with $L(\gamma)=d_{G_1}(w_j,C)=d_{G_1}\big(w_j,\mathfrak{C}(G_1)\big)$ and the w_0,w_k-walk

$$g':=[w_0w_j]\cup\gamma\cup C\cup\gamma\cup[w_jw_k].$$

One can check that $\Gamma_1 g'$ is a w_0',w_k'-walk in $G_1\times P_2$, and so Lemma 5.1.12 gives

$$\begin{aligned}d_{G_1\times P_2}(w_0',w_k')&\le L(\Gamma_1 g')=L(g')=k+2d_{G_1}\big(w_j,\mathfrak{C}(G_1)\big)+L(C)\\&\le k+2d_{G_1}\big(w_j,\mathfrak{C}(G_1)\big)+4\delta(G_1).\end{aligned}$$

□

Consider the set $\mathbb{T}_v(G)$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $V(G)$, and denote by $\delta_v(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_v(G)$ is λ-thin.

Theorem 5.1.16. *For every graph G we have $\delta_v(G) \le \delta(G) \le 4\delta_v(G) + 1/2$. Hence, G is hyperbolic if and only if $\delta_v(G) < \infty$. Furthermore, if G is hyperbolic, then $\delta_v(G)$ is always a multiple of $1/2$ and there exist a geodesic triangle $T = \{x, y, z\} \in \mathbb{T}_v(G)$ and $p \in [xy] \cap J(G)$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta_v(G)$.*

Proof. The inequality $\delta_v(G) \le \delta(G)$ is direct.

Consider the set $\mathbb{T}'_v(G)$ of geodesic triangles T in G such that the three vertices of the triangle T belong to $V(G)$, and denote by $\delta'_v(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}'_v(G)$ is λ-thin. The argument in the proof of [105, Lemma 2.1] gives that $\delta'_v(G) = \delta_v(G)$.

In order to prove the upper bound of $\delta(G)$, assume first that G is hyperbolic. We can assume $\delta'_v(G) < \infty$, since otherwise the inequality is direct. By Theorem 1.3.13, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$. Assume that $x, y, z \in J(G) \setminus V(G)$ (otherwise, the argument is simpler). Let $x_1, x_2, y_1, y_2, z_1, z_2 \in T \cap V(G)$ such that $x \in [x_1, x_2], y \in [y_1, y_2], z \in [z_1, z_2]$ and $x_2, y_1 \in [xy], y_2, z_1 \in [yz], z_2, x_1 \in [xz]$. Since $H := \{x_2, y_1, y_2, z_1, z_2, x_1\}$ is a geodesic hexagon with vertices in $V(G)$, it is $4\delta'_v(G)$-thin and every point $w \in [y_1, y_2] \cup [y_2 z_1] \cup [z_1, z_2] \cup [z_2 x_1] \cup [x_1, x_2]$ verifies $d(w, [xz] \cup [zy]) \le 1/2$, we have

$$\begin{aligned}\delta(G) = d(p, [xz] \cup [zy]) &\le d(p, [y_1, y_2] \cup [y_2 z_1] \cup [z_1, z_2] \cup [z_2 x_1] \cup [x_1, x_2]) + 1/2 \\ &\le 4\delta'_v(G) + 1/2 = 4\delta_v(G) + 1/2.\end{aligned}$$

Assume now that G is not hyperbolic. Therefore, for each $M > 0$ there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) \ge M$. The previous argument gives $M \le 4\delta_v(G) + 1/2$ and, since M is arbitrary, we deduce $\delta_v(G) = \infty = \delta(G)$.

Finally, consider any geodesic triangle $T = \{x, y, z\}$ in $\mathbb{T}_v(G)$. Since $d(p, [xz] \cup [zy]) = d(p, ([xz] \cup [zy]) \cap V(G))$, $d(p, [xz] \cup [zy])$ attains its maximum value when $p \in J(G)$. Hence, $\delta(T)$ is a multiple of $1/2$ for every geodesic triangle $T \in \mathbb{T}_v(G)$. Since the set of non-negative numbers that are multiple of $1/2$ is a discrete set, if G is hyperbolic, then $\delta(G)$ is a multiple of $1/2$ and there exist a geodesic triangle $T = \{x, y, z\} \in \mathbb{T}_v(G)$ and $p \in [xy] \cap J(G)$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta_v(G)$. This finishes the proof. □

Theorem 5.1.17. *If G_1 is a non-hyperbolic graph, then $G_1 \times P_2$ is not hyperbolic.*

Proof. Since G_1 is not hyperbolic, by Theorem 5.1.16, given any $R > 0$ there is a geodesic triangle $T = \{x, y, z\}$ that is a cycle, with $x, y, z \in V(G_1)$ and such that T is not R-thin. Therefore, there exists some point $m \in T$, let us assume that $m \in [xy]$, such that $d_{G_1}(m, [yz] \cup [zx]) > R$.

Seeking for a contradiction let us assume that $G_1 \times P_2$ is δ-hyperbolic.

Suppose that for some $R > \delta$, there is a geodesic triangle $T = \{x, y, z\}$ that is an even cycle in G_1, with $x, y, z \in V(G_1)$ and such that T is not R-thin. Consider the (closed) path $\Lambda = [xy] \cup [yz] \cup [zx]$. Then, since T has even length, the path $\Gamma_1\Lambda$ defines a cycle in $G_1 \times P_2$. Let γ_1, γ_2, γ_3 be the paths in $\Gamma_1\Lambda$ corresponding to $[xy], [yz], [zx]$, respectively. By Corollary 5.1.4, the curves γ_1, γ_2 and γ_3 are geodesics, and $d_{G_1\times P_2}\big((\Gamma_1\Lambda)'(m), \gamma_2 \cup \gamma_3\big) > \delta$, leading to contradiction.

Suppose that for every $R > 0$, there is a geodesic triangle $T = \{x, y, z\}$ which is an odd cycle, with $x, y, z \in V(G_1)$ and such that T is not R-thin.

Let $T_1 = \{x, y, z\}$ be a geodesic triangle as above and let us assume that $\operatorname{diam}(T_1) = D > 8\delta$.

Let $T_2 = \{x', y', z'\}$ be another geodesic triangle as above such that T_2 is not $3(D + 8\delta)$-thin, this is, there is a point m in one of the sides, let us call it σ, of T_2 such that $d_{G_1}(m, T_2 \backslash \sigma) > 3(D + 8\delta)$.

Let $g = [w_0 w_k]$ with $w_0 \in T_1$ and $w_k \in T_2$ be a shortest geodesic in G_1 joining T_1 and T_2 (if T_1 and T_2 intersect, just assume that g is a single vertex, $w_0 = w_k$, in the intersection).

Let us assume that $w_0 \in [xz]$ and $w_k \in [x'z']$. Then, let us consider the cycle C in G_1 given by the union of the geodesics in T_1, g, the geodesics in T_2 and the inverse of g from w_k to w_0, this is,

$$C := [w_0x] \cup [xy] \cup [yz] \cup [zw_0] \cup [w_0w_k] \cup [w_kx'] \cup [x'y'] \cup [y'z'] \cup [z'w_k] \cup [w_kw_0].$$

Since T_1, T_2 are odd cycles, C is an even cycle. Therefore, $\Gamma_1 C$ defines a cycle in $G_1 \times P_2$. Moreover, by Remark 5.1.13, $\Gamma_1 C$ is a geodesic decagon in $G_1 \times P_2$ with sides $\gamma_1 = (\Gamma_1 C)'([w_0x])$, $\gamma_2 = (\Gamma_1 C)'([xy])$, $\gamma_3 = (\Gamma_1 C)'([yz])$, $\gamma_4 = (\Gamma_1 C)'([zw_0])$, $\gamma_5 = (\Gamma_1 C)'([w_0w_k])$, $\gamma_6 = (\Gamma_1 C)'([w_kx'])$, $\gamma_7 = (\Gamma_1 C)'([x'y'])$, $\gamma_8 = (\Gamma_1 C)'([y'z'])$, $\gamma_9 = (\Gamma_1 C)'([z'w_k])$ and $\gamma_{10} = (\Gamma_1 C)'([w_kw_0])$.

