Programa de Doctorado en Matemáticas

Universidad de Málaga

Facultad de Ciencias
Departamento de Análisis Matemático

Tesis Doctoral

Multiplicadores y operadores integrales en espacios de funciones analíticas

(Multipliers and integration operators in spaces of analytic functions)

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13 de Septiembre de 2013

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HACEN CONSTAR:

Que **Christos Chatzifountas** ha realizado bajo nuestra dirección el trabajo de investigación correspondiente a su Tesis Doctoral titulada **Mutiplicadores y operadores integrales en espacios de funciones analíticas** ("Multipliers and integration operators in spaces of analytic functions").

Revisado el presente trabajo, estimamos que puede ser presentado al tribunal que ha de juzgarlo.

Para que así conste a los efectos oportunos, autorizamos la presentación de esta Tesis Doctoral en la Universidad de Málaga.

En Málaga, a 15 de septiembre de 2013

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Resumen

En esta tesis estudiamos la acción de determinados operadores en ciertos espacios clásicos de funciones analíticas en el disco unidad del plano complejo.

Sean $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ el disco unidad y $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ la circunferencia unidad del plano complejo $\mathbb{C}$. El espacio de todas las funciones analíticas en $\mathbb{D}$ será denotado por $\mathcal{H}ol(\mathbb{D})$. Si $0 < r < 1$ y $f \in \mathcal{H}ol(\mathbb{D})$, definimos

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

y

$$M_\infty(r, f) = \max_{0\le\theta\le2\pi} |f(re^{i\theta})|.$$

Para $0 < p \le \infty$, el espacio de Hardy H^p está formado por las funciones $f \in \mathcal{H}ol(\mathbb{D})$ tales que

$$\|f\|_{H^p} = \lim_{r\to1^-} M_p(r, f) < \infty.$$

Referimos a [27] y [39] para la teoría de los espacios de Hardy y operadores entre ellos.

Si $0 < p < \infty$ y $\alpha > -1$, el espacio de Bergman A^p_α es el formado por las funciones $f \in \mathcal{H}ol(\mathbb{D})$ tales que

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha + 1)\int_{\mathbb{D}} (1 - |z|^2)^\alpha |f(z)|^p \, dA(z)\right)^{1/p} < \infty.$$

El espacio A^p_0 es denotado simplemente por A^p. Aquí, $dA(z) = \frac{1}{\pi} dx\, dy$ denota la medida de Lebesgue normalizada en $\mathbb{D}$. Mencionamos [29, 49, 75] como excelentes referencias generales para la teoría de los espacios de Bergman.

Si $1 < p < \infty$ y $\frac{1}{p} + \frac{1}{p'} = 1$, el dual de H^p se puede identificar con $H^{p'}$ y la misma situación ocurre con A^p_α y $A^{p'}_\alpha$. Sin embargo, si $p = 1$ la situación difiere de la anterior y el teorema de dualidad de Fefferman asegura que el dual de H^1 se

identifica con $BMOA$ [39, 41], el espacio de las funciones f en H^1 cuyos valores frontera tienen oscilación media acotada en $\mathbb{T}$. La identificación del dual de A^1_α con el espacio de las funciones de Bloch $\mathcal{B}$ [9], formado por las funciones $f \in \mathcal{H}ol(\mathbb{D})$ tales que

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$$

es también conocida [9, 75] y ha sido muy útil para el desarrollo de la teoría de espacios de Bergman. Indiquemos que se tiene que $H^\infty \subset BMOA \subset \mathcal{B}$.

El espacio de tipo Dirichlet $\mathcal{D}^p_\alpha$ ($0 < p < \infty$, $\alpha > -1$) es el formado por las funciones $f \in \mathcal{H}ol(\mathbb{D})$ tales que $f' \in A^p_\alpha$. Por tanto, si f es una función analítica en $\mathbb{D}$, se tiene que $f \in \mathcal{D}^p_\alpha$ si y sólo si

$$\|f\|^p_{\mathcal{D}^p_\alpha} \stackrel{\text{def}}{=} |f(0)|^p + \|f'\|^p_{A^p_\alpha} < \infty.$$

Es bien sabido que si $p < \alpha+1$ entonces $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$ (véase, e. g. [32, Theorem 6]). Sin embargo, si $p > \alpha+2$ entonces $\mathcal{D}^p_\alpha \subset H^\infty$. Por tanto, diremos que $\mathcal{D}^p_\alpha$ es un "espacio de Dirichlet propio" cuando $\alpha+1 \le p \le \alpha+2$. Se tiene la siguiente cadena de inclusiones

$$\mathcal{D}^p_\alpha \subset H^p \subset A^p_\beta, \quad 0 < p < \infty,\ -1 < \alpha < p-1,\ \beta > -1.$$

Entre los espacios de tipo Dirichlet $\mathcal{D}^p_\alpha$, los espacios $\mathcal{D}^p_{p-1}$ ($0 < p < \infty$) tienen una importancia especial por su estrecha relación con los espacios de Hardy. Indiquemos aquí simplemente que se tiene que $\mathcal{D}^2_1 = H^2$ y que $\mathcal{D}^p_{p-1} \subset H^p$ para $0 < p < 2$, mientras que $H^p \subset \mathcal{D}^p_{p-1}$ para $2 < p < \infty$ [32, 53, 71].

Para $g \in \mathcal{H}ol(\mathbb{D})$, el operador de multiplicación M_g está definido por

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z)f(z), \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D}.$$

Si X e Y son dos espacios normados (o de Fréchet) de funciones analíticas en $\mathbb{D}$ que están continuamente sumergidos en $\mathcal{H}ol(\mathbb{D})$, $M(X,Y)$ denotará al espacio de los multiplicadores de X en Y,

$$M(X,Y) = \{g \in \mathcal{H}ol(\mathbb{D}) :\ fg \in Y, \quad \text{para toda } f \in X\},$$

y $||M_g||_{(X\to Y)}$ denotará la norma del operador M_g. Aquí debemos resaltar que, utilizando el teorema del grafo cerrado, se deduce que si X e Y son completos se tiene que si $M_g(X) \subset Y$ entonces M_g es continuo de X en Y. Si $X = Y$, escribimos simplemente $M(X)$ en lugar de $M(X, X)$.

Es bien sabido que $M(X) \subset H^\infty$ [4, 71]. Este hecho implica fácilmente que, para todo $p > 0$ y todo $\alpha > -1$ se tiene que

$$M(H^p) = M(A^p_\alpha) = H^\infty.$$

Sin embargo, para un buen número de espacios X se tiene que $M(X)$ es un subespacio propio de H^∞ y también debemos resaltar que para $X \neq Y$ el espacio de multiplicadores $M(X, Y)$ puede no estar contenido en H^∞.

Para estudiar la acción del operador de multipicación en un cierto espacio X es a menudo conveniente considerar la acción de dos operadores de integración muy íntimimante relacionados con el de multiplicación y que tienen interés propio, los operadores I_g y T_g, definidos como sigue

$$I_g(f)(z) \stackrel{\text{def}}{=} \int_0^z g(\xi) f'(\xi)\, d\xi, \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D},$$

$$T_g(f)(z) \stackrel{\text{def}}{=} \int_0^z g'(\xi) f(\xi)\, d\xi, \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D}.$$

La relación

$$f(z)g(z) = f(0)g(0) + T_g(f)(z) + I_g(f)(z), \quad f, g \in \mathcal{H}(\mathbb{D})$$

permite comparar, trasladar y utilizar los resultados obtenidos para estos dos operadores integrales al estudio del operador de multiplicación M_g.

El operador T_g ha recibido otros muchos nombres en la literatura, entre ellos el de operador de tipo Volterra (la elección $g(z) = z$ da lugar al operador de Volterra) u operador de Cesáro generalizado (el operador clásico de Cesáro se obtiene cuando $g(z) = \log \frac{1}{1-z}$.) El estudio de operadores integrales juega un papel determinante en el desarrollo y comprensión de otros temas, como la teoría de funciones univalentes [65], teoremas de factorización [2, 62] o análisis armónico [17]. Recordemos que Pommerenke probó en [65] que T_g es acotado en H^2 si y sólo si $g \in BMOA$. Este resultado fue generalizado en [7] por Aleman y Siskakis que probaron que la condición $g \in BMOA$ caracteriza a las funciones $g \in \mathcal{H}ol(\mathbb{D})$ para las que T_g está acotado en H^p $(1 \leq p < \infty)$. Aleman y Siskakis también caracterizaron las funciones g para las que T_g es compacto en H^p $(1 \leq p < \infty)$ y aquéllas para las que T_g pertenece a la clase de Schatten $S_p(H^2)$ $(0 < p < \infty)$. Aleman y Cima [2] consideraron la cuestión de caracterizar las funciones g para los que T_g aplica H^p en H^q para distintos valores de p y q y subsecuentemente se han estudiado distintos aspectos de la acción del operador T_g en espacios de Bergman y espacios de Bergman con pesos y en $BMOA$ [8, 3, 6, 58, 62, 68].

Los operadores de multiplicación M_g y los operadores integrales T_g e I_g actuando sobre espacios de tipo Dirichlet han sido extensivamente estudiados en muchos trabajos (vésase, por ejemplo, [11, 36, 44, 45, 71, 72]). Especial atención merecen por su naturaleza singular y difícil comprensión los espacios $\mathcal{D}^p_{p-1}$ $(0 < p < \infty)$. En [45, 36] se prueba que no existen multiplicadores no triviales entre $\mathcal{D}^p_{p-1}$ y $\mathcal{D}^q_{q-1}$ si $q \neq p$, lo que viene fuertemente motivado por el hecho de que los espacios $\mathcal{D}^p_{p-1}$ no están anidados. Sin embargo, si X es un subespacio del espacio de Bloch $\mathcal{B}$, se tiene que

$$\mathcal{D}^p_{p-1} \cap X \subset \mathcal{D}^q_{q-1} \cap X, \quad 0 < p < q < \infty,$$

lo que conlleva a que, bajo estas hipótesis, el espacio de los multiplicadores de $\mathcal{D}^p_{p-1} \cap X$ en $\mathcal{D}^q_{q-1} \cap X$ no sea trivial y hace que la siguiente pregunta, que es el objeto de nuestro trabajo en el capítulo 2 de la tesis, surja de forma natural:

Si $0 < p, q < \infty$ y X es un subespacio de $\mathcal{B}$, ¿Cuáles son las funciones $g \in \mathcal{H}ol(\mathbb{D})$ tales que M_g es acotado de $\mathcal{D}^p_{p-1} \cap X$ en $\mathcal{D}^q_{q-1} \cap X$?

Podemos plantear la misma pregunta con los operadores integrales T_g e I_g en lugar del operador de multiplicación M_g. Estudiamos estas cuestiones cuando X es H^∞, $BMOA$ y $\mathcal{B}$.

Empecemos considerando el caso $X = \mathcal{B}$. El espacio $M(\mathcal{B})$ de los multiplicadores de $\mathcal{B}$ en sí mismo es conocido (véase [10, 15, 74]). En efecto, se tiene que

$$M(\mathcal{B}) = \mathcal{B}_{\log} \cap H^\infty,$$

donde $\mathcal{B}_{\log}$ es el subespacio de $\mathcal{B}$ formado por las funciones $g \in \mathcal{H}ol(\mathbb{D})$ tales que

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)|g'(z)| \log \frac{e}{1 - |z|^2} < \infty.$$

Utilizando técnicas semejantes a las de Brown y Shields en [15] y estimaciones de las medias integrales de las funciones en $\mathcal{B}$ y en los espacios de tipo Dirichlet, hemos probado que para $p > 1$ el espacio $M(\mathcal{B} \cap \mathcal{D}^p_{p-1})$ coincide con $M(\mathcal{B})$. De hecho, tenemos:

Si $1 < q$ *y* $0 < p \leq q < \infty$*, entonces,* $M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) = M(\mathcal{B})$.

Por el contrario, hemos probado también que si $0 < q < p < \infty$, entonces $M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) = \{0\}$. Para probar este resultado hemos tenido que distinguir varios casos y las demostración en cada uno de dichos casos han seguido ideas distintas. Así, en los casos $2 < q < \infty$ y $0 < q \leq 2 < p < \infty$ hemos hecho uso de propiedades de los conjuntos de ceros de las funciones en los espacios bajo estudio [42, 43]. Por su parte, nuestro trabajo en el caso $0 < p < 2$ ha utilizado series de potencias aleatorias y, concretamente, de la desigualdad de Khinchine [27, Appendix A].

El caso $0 < p \leq q \leq 1$ ha resultado más complicado y no está totalmente cerrado. Señalemos que la inclusión

$$M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) \subset M(\mathcal{B}),$$

es cierta cualesquiera que sean p, q. Por otra parte, hemos demostrado que

$$M(\mathcal{B}) \setminus \mathcal{D}^q_{q-1} \neq \emptyset, \quad \text{si, } 0 < q < 1,$$

resultado que nos permite deducir para $0 < p \leq q \leq 1$

$$M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) \subsetneq M(\mathcal{B}),$$

en contraste con lo que sucede si $q > 1$.

Para construir una función $f \in M(\mathcal{B}) \setminus \mathcal{D}^q_{q-1}$ $(0 < q \leq 1)$ hemos tenido que obtener un refinamiento de un resultado de Fournier [33] sobre interpolación de coeficientes de Taylor (a lo largo de sucesiones lagunares) de funciones en H^∞. Esta extensión del teorema de Fournier también ha sido utilizada para obtener algunos de los resultados que probamos en el caso $X = H^\infty$ que pasamos a considerar.

Es fácil demostrar el siguiente resultado para el caso $p \leq q$:
Si $0 < p \leq q < \infty$*, entonces* $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = H^\infty \cap \mathcal{D}^q_{q-1}$.

Centrándonos en el caso $0 < q < p$, señalemos que si $2 \leq q < p$ entonces se tiene que $H^\infty \cap \mathcal{D}^p_{p-1} = H^\infty \cap \mathcal{D}^q_{q-1} = H^\infty$ con lo que tenemos:

$$M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = H^\infty, \quad 2 \leq q < p.$$

Cuando $0 < q < p$ y $0 < q < 2$ la cuestión es más complicada. Es bien sabido (véase [40, Theorem 1] y [71]) que, para $0 < q < 2$, existe $f \in H^\infty \setminus \mathcal{D}^q_{q-1}$.

Utilizando la antes mencionada extensión del teorema de Fournier, hemos mejorado este resultado probando constructivamente lo siguiente:

Si $0 < q < \min\{p, 2\}$*, entonces existe una función* f *tal que*

$$f \in \left(H^\infty \cap \mathcal{D}^p_{p-1}\right) \setminus \left(H^\infty \cap \mathcal{D}^q_{q-1}\right).$$

Estas funciones han jugado un papel fundamental para demostrar el siguiente resultado:

Si $0 < q < 1$ *y* $0 < q < p < \infty$ *entonces* $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = \{0\}$.

Por otra parte, utilizando [40, Theorem 1] que asegura que si $0 < q < 2$, entonces existe $f \in H^\infty$ tal que

$$\int_0^1 (1-r)^{q-1} |f'(re^{i\theta})|^q \, dr = \infty, \quad \text{para casi todo } \theta \in \mathbb{R},$$

hemos obtenido el siguiente resultado para el rango $1 \le q < 2 \le p$.

Si $1 \le q < 2 \le p$ *entonces* $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = \{0\}$.