Since we are assuming that $G_1 \times P_2$ is δ-hyperbolic, then for every $1 \le i \le 10$ and every point $p \in \gamma_i$, $d_{G_1\times P_2}(p, C \backslash \gamma_i) \le 8\delta$.

Let $p := (\Gamma_1 C)'(m)$.

Case 1. Suppose that $d_{G_1}(m, T_1 \cup g) > 8\delta$.

By assumption, $d_{G_1}(m, T_2 \backslash \sigma) > 8\delta$. If $\sigma = [x'y']$ (resp. $\sigma = [y'z']$), then $p \in \gamma_7$ (resp. $p \in \gamma_8$) and, by Corollary 5.1.4, $d_{G_1\times P_2}(p, C \backslash \gamma_7) > 8\delta$ (resp. $d_{G_1\times P_2}(p, C \backslash \gamma_8) > 8\delta$) leading to contradiction. If $\sigma = [x'z']$, since $[x'z'] = [x'w_k] \cup [w_kz']$, let us assume $m \in [x'w_k]$. Then, since $d_{G_1}(m, w_k) > 8\delta$, it follows that $d_{G_1}(m, [w_kz']) > 8\delta$. Thus, $p \in \gamma_6$ and, by Corollary 5.1.4, $d_{G_1\times P_2}(p, C \backslash \gamma_6) > 8\delta$ leading to contradiction.

Case 2. Suppose that $d_{G_1}(m, T_1 \cup g) \le 8\delta$ and $L(g) \le 8\delta$. Then, for every point q in $T_1 \cup g$, $d_{G_1}(m, q) \le 8\delta + D + 8\delta$. In particular, $d_{G_1}(m, w_k) \le 8\delta + D + 8\delta$. Therefore, $m \in [x'z']$ and let us assume that $m \in [x'w_k]$. Since $d_{G_1}(m, x') \ge d_{G_1}(m, [x'y'] \cup [y'z']) > 3(D + 8\delta)$, there is a point $m' \in [x'm] \subset [x'w_k]$ such that $d_{G_1}(m, m') = 2(D + 8\delta)$. Then, $d_{G_1}(m', T_1 \cup g) \ge 2(D + 8\delta) - D - 8\delta - 8\delta = D > 8\delta$. Also, it is trivial to check that $d_{G_1}(m', [x'y'] \cup [y'z']) > 3(D + 8\delta) - 2(D + 8\delta) > 8\delta$ and since $[x'z']$ is a geodesic, $d_{G_1}(m', [z'w_k]) > 8\delta$. Thus, if

$p' := (\Gamma_1 C)'(m')$, then $p' \in \gamma_6$ and, by Corollary 5.1.4, $d_{G_1\times P_2}(p', C\setminus\gamma_6) > 8\delta$ leading to contradiction.

Case 3. Suppose that $d_{G_1}(m, T_1 \cup g) \le 8\delta$ and $L(g) > 8\delta$. Since g is a shortest geodesic in G_1 joining T_1 and T_2, this implies that $d_{G_1}(T_1, T_2) > 8\delta$ and $d_{G_1}(m, [w_0 w_k]) \le 8\delta$. Moreover, $d_{G_1}(m, w_k) \le 16\delta$. Otherwise, there is a point $q \in [w_0 w_k]$ such that $d_{G_1}(m, q) \le 8\delta$ and $d_{G_1}(q, w_k) > 8\delta$ which means that $d_{G_1}(q, w_0) < d_{G_1}(w_0, w_k) - 8\delta$ and $d_{G_1}(m, w_0) < d_{G_1}(w_0, w_k)$ leading to contradiction.

Since $d_{G_1}(m, w_k) \le 16\delta$, $m \in [x'z']$. Let us assume that $m \in [x'w_k]$. Since $d_{G_1}(m, [x'y'] \cup [y'z']) > 3(D+8\delta)$, there is a point $m' \in [x'm] \subset [x'w_k]$ such that $d_{G_1}(m, m') = 2(D+8\delta)$. Let us see that $d_{G_1}(m', [w_0 w_k]) > 8\delta$. Suppose there is some $q \in [w_0 w_k]$ such that $d_{G_1}(m', q) \le 8\delta$. Since $m' \in T_2$ and g is a shortest geodesic joining T_1 and T_2, $d_{G_1}(q, w_k) \le 8\delta$. However, $32\delta < 2(D+8\delta) = d_{G_1}(m', m) \le d_{G_1}(m', q) + d_{G_1}(q, w_k) + d_{G_1}(w_k, m) \le 8\delta + 8\delta + 16\delta$ which is a contradiction. Hence, $d_{G_1}(m', [w_0 w_k]) > 8\delta$. Also, it is trivial to check that $d_{G_1}(m', [x'y'] \cup [y'z']) > 3(D+8\delta) - 2(D+8\delta) > 8\delta$ and since $[x'z']$ is a geodesic, $d_{G_1}(m', [z'w_k]) > 8\delta$. Thus, if $p' := (\Gamma_1 C)'(m')$, then $p' \in \gamma_6$ and, by Corollary 5.1.4, $d_{G_1\times P_2}(p', C\setminus\gamma_6) > 8\delta$ leading to contradiction. □

Proposition 5.1.7, Lemma 5.1.11 and Theorems 5.1.9, 5.1.10 and 5.1.17 have the following consequence.

Corollary 5.1.18. *If G_1 is a non-hyperbolic graph and G_2 is some non-trivial graph, then $G_1 \times G_2$ is not hyperbolic.*

Proposition 5.1.7 and Corollary 5.1.18 provide a necessary condition for the hyperbolicity of $G_1 \times G_2$.

Theorem 5.1.19. *Let G_1, G_2 be non-trivial graphs. If $G_1 \times G_2$ is hyperbolic, then one factor graph is hyperbolic and the other one is bounded.*

Theorems 5.1.9 and 5.1.10 show that this necessary condition is also sufficient if either G_2 has some odd cycle or G_1 and G_2 do not have odd cycles (when G_1 is a hyperbolic graph and G_2 is a bounded graph). We deal now with the other case, when G_1 has some odd cycle and G_2 does not have odd cycles.

Theorem 5.1.20. *Let G_1 be a graph with some odd cycle and G_2 a non-trivial bounded graph without odd cycles. Assume that G_1 satisfies the following property: for each $M > 0$ there exist a geodesic g joining two minimal cycles of G_1 and a vertex $u \in g \cap V(G_1)$ with $d_{G_1}\big(u, \mathfrak{C}(G_1)\big) \ge M$. Then $G_1 \times G_2$ is not hyperbolic.*

Proof. If G_1 is not hyperbolic, then Corollary 5.1.18 gives that $G_1 \times G_2$ is not hyperbolic. Assume now that G_1 is hyperbolic. By Theorem 1.3.6 and Lemma 5.1.11, we can assume that $G_2 = P_2$ and $V(P_2) = \{v_1, v_2\}$.