El caso $1 \le q < p < 2$ queda abierto y podemos señalar que si la respuesta a la siguiente cuestión abierta fuese afirmativa se tendría que el espacio $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$ sería trivial para estos valores de los parámetros.

Cuestión. *Supongamos que* $0 < q < p < 2$*. ¿Existe* $f \in H^\infty \cap \mathcal{D}^p_{p-1}$ *satisfaciendo*

$$\int_0^1 (1-r)^{q-1} |f'(re^{i\theta})|^q \, dr = \infty, \quad \textit{para casi todo } \theta \in \mathbb{R}?$$

Pasemos ahora a considerar el caso $X = BMOA$. Recordemos que

$$H^\infty \subsetneq BMOA \subsetneq \mathcal{B}, \quad \text{y} \quad H^\infty \subsetneq BMOA \subsetneq \cap_{0<p<\infty} H^p$$

y que $BMOA$ puede caracterizarse en términos de medidas de Carleson:

Una función $f \in \mathcal{H}ol(\mathbb{D})$ *pertenece a* $BMOA$ *si y sólo si la medida de Borel* μ_f *en* $\mathbb{D}$ *definida por* $d\mu_f(z) = (1 - |z|^2)|f'(z)|^2 \, dA(z)$ *es una medida de Carleson.*

Los multiplicadores del espacio $BMOA$ fueron caracterizados en [56] (véase también [68] y [73]). En efecto, se tiene

$$M(BMOA) = H^\infty \cap BMOA_{\log}.$$

Aquí, $BMOA_{\log}$ es el espacio de las funciones $g \in H^1$ para las que existe una constante positiva C tal que

$$\int_{S(I)} (1-|z|^2)|g'(z)|^2\,dA(z) \le C|I| \left(\log \frac{2}{|I|}\right)^{-2}, \quad \text{para todo intervalo } I \subset \partial\mathbb{D}.$$

Siguiendo la terminología de [73], esta caracterización podemos expresarla como sigue:

$BMOA_{\log}$ es el espacio formado for las funciones $g \in H^1$ para las que la medida de Borel μ_g en $\mathbb{D}$ definida por $d\mu_g(z) = (1-|z|^2)|g'(z)|^2\,dA(z)$ es una "medida de Carleson 2-logarítmica".

Para poder hacer un estudio apropiado de los espacios de multiplicadores $M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$, hemos tenido antes que obtener una serie de resultados sobre el espacio $BMOA_{\log}$.

Así, hemos probado directamente la inclusión $BMOA_{\log} \subsetneq \mathcal{B}_{\log} \subsetneq BMOA$ y también hemos obtenido condiciones simples en una función $f \in \mathcal{H}ol(\mathbb{D})$ que implican su pertenencia a $BMOA_{\log}$. Como corolario de estos resultados hemos probado el siguiente resultado sobre series de potencias lagunares en $BMOA_{\log}$.

Sea $f \in \mathcal{H}ol(\mathbb{D})$ dada por una serie de potencias lagunar,

$$f(z) = \textstyle\sum_{k=0}^{\infty} a_k z^{n_k} \ \ (z \in \mathbb{D}) \text{ con } n_{k+1} \ge \lambda n_k \text{ para todo } k, \text{ siendo } \lambda > 1.$$

Si $\sum_{k=0}^{\infty} |a_k|^2 (\log n_k)^3 < \infty$, entonces $f \in BMOA_{\log} \cap H^\infty$.

También hemos considerado series de potencias aleatorias del tipo

$$f_t(z) = \sum_{n=0}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D}, \quad 0 \le t \le 1,$$

donde $f(z) = \sum_{n=0}^{\infty} a_n z^n$ es analítica en $\mathbb{D}$ y $\{r_n\}_{n=0}^{\infty}$ es la sucesión de la funciones de Rademacher. Entre otros resultados, hemos obtenido una condición precisa en los coeficientes a_n de f que implica que f_t pertenece a $BMOA_{\log}$ casi seguramente:

(i) Si $\sum_{n=1}^{\infty} |a_n|^2 (\log n)^3 < \infty$ entonces para casi todo $t \in [0,1]$, la función

$$f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

pertenece a $BMOA_{\log} \cap H^\infty$.

(ii) Además, (i) es muy preciso en el siguiente sentido: Dada una sucesión decreciente de números positivos $\{\delta_n\}_{n=1}^{\infty}$ *con* $\delta_n \to 0$*, cuando* $n \to \infty$*, existe una sucesión de números positivos* $\{a_n\}_{n=1}^{\infty}$ *con* $\sum_{n=1}^{\infty} a_n^2 \delta_n (\log n)^3 < \infty$ *tal que, para casi todo* t *la función* f_t *definida por* $f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n$ $(z \in \mathbb{D})$ *no pertenece a* $\mathcal{B}_{\log}$.

Tras obtener estos resultados para el espacio $BMOA_{\log}$ podemos pasar a exponer nuestros resultados sobre los multiplicadores de $\mathcal{D}_{p-1}^p \cap BMOA$ en $\mathcal{D}_{q-1}^q \cap BMOA$ $(0 < p, q < \infty)$.

Si $\lambda \geq 2$ entonces $BMOA \subset \mathcal{D}_{\lambda-1}^{\lambda}$. Por tanto, se tiene trivialmente que

$$\begin{aligned} M(\mathcal{D}_{p-1}^p \cap BMOA, \mathcal{D}_{q-1}^q \cap BMOA) =& M(BMOA) \\ =& BMOA_{\log} \cap H^{\infty}, \quad 2 \leq p, q < \infty. \quad (\star) \end{aligned}$$

Hemos conseguido probar que esta igualdad sigue siendo válida para otros valores de p y q:

Si $1 < q < \infty$ *y* $0 < p \leq q < \infty$*, entonces*

$$M(\mathcal{D}_{p-1}^p \cap BMOA, \mathcal{D}_{q-1}^q \cap BMOA) = M(BMOA) = BMOA_{\log} \cap H^{\infty}. \quad (\diamond)$$

Por otra parte, también hemos demostrado que *cuando* $q < p$ *entonces* 0 *es el único multiplicador de* $\mathcal{D}_{p-1}^p \cap BMOA$ *en* $\mathcal{D}_{q-1}^q \cap BMOA$*, excepto en los casos cubiertos en* $(\star)$.

Para tratar el caso restante, $0 < p \leq q \leq 1$, hemos utilizado los resultados arriba mencionados sobre series de potencias lagunares y series de potencias aleatorias en el espacio $\mathcal{D}_{p-1}^p \cap BMOA$. Hemos probado los siguientes resultados:

Sea $\{a_n\}_{n=0}^{\infty}$ *una sucesión de complejos satisfaciendo*

$$\sum_{n=1}^{\infty} |a_n|^2 (\log n)^3 < \infty.$$

Para $t \in [0, 1]$ *definimos*

$$f_t(z) = \sum_{n=0}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

donde las r_n son las funciones de Rademacher. Entonces, para casi todo $t \in [0,1]$, la función f_t satisface las siguientes condiciones:

(i) $\int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^2 \left[M_\infty(r, f_t{}')\right]^2 \, dr < \infty$.

(ii) $f_t \in BMOA_{\log} \cap H^\infty$.

(iii) $f_t \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$ *si* $0 < p \leq q$ *y* $q > \frac{1}{2}$.

Además, si $0 < q < \frac{1}{2}$ *entonces existe una sucesión* $\{a_n\}$ *con* $\sum_{n=1}^\infty |a_n|^2 (\log n)^3 < \infty$ *y tal que* $f_t \notin \mathcal{D}^q_{q-1}$*, para casi todo* t*. Por tanto, para esta sucesión* $\{a_n\}$ *y para casi todo* t *tenemos:*

(a) $f_t \in M(BMOA)$.

(b) Si $0 < p \leq \lambda$ *y* $\lambda > \frac{1}{2}$ *entonces* $f_t \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^\lambda_{\lambda-1} \cap BMOA)$.

(c) $f_t \notin M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$*, si* $0 < p \leq q$.

Este resultado demuestra que ($\diamond$) no es cierto para $q < 1/2$.

Hemos obtenido resultados similares sobre multiplicadores de $\mathcal{D}^p_{p-1} \cap BMOA$ en $\mathcal{D}^q_{q-1} \cap BMOA$ dados por series de potencias lagunares, obteniendo en particular otra demostración de la imposibilidad de extender ($\diamond$) al rango $q < 1/2$.

En la sección 5 del capítulo 2 estudiamos los operadores T_g e I_g actuando en los espacios del tipo $\mathcal{D}^p_{p-1} \cap X$ con $X \subset \mathcal{B}$. No incluimos aquí los resultados obtenidos para no alargar demasiado el resumen.

Los resultados hasta aquí expuestos conforman el capítulo 2 de la memoria y buena parte de ellos están incluidos en el artículo [19].

El capítulo 3 de la memoria está dedicado a estudiar una nueva clase de operadores integrales asociados a ciertas matrices de Hankel actuando sobre los espacios de Hardy. La mayor parte de estos resultados se encuentran en [20].

Si μ es una medida de Borel finita en $[0,1)$ y $n = 0, 1, 2, \ldots$, definimos

$$\mu_n = \int_{[0,1)} t^n \, d\mu(t).$$

$\mathcal{H}_\mu$ es la matriz de Hankel $(\mu_{n,k})_{n,k\geq 0}$ con $\mu_{n,k} = \mu_{n+k}$. Esta matriz puede verse como un operador en espacios de funciones analíticas en $\mathbb{D}$ por su acción sobre los coeficientes de Taylor:

$$a_n \mapsto \sum_{k=0}^{\infty} \mu_{n,k} a_k, \quad n = 0, 1, 2, \cdots .$$

De forma más precisa, si $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbb{D})$ definimos

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

siempre que el miembro de la derecha tenga sentido y defina una función analítica en $\mathbb{D}$.

Notemos que para μ la medida de Lebesgue en $[0, 1)$, $\mathcal{H}_\mu$ se reduce a la matriz de Hilbert clásica $\mathcal{H} = \left((n + k + 1)^{-1}\right)_{n,k\geq 0}$ que induce el operador clásico de Hilbert $\mathcal{H}$ que ha sido extensamente estudiado. Recordemos que $\mathcal{H}$ está acotado de H^p en H^p, si y sólo si $1 < p < \infty$ [23, Theorem 1.1] y que la norma de $\mathcal{H}$ como operador de H^p en H^p fue calculada en [24]. Por otra parte, el operador $\mathcal{H}$ está acotado de A^p en A^p si y sólo si $2 < p < \infty$ [22], pero $\mathcal{H}$ no está ni siquiera definido en A^2. En efecto, se demuestra en [24] que existen funciones $f \in A^2$ tales que la serie que define $\mathcal{H}(f)(0)$ es divergente.

Galanopoulos y Peláez [38] describieron las medidas μ para las que el operador de Hilbert generalizado $\mathcal{H}_\mu$ está bien definido y es acotado en H^1.

En esta memoria estudiamos la acción de las matrices de Hilbert generalizadas $\mathcal{H}_\mu$ en los distintos espacios de Hardy H^p $(0 < p < \infty)$. Hemos obtenido una descripción de las medidas μ para las que $\mathcal{H}_\mu$ está bien definido en H^p y además admite la siguiente representación integral

$$\mathcal{H}_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1 - tz}\, d\mu(t), \quad z \in \mathbb{D}, \quad \text{for all } f \in H^1.$$

Por simplicidad, denotaremos

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1 - tz}\, d\mu(t),$$

siempre que el miembro de la derecha tenga sentido y defina una función en $\mathcal{H}ol(\mathbb{D})$.

En concreto, hemos probado:

(1) *Supongamos que* $0 < p \leq 1$ *y sea* μ *una medida de Borel positiva en* $[0,1)$. *Entonces las siguientes condiciones son equivalentes:*

(i) μ *una* $\frac{1}{p}$*-medida de Carleson.*

(ii) $I_\mu(f)$ *define una función analítica en* $\mathbb{D}$ *para cualquier* $f \in H^p$.

Además, si (i) y (ii) se verifican y $f \in H^p$, *entonces* $\mathcal{H}_\mu(f)$ *define también una función analítica en* $\mathbb{D}$, *y* $\mathcal{H}_\mu(f) = I_\mu(f)$, *para toda* $f \in H^p$.

(2) *Supongamos que* $1 < p < \infty$ *y sea* μ *una medida de Borel positiva en* $[0,1)$. *Entonces:*

(i) $I_\mu(f)$ *define una función analítica en* $\mathbb{D}$ *para toda* $f \in H^p$ *si y sólo si*

$$\int_0^1 \left(\int_0^{1-s} \frac{d\mu(t)}{1-t} \right)^{p'} ds < \infty. \quad (\clubsuit)$$

(ii) Si μ *satisface* $(\clubsuit)$, *entonces* $\mathcal{H}_\mu(f)$ *define también una función analítica en* $\mathbb{D}$, *para toda* $f \in H^p$, *y*

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \textit{para toda } f \in H^p.$$

Utilizando estos resultados hemos caracterizado para cualesquiera p, q con $0 < p, q < \infty$, las medidas μ para las que $\mathcal{H}_\mu$ es un operador acotado (compacto) de H^p en H^q. Así, hemos probado los siguientes resultados (sólo enunciaremos aquí los relativos a acotación y omitiremos los correspondientes a compacidad).

(3) *Supongamos que* $0 < p \leq 1$ *y sea* μ *una medida de Borel positiva en* $[0,1)$ *que es una* $\frac{1}{p}$*-medida de Carleson.*

(i) Si $0 < q < 1$, *entonces* $\mathcal{H}_\mu$ *es un operador acotado de* H^p *en* B_q.

(ii) $\mathcal{H}_\mu$ *es un operador acotado de* H^p *en* H^1 *si y sólo si* μ *es una* 1*-logarítmica* $\frac{1}{p}$*-medida de Carleson.*

(iii) Si $q > 1$ *entonces* $\mathcal{H}_\mu$ *es un operador acotado de* H^p *en* H^q *si y sólo si* μ *es una* $\frac{1}{p} + \frac{1}{q'}$*-medida de Carleson.*

Aquí, para $q < 1$, B_q es el espacio formado por las funciones $f \in \mathcal{H}ol(\mathbb{D})$ tales que

$$\int_0^1 (1-r)^{\frac{1}{q}-2} M_1(r,f)\, dr < \infty.$$

El espacio B_q es la "envolvente de Banach" de H^q, es decir, B_q es un espacio de Banach que contiene a H^q como un subespacio denso y los dos espacios tienen los mismos funcionales lineales continuos [26].

(4) *Supongamos que* $1 < p < \infty$ *y sea* μ *una medida de Borel positiva en* $[0, 1)$ *que satisface* (♣).

   (i) Si $0 < p \leq q < \infty$*, entonces* $\mathcal{H}_\mu$ *es un operador acotado de* H^p *en* H^q *si y sólo so* μ *es una* $\frac{1}{p} + \frac{1}{q'}$*-medida de Carleson.*

   (ii) Si $1 < q < p$*, entonces* $\mathcal{H}_\mu$ *es un operador acotado de* H^p *en* H^q *si y sólo si la función definida por* $s \mapsto \int_0^{1-s} \frac{d\mu(t)}{1-t}$ $(s \in [0, 1))$ *pertenece a* $L^{\left(\frac{pq'}{p+q'}\right)'}([0, 1))$.

   (iii) $\mathcal{H}_\mu$ *es un operador acotado de* H^p *en* H^1 *si y sólo si la función definida por* $s \mapsto \int_0^{1-s} \frac{\log \frac{1}{1-t} d\mu(t)}{1-t}$ $(s \in [0, 1))$ *pertenece a* $L^{p'}([0, 1))$.