Fix $M > 0$ and choose a geodesic $g = [w_0 w_k] = [w_0, w_1] \cup [w_1, w_2] \cup \cdots \cup [w_{k-1}, w_k]$ joining two minimal cycles in G_1 and $0 < r < k$ with $d_{G_1}\big(w_r, \mathfrak{C}(G_1)\big) \ge M$.

Define the paths g_1 and g_2 in $G_1 \times P_2$ as $g_1 := \Gamma_1 g$ and $g_2 := \Gamma_2 g$. Since $L(g_1) = L(g_2) = L(g) = d_{G_1}(w_0, w_k)$, we have

$$d_{G_1\times P_2}\big(g_1(w_0), g_1(w_k)\big) \le L(g_1) = d_{G_1}(w_0, w_k),\ d_{G_1\times P_2}\big(g_2(w_0), g_2(w_k)\big) \le L(g_2) = d_{G_1}(w_0, w_k).$$

Corollary 5.1.4 gives that

$$d_{G_1\times P_2}\big(g_1(w_0), g_1(w_k)\big) \ge d_{G_1}(w_0, w_k), \qquad d_{G_1\times P_2}\big(g_2(w_0), g_2(w_k)\big) \ge d_{G_1}(w_0, w_k).$$

Hence, g_1 and g_2 are geodesics in $G_1 \times P_2$. Choose geodesics $g_3 = [g_1(w_0)g_2(w_0)]$ and $g_4 = [g_1(w_k)g_2(w_k)]$ in $G_1 \times P_2$. Since $d_{P_2}(v_1, v_2) = 1$ is odd, Proposition 5.1.3 gives

$$\begin{aligned} d_{G_1\times P_2}\big(g_1(w_0), g_2(w_0)\big) &= \min\big\{L(\sigma) \mid \sigma \text{ is a } w_0, w_0\text{-walk } \big\} \\ &= \min\big\{L(\sigma) \mid \sigma \text{ cycle of odd length containing } w_0\big\}. \end{aligned}$$

Since w_0 belongs to a minimal cycle, $L(g_3) \le 4\delta(G_1)$ by Lemma 5.1.12. In a similar way, we obtain $L(g_4) \le 4\delta(G_1)$.

Consider the geodesic quadrilateral $Q := \{g_1, g_2, g_3, g_4\}$ in $G_1 \times P_2$. Thus $d_{G_1\times P_2}\big(g_1(w_r), g_2 \cup g_3 \cup g_4\big) \le 2\delta(G_1 \times P_2)$. Since $\max\big\{L(g_3), L(g_4)\big\} \le 4\delta(G_1)$, we deduce $d_{G_1\times P_2}\big(g_1(w_r), g_2\big) \le 2\delta(G_1 \times P_2) + 4\delta(G_1)$.

Let $0 \le j \le k$ with $d_{G_1\times P_2}\big(g_1(w_r), g_2\big) = d_{G_1\times P_2}\big(g_1(w_r), g_2(w_j)\big)$. Let us define $w_r' := g_1(w_r)$ and $w_j' := g_2(w_j)$. Thus Lemma 5.1.14 gives

$$\sqrt{M} \le \sqrt{d_{G_1}\big(w_r, \mathfrak{C}(G_1)\big)} \le d_{G_1\times P_2}(w_r', w_j') = d_{G_1\times P_2}(w_r', g_2) \le 2\delta(G_1 \times P_2) + 4\delta(G_1),$$

and since M is arbitrarily large, we deduce that $G_1 \times P_2$ is not hyperbolic. □

Lemma 5.1.21. *Let G_1 be a hyperbolic graph and suppose there is some constant $K > 0$ such that for every vertex $w \in G_1$, $d_{G_1}(w, \mathfrak{C}(G_1)) \le K$. Then, $G_1 \times P_2$ is hyperbolic.*

Proof. Denote by v_1 and v_2 the vertices of P_2. Let $i : V(G_1) \to V(G_1 \times P_2)$ defined as $i(w) := (w, v_1)$ for every $w \in G_1$.

For every pair of vertices $x, y \in V(G_1)$, by Corollary 5.1.4, $d_{G_1}(x, y) \le d_{G_1\times P_2}(i(x), i(y))$. By Corollary 5.1.15,

$$d_{G_1\times P_2}(i(x), i(y)) \le d_{G_1}(x, y) + 2d_{G_1}\big(x, \mathfrak{C}(G_1)\big) + 4\delta(G_1) \le d_{G_1}(x, y) + 2K + 4\delta(G_1).$$

Therefore, $i : V(G_1) \to V(G_1 \times P_2)$ is a $\big(1, 2K + 4\delta(G_1)\big)$-quasi-isometric embedding.

Notice that for every $(w, v_1) \in V(G_1 \times P_2)$, $(w, v_1) = i(w)$. Also, for any $(w, v_2) \in V(G_1 \times P_2)$, since G_1 is connected, there is some edge $[w, w'] \in E(G_1)$ and we have $[(w, v_2), (w', v_1)] \in E(G_1 \times P_2)$. Therefore, $i : V(G_1) \to V(G_1 \times P_2)$ is 1-full.

Thus, by Lemma 5.1.8, G_1 and $G_1 \times P_2$ are quasi-isometric and, by Theorem 1.3.6, $G_1 \times P_2$ is hyperbolic. □

Theorem 5.1.9 and Lemmas 5.1.11 and 5.1.21 have the following consequence.

Theorem 5.1.22. *Let G_1 be a hyperbolic graph and G_2 some non-trivial bounded graph. If there is some constant $K > 0$ such that for every vertex $w \in G_1$, $d_{G_1}(w, \mathfrak{C}(G_1)) \le K$, then $G_1 \times G_2$ is hyperbolic.*

We will finish this section with a characterization of the hyperbolicity of $G_1 \times G_2$, under an additional hypothesis. Since the proof of this result is long and technical, in order to make the arguments more transparent, we collect some results we need along the proof in technical lemmas.

Let J be a finite or infinite index set. Now, given a graph G_1, we define some graphs related to G_1 which will be useful in the following results. Let $B_j := B_{G_1}(w_j, K_j)$ with $w_j \in V(G_1)$ and $K_j \in \mathbb{Z}^+$, for any $j \in J$, such that $\sup_j K_j = K < \infty$, $\overline{B}_{j_1} \cap \overline{B}_{j_2} = \emptyset$ if $j_1 \neq j_2$, and every odd cycle C in G_1 satisfies $C \cap B_j \neq \emptyset$ for some $j \in J$. Denote by G_1' the subgraph of G_1 induced by $V(G_1) \setminus (\cup_j B_j)$. Let $N_j := \partial B_j = \{w \in V(G_1) : d_{G_1}(w, w_j) = K_j\}$. Denote by G_1^* the graph with $V(G_1^*) = V(G_1') \cup (\cup_j \{w_j^*\})$, where w_j^* are additional vertices, and $E(G_1^*) = E(G_1') \cup (\cup_j \{[w, w_j^*] : w \in N_j\})$. We have $G_1' = G_1 \cap G_1^*$.

Lemma 5.1.23. *Let G_1 be a graph as above. Then, there exists a quasi-isometry $g : G_1 \to G_1^*$ with $g(w_j) = w_j^*$ for every $j \in J$.*

Proof. Let $f : V(G_1) \to V(G_1^*)$ defined as $f(u) = u$ for every $u \in V(G_1')$, and $f(u) = w_i^*$ for every $u \in V(B_i)$. It is clear that $f : V(G_1) \to V(G_1^*)$ is 0-full.

Now, we focus on proving that $f : V(G_1) \to V(G_1^*)$ is a $(K, 2K)$-quasi-isometric embedding. For every $u, v \in V(G_1)$, it is clear that $d_{G_1^*}(f(u), f(v)) \le d_{G_1}(u, v)$.

In order to prove the other inequality, let us fix $u, v \in V(G_1)$ and let us consider a geodesic γ in G_1^* joining $f(u)$ and $f(v)$.

Assume that $u, v \in V(G_1')$. If $L(\gamma) = d_{G_1}(u, v)$, then $d_{G_1}(u, v) = d_{G_1^*}(f(u), f(v))$. If $L(\gamma) < d_{G_1}(u, v)$, then γ meets some w_j^*. Since γ is a compact set, it intersects just a finite number of w_j^*'s, which we denote by $w_{j_1}^*, \dots w_{j_r}^*$. We consider γ as an oriented curve from $f(u)$ to $f(v)$; thus we can assume that γ meets $w_{j_1}^*, \dots w_{j_r}^*$ in this order.

Let us define the following vertices in γ

$$w_i^1 = [f(u) w_{j_i}^*] \cap N_{j_i}, \qquad w_i^2 = [w_{j_i}^* f(v)] \cap N_{j_i},$$

for every $1 \le i \le r$. Note that $[w_i^2 w_{i+1}^1] \subset G_1'$ for every $1 \le i < r$ (it is possible to have $w_i^2 = w_{i+1}^1$).

Since $d_{G_1^*}(w_i^1, w_i^2) = 2$ and $d_{G_1}(w_i^1, w_i^2) \le 2K$, we have $d_{G_1^*}(w_i^1, w_i^2) \ge \frac{1}{K} d_{G_1}(w_i^1, w_i^2)$ for every $1 \le i \le r$. Thus,

$$\begin{aligned} d_{G_1^*}(f(u), f(v)) &= d_{G_1^*}(f(u), w_1^1) + \sum_{i=1}^{r} d_{G_1^*}(w_i^1, w_i^2) + \sum_{i=1}^{r-1} d_{G_1^*}(w_i^2, w_{i+1}^1) + d_{G_1^*}(w_r^2, f(v)) \\ &\ge d_{G_1}(u, w_1^1) + \frac{1}{K} \sum_{i=1}^{r} d_{G_1}(w_i^1, w_i^2) + \sum_{i=1}^{r-1} d_{G_1}(w_i^2, w_{i+1}^1) + d_{G_1}(w_r^2, v) \\ &\ge \frac{1}{K} \Big(d_{G_1}(u, w_1^1) + \sum_{i=1}^{r} d_{G_1}(w_i^1, w_i^2) + \sum_{i=1}^{r-1} d_{G_1}(w_i^2, w_{i+1}^1) + d_{G_1}(w_r^2, v) \Big) \\ &\ge \frac{1}{K}\, d_{G_1}(u, v). \end{aligned}$$

Assume that $f(u) = f(v)$. Therefore, there exists j with $u, v \in B_j$ and

$$d_{G_1^*}(f(u), f(v)) = 0 > d_{G_1}(u, v) - 2K.$$

Assume now that u and/or v does not belong to $V(G_1')$ and $f(u) \ne f(v)$. Let u_0, v_0 be the closest vertices in $V(G_1') \cap \gamma$ to $f(u), f(v)$, respectively (it is possible to have $u_0 = f(u)$ or $v_0 = f(v)$). Since $u_0, v_0 \in V(G_1')$, $u_0 = f(u_0), v_0 = f(v_0)$, we have $d_{G_1}(u, u_0) < 2K$ and $d_{G_1}(v, v_0) < 2K$. Hence,

$$\begin{aligned} d_{G_1^*}(f(u), f(v)) &= d_{G_1^*}(f(u), u_0) + d_{G_1^*}(u_0, v_0) + d_{G_1^*}(v_0, f(v)) \\ &\ge d_{G_1^*}(f(u_0), f(v_0)) \\ &\ge \frac{1}{K}\, d_{G_1}(u_0, v_0) \\ &\ge \frac{1}{K} \Big(d_{G_1}(u, v) - d_{G_1}(u, u_0) - d_{G_1}(v, v_0) \Big) \\ &> \frac{1}{K}\, d_{G_1}(u, v) - 4. \end{aligned}$$

If $K \ge 2$, then $d_{G_1^*}(f(u), f(v)) > \frac{1}{K} d_{G_1}(u, v) - 2K$. If $K = 1$, then $d_{G_1}(u, u_0) \le 1, d_{G_1}(v, v_0) \le 1$, and $d_{G_1^*}(f(u), f(v)) \ge d_{G_1}(u, v) - 2$.

Finally, we conclude that $f : V(G_1) \to V(G_1^*)$ is a $(K, 2K)$-quasi-isometric embedding. Thus, Lemma 5.1.8 provides a quasi-isometry $g : G_1 \to G_1^*$ with the required property. □

Definition 5.1.24. *Given a graph G_1 and some index set J let $\mathcal{B}_J = \{B_j\}_{j \in J}$ be a family of balls where $B_j := B_{G_1}(w_j, K_j)$ with $w_j \in V(G_1)$, $K_j \in \mathbb{Z}^+$ for any $j \in J$, $\sup_j K_j = K < \infty$ and $\overline{B}_{j_1} \cap \overline{B}_{j_2} = \emptyset$ if $j_1 \ne j_2$. Suppose that every odd cycle C in G_1 satisfies that $C \cap B_j \ne \emptyset$ for some $j \in J$. If there is some constant $M > 0$ such that for every $j \in J$, there is an odd cycle C_j such that $C_j \cap B_j \ne \emptyset$ with $L(C_j) < M$, then we say that $\mathcal{B}_J$ is M-*regular.

Remark 5.1.25. *If J is finite, then there exists $M > 0$ such that $\{B_j\}_{j \in J}$ is M-regular.*

Denote by G^* the graph with $V(G^*) = V(G_1' \times P_2) \cup (\cup_j \{w_j^*\})$, where G_1' is a graph as above and w_j^* are additional vertices, and $E(G^*) = E(G_1' \times P_2) \cup (\cup_j \{[w, w_j^*] : \pi_1(w) \in N_j\})$.

Lemma 5.1.26. *Let G_1 be a graph as above and P_2 with $V(P_2) = \{v_1, v_2\}$. If G_1 is hyperbolic and $\mathcal{B}_J$ as above is M-regular, then there exists a quasi-isometry $f : G_1 \times P_2 \to G^*$ with $f(w_j, v_i) = w_j^*$ for every $j \in J$ and $i \in \{1, 2\}$.*

Proof. Let $F : V(G_1 \times P_2) \to V(G^*)$ defined as $F(v, v_i) = (v, v_i)$ for every $v \in V(G_1')$, and $F(v, v_i) = w_j^*$ for every $v \in V(B_j)$. It is clear that $F : V(G_1 \times P_2) \to V(G^*)$ is 0-full. Recall that we denote by $\pi_1 : G_1 \times P_2 \to G_1$ the projection map. Define $\pi^* : G^* \to G_1$ as $\pi^* = \pi_1$ on $G_1' \times P_2$ and $\pi^*(x) = w_j$ for every x with $d_{G^*}(x, w_j^*) < 1$ for some $j \in J$.