   (iv) Si $0 < q < 1$*, entonces* $\mathcal{H}_\mu$ *es un operador acotado de* H^p *en* B_q*.*

También hemos obtenido una descripción de las medidas de Borel positivas μ en $[0, 1)$ para las que $\mathcal{H}_\mu$ pertenece a la clase de Schatten $\mathcal{S}_p(H^2)$ en términos de los momentos, simplificando un resultado de Peller [64, p. 239, Corollary 2. 2]. En concreto, hemos probado que, para $1 < p < \infty$, $\mathcal{H}_\mu \in \mathcal{S}_p(H^2)$ si y sólo si

$$\sum_{n=0}^{\infty} (n + 1)^{p-1} \mu_n^p < \infty.$$

Introduction

This thesis is devoted to study certain operators acting on classical spaces of analytic functions in the unit disc.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle.

We shall also let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$, the Hardy space H^p consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty.$$

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1) \int_{\mathbb{D}} (1-|z|^2)^\alpha |f(z)|^p \, dA(z) \right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p. Here, $dA(z) = \frac{1}{\pi} dx\, dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$.

The space $\mathcal{D}^p_\alpha$ $(0 < p < \infty,\ \alpha > -1)$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $f' \in A^p_\alpha$. Hence, if f is analytic in $\mathbb{D}$, then $f \in \mathcal{D}^p_\alpha$ if and only if

$$\|f\|^p_{\mathcal{D}^p_\alpha} \stackrel{\text{def}}{=} |f(0)|^p + \|f'\|^p_{A^p_\alpha} < \infty.$$

If $p < \alpha + 1$ then it is well known that $\mathcal{D}^p_\alpha = A^p_{\alpha - p}$ (see, e. g. Theorem 6 of [32]). On the other hand, if $p > \alpha + 2$ then $\mathcal{D}^p_\alpha \subset H^\infty$. Therefore $\mathcal{D}^p_\alpha$ becomes a "proper Dirichlet space" when $\alpha + 1 \leq p \leq \alpha + 2$. The following chain of embeddings holds

$$\mathcal{D}^p_\alpha \subset H^p \subset A^p_\beta, \quad 0 < p < \infty, \; -1 < \alpha < p - 1, \; \beta > -1.$$

We shall be mainly interested in the spaces $\mathcal{D}^p_{p-1}$ $(0 < p < \infty)$ which are very closely related with Hardy spaces. Let us simply note here that $\mathcal{D}^2_1 = H^2$ and that $\mathcal{D}^p_{p-1} \subset H^p$ for $0 < p < 2$, while $H^p \subset \mathcal{D}^p_{p-1}$ for $2 < p < \infty$.

Let us recall also that the Bloch space $\mathcal{B}$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \, |f'(z)| < \infty.$$

For $g \in \mathcal{H}ol(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z) f(z), \quad f \in \mathcal{H}ol(\mathbb{D}), \; z \in \mathbb{D}.$$

If X and Y are two normed (or Fréchet) spaces of analytic functions in $\mathbb{D}$ which are continuously embedded in $\mathcal{H}ol(\mathbb{D})$, $M(X, Y)$ will denote the space of multipliers from X to Y,

$$M(X, Y) = \{g \in \mathcal{H}ol(\mathbb{D}) : \; fg \in Y, \quad \text{for all } f \in X\},$$

and $||M_g||_{(X \to Y)}$ will denote the norm of the operator M_g. If $X = Y$ we simply write $M(X)$ instead of $M(X, X)$. Let us remark that for a reasonable space $X \subset \mathcal{H}ol(\mathbb{D})$, $M(X)$ only contains bounded functions. This easily implies that, for all $p > 0$ and all $\alpha > -1$

$$M(H^p) = M(A^p_\alpha) = H^\infty.$$

However, for a lot of important spaces X we have that $M(X)$ is a proper subspace of H^∞. Also, when $X \neq Y$ the space of multipliers $M(X, Y)$ need not be contained in H^∞.

In order to study the action of the multiplication operator on a certain space X it is often convenient to consider two closely related operators, the operators I_g y T_g, defined as follows

$$I_g(f)(z) \stackrel{\text{def}}{=} \int_0^z g(\xi) f'(\xi) \, d\xi, \quad f \in \mathcal{H}ol(\mathbb{D}), \; z \in \mathbb{D}, \tag{0.0.1}$$

$$T_g(f)(z) \stackrel{\text{def}}{=} \int_0^z g'(\xi)f(\xi)\,d\xi, \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D}. \tag{0.0.2}$$

The relation

$$f(z)g(z) = f(0)g(0) + T_g(f)(z) + I_g(f)(z), \quad f, g \in \mathcal{H}(\mathbb{D})$$

allows us to relate results obtained for the integral operators with results on the multiplication operator.

Multipliers and integration operators acting on Dirichlet type spaces have been extensively studied in $\mathcal{D}^p_\alpha$ in [44, 45, 36], where among other results it is proved that

$$M(\mathcal{D}^p_{p-1}, \mathcal{D}^q_{q-1}) = \{0\}, \quad 0 < p, q < \infty,\ p \neq q.$$

However, if X is a subspace of the Bloch space then

$$\mathcal{D}^p_{p-1} \cap X \subset \mathcal{D}^q_{q-1} \cap X, \quad 0 < p < q < \infty.$$

This implies that under this assumption then for many spaces X any polynomial is a multiplier from $\mathcal{D}^p_{p-1} \cap X$ to $\mathcal{D}^q_{q-1} \cap X$. Then the following question arises naturally:

If $0 < p$, $q < \infty$ and X is a subspace of the Bloch space $\mathcal{B}$, what are the multipliers from $\mathcal{D}^p_{p-1} \cap X$ in $\mathcal{D}^q_{q-1} \cap X$? The same question can be formulated for the integral operators T_g and I_g instead of the multiplication operators M_g. Chapter 2 of this thesis is devoted to study these questions taking as X some of the most important subspaces of $\mathcal{B}$ such as $\mathcal{B}$, H^∞ and $BMOA$. Most of the results contained in this chapter are included in the article [19].

Chapter 3 is devoted to study another class of integral operators associated with certain Hankel matrices acting on Hardy spaces.

If μ is a finite positive Borel measure on $[0, 1)$ and $n = 0, 1, 2, \ldots,$ we let μ_n denote the moment of order n of μ, that is,

$$\mu_n = \int_{[0,1)} t^n\, d\mu(t),$$

and we define $\mathcal{H}_\mu$ to be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathcal{H}_\mu$ can be viewed as an operator on spaces of analytic functions by its action on the Taylor coefficients:

$$a_n \mapsto \sum_{k=0}^{\infty} \mu_{n,k} a_k, \quad n = 0, 1, 2, \cdots.$$

To be precise, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbb{D})$ we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

Notice that when μ is the Lebesgue measure in $[0,1)$, $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = \left((n+k+1)^{-1}\right)_{n,k\geq 0}$ which induces the extensively studied Hilbert operator.

Galanopoulos and Peláez [38] described the measures μ for which the generalized Hilbert operator $\mathcal{H}_\mu$ is well defined and bounded in H^1.

In this thesis we study the generalized Hilbert matrices $\mathcal{H}_\mu$ acting in the distinct Hardy spaces H^p $(0 < p < \infty)$. We have obtained a description of the measures μ such that $\mathcal{H}_\mu$ is well defined and admits the following integral representation in H^p $(0 < p < \infty)$

$$\mathcal{H}_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t), \quad z \in \mathbb{D}, \quad \text{for all } f \in H^1.$$

Using this in a basic way, we have also characterized, for any p, q with $0 < p, q < \infty$, the measures μ for which $\mathcal{H}_\mu$ is a bounded (compact) operator from H^p to H^q. Furthermore, we have obtained a description of those μ for which $\mathcal{H}_\mu$ belongs to the Schatten class $\mathcal{S}_p(H^2)$.