Now, we focus on proving that $F : V(G_1 \times P_2) \to V(G^*)$ is a quasi-isometric embedding. For every $(w, v_i), (w', v_{i'}) \in V(G_1 \times P_2)$, one can check

$$d_{G^*}(F(w, v_i), F(w', v_{i'})) \le d_{G_1 \times P_2}((w, v_i), (w', v_{i'})).$$

In order to prove the other inequality, let us fix $(w, v_i), (w', v_{i'}) \in V(G_1' \times P_2)$ (the inequalities in other cases can be obtained from the one in this case, as in the proof of Lemma 5.1.23). Consider a geodesic $\gamma := [F(w, v_i)F(w', v_{i'})]$ in G^*. If $L(\gamma) = d_{G_1 \times P_2}((w, v_i), (w', v_{i'}))$, then

$$d_{G^*}(F(w, v_i), F(w', v_{i'})) = d_{G_1 \times P_2}((w, v_i), (w', v_{i'})).$$

If $L(\gamma) < d_{G_1 \times P_2}((w, v_i), (w', v_{i'}))$, then $\pi^*(\gamma)$ meets some B_j. Since γ is a compact set, $\pi^*(\gamma)$ intersects just a finite number of B_j's, which we denote by $B_{j_1}, \dots B_{j_r}$. We consider γ as an oriented curve from $F(w, v_i)$ to $F(w', v_{i'})$; thus we can assume that $\pi^*(\gamma)$ meets $B_{j_1}, \dots B_{j_r}$ in this order.

Let us define the following set of vertices in γ

$$\{w_i^1, w_i^2\} := \gamma \cap (N_{j_i} \times P_2),$$

for every $1 \le i \le r$, such that $d_{G_1 \times P_2}((w, v_i), w_i^1) < d_{G_1 \times P_2}((w, v_i), w_i^2)$. Note that $[w_i^2 w_{i+1}^1] \subset G_1' \times P_2$ for every $1 \le i < r$ and $d_{G_1 \times P_2}(w_i^2, w_{i+1}^1) \ge 1$ since $\overline{B}_{j_i} \cap \overline{B}_{j_{i+1}} = \emptyset$.

If $d_{G_1}(\pi(w_i^1), \pi(w_i^2)) = d_{G_1 \times P_2}(w_i^1, w_i^2)$ for some $1 \le i \le r$, then $d_{G_1 \times P_2}(w_i^1, w_i^2) \le 2K$. Since
$d_{G_1 \times P_2}(w_i^2, w_{i+1}^1) \ge 1$ for $1 \le i < r$, we have that $d_{G_1 \times P_2}(w_i^1, w_i^2) \le 2K\, d_{G_1 \times P_2}(w_i^2, w_{i+1}^1)$ in this case.

If $d_{G_1}(\pi_1(w_i^1), \pi_1(w_i^2)) < d_{G_1 \times P_2}(w_i^1, w_i^2)$ for some $1 \le i \le r$, then $d_{G_1}(\pi_1(w_i^1), \pi_1(w_i^2)) + d_{G_1 \times P_2}(w_i^1, w_i^2)$ is odd.

Since $\mathcal{B}_J$ is M-regular, consider an odd cycle C with $C \cap B_{j_i} \neq \emptyset$ and $L(C) < M$, and let $b_i \in C \cap B_{j_i}$ and $[\pi_1(w_i^1)b_i], [b_i\pi_1(w_i^2)]$ geodesics in G_1. Thus, $[\pi_1(w_i^1)b_i] \cup [b_i\pi_1(w_i^2)]$ and $[\pi_1(w_i^1)b_i] \cup C \cup [b_i\pi_1(w_i^2)]$ have different parity which means that one of them has different parity from $[\pi_1(w_i^1)\pi_1(w_i^2)]$. Then, $d_{G_1 \times P_2}(w_i^1, w_i^2) \le L([\pi_1(w_i^1)b_i] \cup C \cup [b_i\pi_1(w_i^2)]) \le$

$4K+M$. Since $d_{G_1\times P_2}(w_i^2,w_{i+1}^1)\ge 1$ for $1\le i<r$, we have that $d_{G_1\times P_2}(w_i^1,w_i^2)\le \Big(4K+M\Big)\, d_{G_1\times P_2}(w_i^2,w_{i+1}^1)$ in this case.

Thus, we have that $d_{G_1\times P_2}(w_i^1,w_i^2)\le 4K+M$ for every $1\le i\le r$ and $d_{G_1\times P_2}(w_i^1,w_i^2)\le \Big(4K+M\Big)\, d_{G_1\times P_2}(w_i^2,w_{i+1}^1)$ for every $1\le i<r$. Therefore,

$$\begin{aligned}
d_{G_1\times P_2}((w,v_i),(w',v_{i'})) &\le d_{G_1\times P_2}((w,v_i),w_1^1)+\sum_{i=1}^{r} d_{G_1\times P_2}(w_i^1,w_i^2)+\sum_{i=1}^{r-1} d_{G_1\times P_2}(w_i^2,w_{i+1}^1)\\
&\quad + d_{G_1\times P_2}(w_r^2,(w',v_{i'}))\\
&\le d_{G_1\times P_2}((w,v_i),w_1^1)+d_{G_1\times P_2}(w_r^2,(w',v_{i'}))+\Big(4K+M+1\Big)\sum_{i=1}^{r-1} d_{G_1\times P_2}(w_i^2,w_{i+1}^1)\\
&\quad + d_{G_1\times P_2}(w_r^1,w_r^2)\\
&= d_{G^*}(F(w,v_i),F(w_1^1))+d_{G^*}(F(w_r^2),F(w',v_{i'}))+\Big(4K+M+1\Big)\sum_{i=1}^{r-1} d_{G^*}(F(w_i^2),F(w_{i+1}^1))\\
&\quad + d_{G_1\times P_2}(w_r^1,w_r^2)\\
&\le \Big(4K+M+1\Big)\Big(d_{G^*}(F(w,v_i),F(w_1^1))+d_{G^*}(F(w_r^2),F(w',v_{i'}))+\sum_{i=1}^{r-1} d_{G^*}(F(w_i^2),F(w_{i+1}^1))\Big)\\
&\quad +4K+M\\
&\le \Big(4K+M+1\Big)\, d_{G^*}(F(w,v_i),F(w',v_{i'}))+4K+M.
\end{aligned}$$

We conclude that $F:V(G_1\times P_2)\to V(G^*)$ is a quasi-isometric embedding. Thus, Lemma 5.1.8 provides a quasi-isometry $f:G_1\times P_2\to G^*$ with the required property. □

Definition 5.1.27. *Given a geodesic metric space X and closed connected pairwise disjoint subsets $\{\eta_j\}_{j\in J}$ of X, we consider another copy X' of X. The* double *DX of X is the union of X and X' obtained by identifying the corresponding points in each η_j and η_j'.*

Definition 5.1.28. *Let us consider $H>0$, a metric space X, and subsets $Y,Z\subseteq X$. The set $V_H(Y):=\{x\in X:\, d(x,Y)\le H\}$ is called the* H-neighborhood *of Y in X. The* Hausdorff distance *of Y to Z is defined by $\mathcal{H}(Y,Z):=\inf\{H>0:\ Y\subseteq V_H(Z),\ Z\subseteq V_H(Y)\}$.*

The following results in [5] and [51] will be useful.

Theorem 5.1.29. *[5, Theorem 3.2] Let us consider a geodesic metric space X and closed connected pairwise disjoint subsets $\{\eta_j\}_{j\in J}$ of X, such that the double DX is a geodesic metric space. Then the following conditions are equivalent:*

(1) *DX is hyperbolic.*

(2) *X is hyperbolic and there exists a constant c_1 such that for every $k, l \in J$ and $a \in \eta_k, b \in \eta_l$ we have $d_X(x, \cup_{j\in J}\eta_j) \le c_1$ for every $x \in [ab] \subset X$.*

(3) *X is hyperbolic and there exist constants c_2, α, β such that for every $k, l \in J$ and $a \in \eta_k, b \in \eta_l$ we have $d_X(x, \cup_{j\in J}\eta_j) \le c_2$ for every x in some (α, β)-quasi-geodesic joining a with b in X.*

Theorem 5.1.30. *[51, p.87] For each $\delta \ge 0$, $a \ge 1$ and $b \ge 0$, there exists a constant $H = H(\delta, a, b)$ with the following property:*

Let us consider a δ-hyperbolic geodesic metric space X and an (a, b)-quasigeodesic g starting in x and finishing in y. If γ is a geodesic joining x and y, then $\mathcal{H}(g, \gamma) \le H$.

This property is known as geodesic stability. Mario Bonk proved in 1996 that geodesic stability was, in fact, equivalent to Gromov hyperbolicity (see [15]).

Theorem 5.1.31. *Let G_1 be a graph and $B_j := B_{G_1}(w_j, K_j)$ with $w_j \in V(G_1)$ and $K_j \in \mathbb{Z}^+$, for any $j \in J$, such that $\sup_j K_j = K < \infty$, $\overline{B}_{j_1} \cap \overline{B}_{j_2} = \emptyset$ if $j_1 \ne j_2$, and every odd cycle C in G_1 satisfies $C \cap B_j \ne \emptyset$ for some $j \in J$. Suppose $\{B_j\}_{j\in J}$ is M-regular for some $M > 0$. Let G_2 be a non-trivial bounded graph without odd cycles. Then, the following statements are equivalent:*

(1) *$G_1 \times G_2$ is hyperbolic.*

(2) *G_1 is hyperbolic and there exists a constant c_1, such that for every $k, l \in J$ and $w_k \in B_k$, $w_l \in B_l$ there exists a geodesic $[w_k w_l]$ in G_1 with $d_{G_1}(x, \cup_{j\in J} w_j) \le c_1$ for every $x \in [w_k w_l]$.*

(3) *G_1 is hyperbolic and there exist constants c_2, α, β, such that for every $k, l \in J$ we have $d_{G_1}(x, \cup_{j\in J} w_j) \le c_2$ for every x in some (α, β)-quasi-geodesic joining w_k with w_l in G_1.*

Proof. Items (2) and (3) are equivalent by geodesic stability in G_1 (see Theorem 5.1.30).

Assume that (2) holds. By Lemma 5.1.23, there exists an (α, β)-quasi-isometry $f : G_1 \to G_1^*$ with $f(w_j) = w_j^*$ for every $j \in J$. Given $k, l \in J, f([w_k w_l])$ is an (α, β)-quasi-geodesic with endpoints w_k^* and w_l^* in G_1^*. Given $x \in f([w_k w_l])$, we have $x = f(x_0)$ with $x_0 \in [w_k w_l]$ and $d_{G_1^*}(x, \cup_{j\in J} w_j^*) \le \alpha d_{G_1}(x_0, \cup_{j\in J} w_j) + \beta \le \alpha c_1 + \beta$. Taking $X = G_1^*, DX = G^*$ and $\eta_j = w_j^*$ for every $j \in J$, Theorem 5.1.29 gives that G^* is hyperbolic. Now, Lemma 5.1.26 gives that $G_1 \times P_2$ is hyperbolic and we conclude that $G_1 \times G_2$ is hyperbolic by Lemma 5.1.11.

Now suppose (1) holds. By Lemma 5.1.11, $G_1 \times P_2$ is hyperbolic and, by Theorem 5.1.17, G_1 is hyperbolic. Then, Lemma 5.1.26 gives that G^* is hyperbolic and taking $X = G_1^*, DX = G^*$ and $\eta_j = w_j^*$ for every $j \in J$, by Theorem 5.1.29, (2) holds. □

Theorem 5.1.31 and Remark 5.1.25 have the following consequence.

Corollary 5.1.32. *Let G_1 be a graph and suppose that there are a positive integer K and a vertex $w \in G_1$, such that every odd cycle in G_1 intersects the open ball $B := B_{G_1}(w, K)$. Let G_2 be a non-trivial bounded graph without odd cycles. Then, $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

5.2 Bounds for the hyperbolicity constant of some direct products

Remark 5.2.1. *Note that if G_1 is a bipartite graph, then* $\operatorname{diam} G_1 = \operatorname{diam} V(G_1)$*. Furthermore, if G_2 is a bipartite graph, then the product $G_1 \times G_2$ has exactly two connected components, which will be denoted by $(G_1 \times G_2)^1$ and $(G_1 \times G_2)^2$, where each one is a bipartite graph and, consequently,* $\operatorname{diam}(G_1 \times G_2)^i = \operatorname{diam} V((G_1 \times G_2)^i)$ *for $i \in \{1, 2\}$.*

Remark 5.2.2. *Let P_m, P_n be two path graphs with $m \geq n \geq 2$. The product $P_m \times P_n$ has exactly two connected components, which will be denoted by $(P_m \times P_n)^1$ and $(P_m \times P_n)^2$. If $u, v \in V((P_m \times P_n)^i)$ for $i \in \{1, 2\}$, then $d_{(P_m \times P_n)^i}(u, v) = \max\left\{d_{P_m}(\pi_1(u), \pi_1(v)), d_{P_n}(\pi_2(u), \pi_2(v))\right\}$ and* $\operatorname{diam}(P_m \times P_n)^i = \operatorname{diam} V((P_m \times P_n)^i) = m - 1$*.*

Furthermore, if $m_1 \leq m$ and $n_1 \leq n$ then $\delta(P_m \times P_n) \geq \delta(P_{m_1} \times P_{n_1})$.

Lemma 5.2.3. *Let P_m, P_n be two path graphs with $m \geq n \geq 3$, and let γ be a geodesic in $P_m \times P_n$ such that there are two different vertices u, v in γ, with $\pi_1(u) = \pi_1(v)$. Then, $L(\gamma) \leq n - 1$.*

Proof. Let $\gamma := [xy]$, and let $V(P_m) = \{v_1, \ldots, v_m\}, V(P_n) = \{w_1, \ldots, w_n\}$ be the sets of vertices in P_m, P_n, respectively, such that $[v_j, v_{j+1}] \in E(P_m)$ and $[w_i, w_{i+1}] \in E(P_n)$ for $1 \leq j < m, 1 \leq i < n$. Seeking for a contradiction, assume that $L(\gamma) > n - 1$. Notice that if $[uv]$ denotes the geodesic contained in γ joining u and v, then π_2 restricted to $[uv]$ is injective. Consider two vertices $u', v' \in \gamma$ such that $[uv] \subseteq [u'v'] \subseteq \gamma$, π_2 is injective in $[u'v']$ and $\pi_2(u') = w_{i_1}$, $\pi_2(v') = w_{i_2}$ with $i_2 - i_1$ maximal under these conditions. Since $L(\gamma) > n-1 \geq i_2 - i_1$, either there is an edge $[v', w]$ in $G_1 \times G_2$ such that $[v', w] \cap (\gamma \setminus [u'v']) \neq \emptyset$ or there is an edge $[u', w']$ in $G_1 \times G_2$ such that $[u', w'] \cap (\gamma \setminus [u'v']) \neq \emptyset$. Also, since $L(\gamma) > n-1$, notice that π_2 is not injective in γ. Moreover, since $i_2 - i_1$ is maximal, if $\pi_2(w) = w_{i_2+1}$, then $w \notin \gamma$, and since $L(\gamma) > n - 1$, $u' \notin \{x, y\}$ and $\pi_2(w') = w_{i_1+1}$. Thus, either $\pi_2(w) = w_{i_2-1}$ or $\pi_2(w') = w_{i_1+1}$.