Chapter 1

Preliminaries

This chapter is devoted to present the main spaces and operators which will be the object of our work.

We shall let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ be the boundary of $\mathbb{D}$. We shall also let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \, dt\right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$, the Hardy space H^p consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty$$

(see [27, 39] for the theory of H^p-spaces). In particular, it is known that whenever $f \in H^p$, $0 < p \leq \infty$, f has finite non-tangential limits a. e. on $\mathbb{T}$. We shall also denote this function defined on $\mathbb{T}$ by f.

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha + 1)\int_{\mathbb{D}} (1 - |z|^2)^\alpha |f(z)|^p \, dA(z)\right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p. Here, $dA(z) = \frac{1}{\pi} dx\, dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We refer to [29, 49, 75] for the theory of these spaces.

The space $\mathcal{D}^p_\alpha$ ($0 < p < \infty$, $\alpha > -1$) consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $f' \in A^p_\alpha$. Hence, if f is analytic in $\mathbb{D}$, then $f \in \mathcal{D}^p_\alpha$ if and only if

$$\|f\|^p_{\mathcal{D}^p_\alpha} \stackrel{\text{def}}{=} |f(0)|^p + \|f'\|^p_{A^p_\alpha} < \infty.$$

If $p < \alpha + 1$ then it is well known that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$ (see, e. g. Theorem 6 of [32]). On the other hand, if $p > \alpha + 2$ then $\mathcal{D}^p_\alpha \subset H^\infty$. Therefore $\mathcal{D}^p_\alpha$ becomes a "proper Dirichlet space" when $\alpha + 1 \leq p \leq \alpha + 2$. The following chain of embeddings holds

$$\mathcal{D}^p_\alpha \subset H^p \subset A^p_\beta, \quad 0 < p < \infty,\ -1 < \alpha < p - 1,\ \beta > -1.$$

We recall that the Bloch space $\mathcal{B}$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)\, |f'(z)| < \infty.$$

We refer to [9, 75] for the theory of Bloch functions.

We shall write I for an interval of $\mathbb{T}$ and $|I|$ for its length. If $\psi \in L^1(\partial\mathbb{D})$, we let ψ_I denote the mean of f over the interval I, that is,

$$\psi_I \stackrel{\text{def}}{=} \frac{1}{|I|} \int_I \psi(e^{i\theta})\, d\theta.$$

The mean oscillation of ψ over I is

$$|\psi - \psi_I|_I = \frac{1}{|I|} \int_I |\psi(e^{i\theta}) - \psi_I|\, d\theta.$$

We say that ψ has bounded mean oscillation or that $\psi \in BMO(\mathbb{T})$ if

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |\psi(e^{i\theta}) - \psi_I|\, d\theta < \infty.$$

We define $BMOA$ as the space of those functions $f \in H^1$ such that the function $e^{i\theta} \mapsto f(e^{i\theta})$ of the boundary values of f belongs to $BMO(\mathbb{T})$. The space $BMOA$ can be equipped with several different equivalent norms [12, 39, 41]. We often work with the one given in terms of Carleson measures.

If $I \subset \mathbb{T}$ is an interval, the *Carleson square* $S(I)$ is defined as

$$S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}.$$

Also, for $a \in \mathbb{D}$, the Carleson box $S(a)$ is defined by

$$S(a) = \Big\{ z \in \mathbb{D} : 1 - |z| \leq 1 - |a|, \, \Big| \frac{\arg(a\bar{z})}{2\pi} \Big| \leq \frac{1 - |a|}{2} \Big\}.$$

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D},$$

or, equivalently, if there exists $C > 0$ such that

$$\mu\left(S(a)\right) \leq C(1 - |a|)^s, \quad \text{for all } a \in \mathbb{D}.$$

If μ satisfies $\lim_{|I| \to 0} \frac{\mu\left(S(I)\right)}{|I|^s} = 0$ or, equivalently, $\lim_{|a| \to 1} \frac{\mu\left(S(a)\right)}{(1 - |a|^2)^s} = 0$, then we say that μ is a *vanishing s-Carleson measure.*

An 1-Carleson measure, respectively, a vanishing 1-Carleson measure, will be simply called a Carleson measure, respectively, a vanishing Carleson measure.

As an important ingredient in his work on interpolation by bounded analytic functions, Carleson [18] (see also Theorem 9.3 of [27]) proved that if $0 < p < \infty$ and μ is a positive Borel measure in $\mathbb{D}$ then $H^p \subset L^p(d\mu)$ if and only if μ is a Carleson measure. This result was extended by Duren [25] (see also [27, Theorem 9.4]) who proved that for $0 < p \leq q < \infty$, $H^p \subset L^q(d\mu)$ if and only if μ is a q/p-Carleson measure.

Now we can give the characterization of $BMOA$ in terms of Carleson measurees: Let $g \in \mathcal{H}ol(\mathbb{D})$, then $g \in BMOA$ if and only if the measure $|g'(z)|^2(1 - |z|^2)dA(z)$ is a Carleson measure, and we equip $BMOA$ with the norm [41]

$$\|g\|^2_{\text{BMOA}} = \sup_{I \in \mathbb{T}} \frac{\int_{S(I)} |g'(z)|^2(1 - |z|^2)\, dA(z)}{|I|} + |g(0)|^2.$$

The following chain of embeddings [12, 39, 41] holds

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

For $w \in \mathbb{D}$, we let φ_w denote the Möbius transformation defined by

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}.$$

Then φ_w is a conformal mapping from the unit disc onto itself and interchanges the origin with w.

Let us denote by $\mathrm{Aut}(\mathbb{D})$ the group of all conformal mappings from $\mathbb{D}$ onto itself. It is known that

$$\mathrm{Aut}(\mathbb{D}) = \{\lambda\varphi_w : w \in \mathbb{D}, |\lambda| = 1\}.$$

A space $X \subset \mathcal{H}(\mathbb{D})$ equipped with a seminorm ρ is called *conformally invariant* or *Möbius invariant* if there exists a constant $C > 0$ such that

$$\sup_{\varphi} \rho(g \circ \varphi) \leq C\rho(g), \quad g \in X,$$

where the supremum is taken on all Möbius transformations φ of $\mathbb{D}$ onto itself. BMOA and $\mathcal{B}$ have the important property of being conformally invariant spaces [41].

We define the *pseudo-hyperbolic metric* ρ by

$$\rho(z, w) = |\varphi_w(z)|, \quad z, w \in \mathbb{D}.$$

Note that $\rho(z, 0) = |z|$.

The Schwarz-Pick Lemma can be stated as follows:

Lemma A. (Schwarz-Pick Lemma). *If f is an analytic function in the unit disc $\mathbb{D}$ such that $f(\mathbb{D}) \subset \mathbb{D}$ then*

$$\rho\big(f(z), f(w)\big) \leq \rho(z, w), \quad z, w \in \mathbb{D}. \tag{1.0.1}$$

Furthermore, if equality holds for some pair of points $z, w \in \mathbb{D}$ with $z \neq w$ then $f \in \mathrm{Aut}(\mathbb{D})$ and in such a case equality holds for every $z, w \in \mathbb{D}$.

The following known formulas [39] will be used throughout this memory.

(i) φ_w is an isometry with respect to the pseudo-hyperbolic metric

$$\rho\big(\varphi_w(z), \varphi_w(\zeta)\big) = \rho(z, \zeta) \qquad \text{for all z, } \zeta \in \mathbb{D}. \tag{1.0.2}$$

(ii) The quantities $1-|z|^2$ and $1-|\varphi_w(z)|^2$ are related by the equation

$$1-|\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^2} = (1-|z|^2)|\varphi_w'(z)|.$$

(iii) φ_w is involutive: $\varphi_w\big(\varphi_w(z)\big) = z$.