Hence, let us assume that there is an edge $[v', w]$ in $G_1 \times G_2$ such that $[v', w] \cap (\gamma \setminus [u'v']) \neq \emptyset$ with $\pi_2(w) = w_{i_2-1}$ (otherwise, if there is an edge $[u', w']$ in $G_1 \times G_2$ such that $[u', w'] \cap (\gamma \setminus [u'v']) \neq \emptyset$ with $\pi_2(w') = w_{i_1+1}$, the proof is similar).

Suppose $\pi_1(v') = v_j$. Let v'' be the vertex in $[u'v']$ such that $\pi_2(v'') = w_{i_2-1}$. Then, by construction of $G_1 \times G_2$, since $v'' \neq w$, it follows that $\{\pi_1(v''), \pi_1(w)\} = \{v_{j-1}, v_{j+1}\}$. Therefore, in particular, $1 < j < m$.

Assume that $v'' = (v_{j-1}, w_{i_2-1})$ (if $v'' = (v_{j+1}, w_{i_2-1})$, then the argument is similar). Therefore, $w = (v_{j+1}, w_{i_2-1})$.

Consider the geodesic

$$\sigma = [(v_{j+1}, w_{i_2-1}), (v_j, w_{i_2-2})] \cup [(v_j, w_{i_2-2}), (v_{j-1}, w_{i_2-3})] \cup [(v_{j-1}, w_{i_2-3}), (v_{j-2}, w_{i_2-4})] \cup \dots$$

Since $\pi_1(u) = \pi_1(v)$, there is a vertex ξ of $V(P_m \times P_n)$ in $[u'v'] \cap \sigma$. Let $s \in [v', w] \cap \gamma$ with $s \neq v'$. Let σ_0 be the geodesic contained in σ joining ξ and w. Let γ_0 be the geodesic contained in γ joining ξ and s. Hence, $L(\sigma_0 \cup [ws]) < L(\sigma_0) + 1 < L(\gamma_0)$ leading to contradiction. □

Theorem 5.2.4. *Let P_m, P_n be two path graphs with $m \geq n \geq 2$. If $n = 2$, then $\delta(P_m \times P_2) = 0$. If $n \geq 3$, then*

$$\min\left\{\frac{m}{2}\,,\, n-1\right\} - 1 \leq \delta(P_m \times P_n) \leq \min\left\{\frac{m}{2}\,,\, n\right\} - \frac{1}{2}\,.$$

Furthermore, if $m \leq 2n-3$ and m is odd, then $\delta(P_m \times P_n) = (m-1)/2$.

Proof. If $m \geq 2$, then $P_m \times P_2$ has two connected components isomorphic to P_m, and $\delta(P_m \times P_2) = 0$.

Assume that $n \geq 3$. By symmetry, it suffices to prove the inequalities for $\delta((P_m \times P_n)^1)$. Hence, Lemma 1.3.7 and Remark 5.2.2 give $\delta((P_m \times P_n)^1) \leq \frac{m-1}{2}$. By Theorem 1.3.13, there exists a geodesic triangle $T = \{x, y, z\} \in \mathbb{T}_1(P_m \times P_n)$ with $p \in \gamma_1 := [xy], \gamma_2 := [xz], \gamma_3 := [yz]$, and $\delta((P_m \times P_n)^1) = \delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3)$. Let $u \in V(\gamma_1)$ such that $d_{(P_m \times P_n)^1}(p, u) \leq 1/2$.

In order to prove $\delta((P_m \times P_n)^1) \leq n - 1/2$, we consider two cases.

Assume first that there is at least a vertex $v \in V((P_m \times P_n)^1) \cap T \setminus \{u\}$ such that $\pi_1(u) = \pi_1(v)$. If $v \notin \gamma_1$, then $v \in \gamma_2 \cup \gamma_3$ and

$$\delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3) \leq 1/2 + d_{(P_m \times P_n)^1}(u, v) \leq n - 1/2.$$

If $v \in \gamma_1$, then $L(\gamma_1) \leq n-1$ by Lemma 5.2.3, and

$$\delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3) \leq d_{(P_m \times P_n)^1}(p, \{x, y\}) \leq (n-1)/2 < n - 1/2.$$

Assume now that there is not a vertex $v \in V((P_m \times P_n)^1) \cap T \setminus \{u\}$ such that $\pi_1(u) = \pi_1(v)$. Then, there exist two different vertices v_1, v_2 in $T \setminus \{u\}$ such that $d_{(P_m \times P_n)^1}(u, v_1) = d_{(P_m \times P_n)^1}(u, v_2) = 1$, and $\pi_1(v_1) = \pi_1(v_2)$. If v_1 or v_2 belongs to $\gamma_2 \cup \gamma_3$, then $\delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3) \leq 3/2 \leq n - 1/2$. Otherwise, $v_1, v_2 \in \gamma_1 \setminus \{u\}$. Lemma 5.2.3 gives $L(\gamma_1) \leq n - 1$, and we have that

$$\delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3) \leq d_{(P_m \times P_n)^1}(p, \{x, y\}) \leq (n-1)/2 < n - 1/2.$$

In order to prove the lower bound, denote the vertices of P_m and P_n by $V(P_m) = \{w_1, w_2, w_3, \dots, w_m\}$ and $V(P_n) = \{v_1, v_2, v_3, \dots, v_n\}$, with $[w_i, w_{i+1}] \in E(P_m)$ for $1 \leq i < m$ and $[v_i, v_{i+1}] \in E(P_n)$ for $1 \leq i < n$.

Let $(P_m \times P_n)^1$ be the connected component of $P_m \times P_n$ containing (w_1, v_{n-1}).