1.1 Operators on spaces of analytic functions

Let X be a space of analytic functions on the disc. For $g \in \mathcal{H}ol(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z)f(z), \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D}.$$

If X and Y are two normed (or Fréchet) spaces of analytic functions in $\mathbb{D}$ which are continuously embedded in $\mathcal{H}ol(\mathbb{D})$, $M(X,Y)$ will denote the space of multipliers from X to Y,

$$M(X,Y) = \{g \in \mathcal{H}ol(\mathbb{D}) :\ fg \in Y, \quad \text{for all } f \in X\},$$

and $||M_g||_{(X\to Y)}$ will denote the norm of the operator M_g. If $X = Y$ we simply write $M(X)$ instead of $M(X,X)$. The next useful lemma [71, Lemma 1.10] asserts that for a reasonable space $X \subset \mathcal{H}ol(\mathbb{D})$, $M(X)$ only contains bounded functions.

Lemma 1. *Assume that X is an space of analytic functions on $\mathbb{D}$ such that the point evaluations are bounded on X. Then $M(X) \subset H^\infty$ and*

$$||g||_{H^\infty} \leq ||M_g||_{(X,X)}.$$

Proof. Take $f \in X$, $f \neq 0$ and $a \notin Z(f) = \{z \in \mathbb{D} : f(z) = 0\}$. Then, there is $C_a > 0$ such that

$$|f(a)| \leq C_a||f||_X.$$

So, for each $n \in \mathbb{N}$ and $g \in M(X)$,

$$|g^n(a)f(a)| \leq C_a||g^nf||_X \leq C_a||M_g||_{(X,X)} \cdot ||g^{n-1}f||_X \leq C_a||M_g||^n_{(X,X)} \cdot ||f||_X,$$

that is

$$|g(a)|f(a)|^{1/n} \leq ||M_g||_{(X,X)}(C_a||f||_X)^{1/n},$$

which implies

$$|g(a)| \leq ||M_g||_{(X,X)}.$$

Since g is continuous on $\mathbb{D}$ and $Z(f)$ is a discrete set, the result follows. □

The previous result implies that

$$M(H^p) = H^\infty, \quad 0 < p \leq \infty$$

and $M(A^p_\alpha) = H^\infty$, $0 < p < \infty$.

For a good number of classical spaces $X \subset \mathcal{H}ol(\mathbb{D})$, $M(X)$ is a proper subspace of H^∞. In order to study the action of the multiplication operator M_g on a certain space is quite often useful to consider the closely related integration operators I_g and T_g, defined as follows

$$I_g(f)(z) \stackrel{\text{def}}{=} \int_0^z g(\xi)f'(\xi)\,d\xi, \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D}, \tag{1.1.1}$$

$$T_g(f)(z) \stackrel{\text{def}}{=} \int_0^z g'(\xi)f(\xi)\,d\xi, \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D}. \tag{1.1.2}$$

The relation

$$M_g(f)(z) = I_g(f)(z) + T_g(f)(z) + f(0)g(0),$$

roughly speaking, implies that whenever two of these operators are bounded, the third one is also bounded. However, for many classical spaces, there are symbols $g \in \mathcal{H}ol(\mathbb{D})$ such that one of them (usually T_g) is bounded, but I_g and M_g do not remain bounded, that is, in the relation $T_g(f) = fg - I_g(f)$ there is some sort of cancellation. For instance, this happens for Hardy spaces, because $T_g : H^p \to H^p$ is bounded if and only if $g \in BMOA$ [2, 8], while, as noted above, M_g is bounded on H^p if and only if $g \in H^\infty$.

1.2 The spaces of Dirichlet type $\mathcal{D}^p_{p-1}$

In this section we present some properties of $\mathcal{D}^p_{p-1}$ spaces, most of them will be used throughout the next chapter. Among all the $\mathcal{D}^p_\alpha$-spaces, the spaces $\mathcal{D}^p_{p-1}$ are those which are "closer" to Hardy spaces. Indeed, the classical Littlewood-Paley formula says that $\mathcal{D}^2_1 = H^2$. Moreover, [53]

$$H^p \subsetneq \mathcal{D}^p_{p-1}, \quad \text{for } 2 < p < \infty, \tag{1.2.1}$$

and by [32, 71]

$$\mathcal{D}^p_{p-1} \subsetneq H^p, \quad \text{for } 0 < p < 2. \tag{1.2.2}$$

We remark that for $p \neq q$ there is no relation of inclusion between $\mathcal{D}^p_{p-1}$ and $\mathcal{D}^q_{q-1}$ (see, e. g., [13] and [43]).

The connection between Hardy and Dirichlet-type spaces $\mathcal{D}^p_{p-1}$ is evident by many facts, for example they have the same univalent (analytic and one to one) functions [13]. Going further, let us turn our attention to Carleson measures:

If X is a subspace of $\mathcal{H}ol(\mathbb{D})$, $0 < q < \infty$, and μ is a positive Borel measure in $\mathbb{D}$, μ is said to be a "*q-Carleson measure for the space X*" or an "*(X, q)-Carleson measure*" if $X \subset L^q(d\mu)$. The q-Carleson measures for the spaces H^p, $0 < p, q < \infty$ are completely characterized. The mentioned results of Carleson and Duren can be stated saying that if $0 < p \leq q < \infty$ then a positive Borel measure μ in $\mathbb{D}$ is a q-Carleson measure for H^p if and only if μ is a q/p-Carleson measure. Luecking [54] and Videnskii [70] solved the remaining case $0 < q < p$. We mention [14] for a complete information on Carleson measures for Hardy spaces.

Regarding the $\mathcal{D}^p_{p-1}$ spaces we have the following: If $0 < p < q$, the q-Carleson measures for $\mathcal{D}^p_{p-1}$ are the same as those for H^p [45]. This statement remains valid also in the diagonal case $q = p$, if $p \leq 2$, but fails for $p > 2$ [44, 36, 72]. In more general terms, the p-Carleson measures for $\mathcal{D}^p_\alpha$ are known excepting the case $\alpha = p-1$ for $p > 2$ [11, 72]. This corresponds to the diagonal case $q = p > 2$. What is known with respect to this case, is that μ being a 1-Carleson measure is a necessary but not a sufficient condition for μ to be a p-Carleson measure for $\mathcal{D}^p_{p-1}$ [44]. Recently, it has been proved in [61, Theorem 5] that the condition

$$\sup_{I \subset \mathbb{T}} \frac{\mu\left(S(I)\right)}{|I| \left(\log \frac{e}{|I|}\right)^{-p/2+1}} < \infty, \tag{1.2.3}$$

implies that μ is a p-Carleson measure for $\mathcal{D}^p_{p-1}$. Moreover, this condition is sharp in a strong sense [61, Proposition 12] and the best sufficient L^∞-type condition known, involving Carleson measures.

On the other hand, $\mathcal{D}^p_{p-1}$ spaces present a lot of differences with H^p spaces. Roughly speaking, in some sense they are much bigger (smaller) than Hardy spaces if $2 < p < \infty$ $(0 < p < 2)$. By (1.2.1) the L^p-means $M_p(r, f)$ are bounded if $f \in \mathcal{D}^p_{p-1}$ and $0 < p < 2$. This is not longer true for $p > 2$ [43]. Indeed, if $f \in \mathcal{D}^p_{p-1}$ and $2 < p < \infty$

$$M_p(r,f) = \mathrm{o}\left(\left(\log\frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}}\right), \quad r \to 1^- \tag{1.2.4}$$

and this result is sharp [61, Lemma 10] (see also [43, 46]).