Assume first that $m \geq 2n-3$. Consider the following curves in $(P_m \times P_n)^1$:

$$\begin{aligned}\gamma_1 &:= [(w_1, v_{n-1}), (w_2, v_n)] \cup [(w_2, v_n), (w_3, v_{n-1})] \cup [(w_3, v_{n-1}), (w_4, v_n)] \cup \cdots \\ &\quad \cdots \cup [(w_{2n-4}, v_n), (w_{2n-3}, v_{n-1})], \\ \gamma_2 &:= [(w_1, v_{n-1}), (w_2, v_{n-2})] \cup [(w_2, v_{n-2}), (w_3, v_{n-3})] \cup \cdots \cup [(w_{n-2}, v_2), (w_{n-1}, v_1)] \\ &\quad \cup [(w_{n-1}, v_1), (w_n, v_2)] \cup \cdots \cup [(w_{2n-4}, v_{n-2}), (w_{2n-3}, v_{n-1})].\end{aligned}$$

Corollary 5.1.4 gives that γ_1, γ_2 are geodesics. If B is the geodesic bigon $B = \{\gamma_1, \gamma_2\}$, then Remark 5.2.2 gives that

$$\delta(P_m \times P_n) \geq \delta(B) \geq d_{(P_m \times P_n)^1}((w_{n-1}, v_1), \gamma_1) = n-2.$$

If m is odd with $m \leq 2n-3$, then $n-(m+1)/2 \geq 1$ and we can consider the curves in $(P_m \times P_n)^1$:

$$\begin{aligned}\gamma_1 &:= [(w_1, v_{n-1}), (w_2, v_n)] \cup [(w_2, v_n), (w_3, v_{n-1})] \cup [(w_3, v_{n-1}), (w_4, v_n)] \cup \cdots \\ &\quad \cdots \cup [(w_{m-1}, v_n), (w_m, v_{n-1})], \\ \gamma_2 &:= [(w_1, v_{n-1}), (w_2, v_{n-2})] \cup [(w_2, v_{n-2}), (w_3, v_{n-3})] \cup \cdots \cup [(w_{(m+1)/2-1}, v_{n-(m+1)/2+1}), \\ &\quad (w_{(m+1)/2}, v_{n-(m+1)/2})] \cup [(w_{(m+1)/2}, v_{n-(m+1)/2}), (w_{(m+1)/2+1}, v_{n-(m+1)/2+1})] \cup \cdots \\ &\quad \cdots \cup [(w_{m-1}, v_{n-2}), (w_m, v_{n-1})].\end{aligned}$$

Corollary 5.1.4 gives that γ_1, γ_2 are geodesics. If $B = \{\gamma_1, \gamma_2\}$, then Remark 5.2.2 gives that

$$\delta(P_m \times P_n) \geq \delta(B) \geq d_{(P_m \times P_n)^1}((w_{(m+1)/2}, v_{n-(m+1)/2}), \gamma_1) = (m-1)/2.$$

By Remark 5.2.2, if m is even with $m-1 \leq 2n-3$, then we have that

$$\delta(P_m \times P_n) \geq \delta(P_{m-1} \times P_n) \geq (m-2)/2.$$

Hence,

$$\delta(P_m \times P_n) \geq \left\{ \begin{array}{ll} n-2, & \text{if } \ m \geq 2n-3 \\ (m-2)/2, & \text{if } \ m \leq 2n-2 \end{array} \right\} = \min\left\{n-2, \frac{m-2}{2}\right\} = \min\left\{\frac{m}{2}, n-1\right\} - 1.$$

Furthermore, if $m \leq 2n-3$ and m is odd, then we have proved $(m-1)/2 \leq \delta(P_m \times P_n) \leq (m-1)/2$. □

Theorem 5.2.5. *If G_1 and G_2 are bipartite graphs with $k_1 := \operatorname{diam} V(G_1)$ and $k_2 := \operatorname{diam} V(G_2)$ such that $k_1 \geq k_2 \geq 1$, then*

$$\max\left\{\min\left\{\frac{k_1-1}{2}, k_2-1\right\}, \delta(G_1), \delta(G_2)\right\} \leq \delta(G_1 \times G_2) \leq \frac{k_1}{2}.$$

Furthermore, if $k_1 \leq 2k_2-2$ and k_1 is even, then $\delta(G_1 \times G_2) = k_1/2$.

Proof. Corollary 5.1.4, Lemma 1.3.7 and Remark 5.2.1 give us the upper bound.

In order to prove the lower bound, we can see that there exist two path graphs P_{k_1+1}, P_{k_2+1} which are isometric subgraphs of G_1 and G_2, respectively. It is easy to check that $P_{k_1+1} \times P_{k_2+1}$ is an isometric subgraph of $G_1 \times G_2$. By Lemma 1.3.3 and Theorem 5.2.4, we have

$$\min\left\{\frac{k_1-1}{2}, k_2-1\right\} \leq \delta(P_{k_1+1} \times P_{k_2+1}) \leq \delta(G_1 \times G_2).$$

Using a similar argument as above, we have $\delta(P_2 \times G_2) \leq \delta(G_1 \times G_2)$ and $\delta(G_1 \times P_2) \leq \delta(G_1 \times G_2)$. Thus, since $(G_1 \times P_2)^i \simeq G_1$ and $(P_2 \times G_2)^i \simeq G_2$ for $i \in \{1, 2\}$, we obtain the first statement.

Furthermore, if $k_1 + 1 \leq 2(k_2 + 1) - 3$ and $k_1 + 1$ is odd, then Theorem 5.2.4 gives $\delta(P_{k_1+1} \times P_{k_2+1}) = k_1/2$, and we conclude $\delta(G_1 \times G_2) = k_1/2$. □

Conclusions and Open Problems

Conclusions

We characterize the strong product of two graphs $G_1 \boxtimes G_2$ which are hyperbolic, in terms of G_1 and G_2: the strong product graph $G_1 \boxtimes G_2$ is hyperbolic if and only if one of the factors is hyperbolic and the other one is bounded. We also prove some sharp relations between $\delta(G_1 \boxtimes G_2)$, $\delta(G_1)$, $\delta(G_2)$ and the diameters of G_1 and G_2 (and we find families of graphs for which the inequalities are attained). Furthermore, we obtain the exact values of the hyperbolicity constant for many strong product graphs.

Furthermore, we characterize the lexicographic product of two graphs $G_1 \circ G_2$ which are hyperbolic, in terms of G_1 and G_2: the lexicographic product graph $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic, unless if G_1 is a trivial graph; if G_1 is trivial, then $G_1 \circ G_2$ is hyperbolic if and only if G_2 is hyperbolic. Also, we obtain that $\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1)+3/2$ if G_1 is not a trivial graph, and we find families of graphs for which the inequalities are attained.

Besides, we characterize the hyperbolic product graphs for the Cartesian sum $G_1 \oplus G_2$: $G_1 \oplus G_2$ is always hyperbolic, unless either G_1 or G_2 is the trivial graph; if G_1 or G_2 is the trivial graph, then $G_1 \oplus G_2$ is hyperbolic if and only if G_2 or G_1 is hyperbolic, respectively. We also obtain the sharp inequalities $1 \leq \delta(G_1 \oplus G_2) \leq 3/2$ for every non-trivial graphs G_1, G_2. Besides, we characterize the Cartesian sums with $\delta(G_1 \oplus G_2) = 1$, with $\delta(G_1 \oplus G_2) = 5/4$ and with $\delta(G_1 \oplus G_2) = 3/2$. Furthermore, we obtain the precise value of the hyperbolicity constant of the Cartesian sum of many graphs.

Finally, we prove that if the direct product $G_1 \times G_2$ is hyperbolic, then one factor is hyperbolic and the other one is bounded. Also, we prove that this necessary condition is, in fact, a characterization in many cases. In other cases, we find characterizations which are not so simple. Furthermore, we obtain good bounds for the hyperbolicity constant of the direct product of some important graphs.

Open Problems

- We have characterized the hyperbolic direct product graphs in several cases, but we would like to obtain a complete characterization.

- We have obtained good bounds of $\delta(G_1 \times G_2)$ for several kinds of graphs, but we would like to compute the precise value of $\delta(G_1 \times G_2)$ for some families of graphs.

- We would like to relate the hyperbolicity with other properties of graphs. In particular, we are interested in the relation between the hyperbolicity and the Cheeger isoperimetric inequality.

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