For a given space X of analytic functions in $\mathbb{D}$, a sequence $\{z_k\}$ is called an X-*zero set*, if there exists a function f in X such that f vanishes precisely on the points z_k and nowhere else. It follows from (1.2.2) that the zero sequence of f satisfies the Blaschke condition whenever $f \in \mathcal{D}^p_{p-1}$, $0 < p \le 2$. However, if $2 < p < \infty$ this does not remain true. If $f \in \mathcal{D}^p_{p-1}$, and $\{z_k\}$ is its zero sequence repeated according to multiplicity and ordered by increasing moduli, then [43]

$$\prod_{k=1}^{N} \frac{1}{|z_k} = o\left((\log N)^{\frac{1}{2}-\frac{1}{p}}\right), \quad \text{as } N \to \infty. \tag{1.2.5}$$

Moreover, this result is sharp in the following sense [43];

Theorem A. *(***Theorem 1.7 of [43]***) Suppose that* $2 < p < \infty$ *and* $0 < \beta < \frac{1}{2} - \frac{1}{p}$ *Then there exists a sequence* $\{\omega_j\} \subset [0,1]$ *such that the function*

$$f(z) = \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{2}-\beta}} e^{2\pi i \omega_j} z^{2^j} \tag{1.2.6}$$

belongs to $\mathcal{D}^p_{p-1}$ *and* $\{z_n\}$*, its zero sequence repeated according to multiplicity and ordered by increasing moduli satisfies that*

$$\prod_{n=1}^{N} \frac{1}{|z_n|} \ne o\left(\log N\right)^{\beta}, \quad \text{as } N \to \infty.$$

Finally, it is worth to mention the following result proved in [40] (see [47] for a refinement) and which will be basic to obtain some of our results.

Theorem B. *Let $0 < p < 2$. Then there exists $f \in H^\infty$ such that*

$$\int_0^1 (1-r)^{p-1}|f'(re^{i\theta})|^p\,dr = \infty, \quad \textit{for almost every } \theta \in \mathbb{R}, \tag{1.2.7}$$

in particular

$$\int_{\mathbb{D}} |f'(z)|^p(1-|z|)^{p-1}dA(z) = \infty. \tag{1.2.8}$$

Let us close this chapter noticing that, as usual, throughout this work, the letter $C = C(\cdot)$ will denote a constant whose value depends on the parameters indicated in the parenthesis (which often will be omitted), and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists $C = C(\cdot) > 0$ such that $a \le Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

Chapter 2

Operators on Dirichlet subspaces of the Bloch space

In this chapter, we shall study multiplication and integration operators on Dirichlet subspaces of the Bloch space and most of our results in it are included in [19].

Multipliers and integration operators acting on the Dirichlet type spaces $\mathcal{D}^p_\alpha$ have been extensively studied in [44, 45, 36], where among other results it is proved that

$$M(\mathcal{D}^p_{p-1}, \mathcal{D}^q_{q-1}) = \{0\}, \quad 0 < p, q < \infty, \ p \neq q. \tag{2.0.1}$$

However, the following result implies that this is not the case when we intersect $\mathcal{D}^p_{p-1}$ with a subspace of the Bloch space $\mathcal{B}$.

Lemma 2. *Suppose that* $0 < p < q < \infty$ *and* $f \in \mathcal{D}^p_{p-1} \cap \mathcal{B}$*. Then* $f \in \mathcal{D}^q_{q-1}$.

Proof. Since $f \in \mathcal{B}$ we have that $\sup_{z\in\mathbb{D}}(1-|z|)|f'(z)| = M < \infty$. Using this we obtain

$$\begin{aligned}\int_{\mathbb{D}}(1-|z|)^{q-1}|f'(z)|^q \, dA(z) &= \int_{\mathbb{D}}\big[(1-|z|)|f'(z)|\big]^{q-p}(1-|z|)^{p-1}|f'(z)|^p \, dA(z)\\ &\leq M^{q-p}\int_{\mathbb{D}}(1-|z|)^{p-1}|f'(z)|^p \, dA(z) < \infty.\end{aligned}$$

Hence, $f \in \mathcal{D}^q_{q-1}$. □ □

In other words, whenever X is a subspace of the Bloch space we have that

$$X \cap \mathcal{D}^p_{p-1} \subset X \cap \mathcal{D}^q_{q-1}, \quad \text{if } \ 0 < p \leq q < \infty, \tag{2.0.2}$$

a fact which implies that, contrary to what happens in (2.0.1), whenever $0 < p \leq q < \infty$, the space of multipliers $M(\mathcal{D}^p_{p-1} \cap X, \mathcal{D}^q_{q-1} \cap X)$ is non-trivial.

If $X \subset \mathcal{B}$, the space $X \cap \mathcal{D}^p_{p-1}$ is equipped with the norm

$$\|f\|_{X\cap\mathcal{D}^p_{p-1}} \stackrel{\text{def}}{=} \|f\|_X + \|f\|_{\mathcal{D}^p_{p-1}}.$$

In this way, $X \cap \mathcal{D}^p_{p-1}$ is complete. Our main aim in this chapter is to obtain a description of the spaces of multipliers $M(\mathcal{D}^p_{p-1} \cap X, \mathcal{D}^q_{q-1} \cap X)$ $(0 < p, q < \infty)$ for some important subspaces X of the Bloch space. In order to present our results we need to define the following spaces.

For $\alpha > 0$, the α-logarithmic-Bloch space $\mathcal{B}_{\log,\alpha}$ consists of those $g \in \mathcal{H}ol(\mathbb{D})$ such that

$$\rho_\alpha(g) \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)|\left(\log\frac{e}{1-|z|^2}\right)^\alpha < \infty.$$

It is clear that

$$\mathcal{B}_{\log,\alpha} \subset \mathcal{B}_{\log,\beta} \subset \mathcal{B}, \quad \alpha \geq \beta > 0. \tag{2.0.3}$$

For simplicity, the space $\mathcal{B}_{\log,1}$ will be denoted by $\mathcal{B}_{\log}$. Spaces of $\mathcal{B}_{\log,\alpha}$ type show up in the study of integration and multiplication operators on the Bloch and related spaces.

2.1 Multipliers from $\mathcal{B} \cap \mathcal{D}^p_{p-1}$ to $\mathcal{B} \cap \mathcal{D}^q_{q-1}$.

The study of the multipliers from $\mathcal{B}\cap\mathcal{D}^p_{p-1}$ to $\mathcal{B}\cap\mathcal{D}^q_{q-1}$ is related to that of the multipliers of the Bloch space into itself. $M(\mathcal{B})$ has been characterized by several authors independently (see [10, 15, 74]). Namely, the following holds

$$M(\mathcal{B}) = \mathcal{B}_{\log} \cap H^\infty. \tag{2.1.1}$$

We have proved that the space $M(\mathcal{B}\cap\mathcal{D}^p_{p-1})$ is nothing else but $M(\mathcal{B})$ for $p > 1$. This is part of the following result.

Theorem 1. *Let* $0 < p, q < \infty$.

(i) If $1 < q$ *and* $0 < p \leq q < \infty$*, then,*

$$M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) = M(\mathcal{B}).$$

(ii) If $0 < q < p < \infty$*, then*

$$M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) = \{0\}.$$

The proof of the first part is based on the proof by Brown and Shields in [15] where they characterize $M(\mathcal{B})$. In order to prove the second part of the theorem we introduce the notion of lacunary power series or power series with Hadamard gaps.

A function $f \in \mathcal{H}ol(\mathbb{D})$ is given by a lacunary power series (also called power series with Hadamard gaps) if it is of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \ \ (z \in \mathbb{D}) \ \text{ with } n_{k+1} \geq \lambda n_k \ \text{ for all } k, \text{ for a certain } \lambda > 1.$$

For simplicity, we shall let $\mathcal{L}$ denote the class of all functions $f \in \mathcal{H}ol(\mathbb{D})$ which are given by a lacunary power series. Several known results on power series with Hadamard gaps will be repeatedly used throughout this section. We collect them in the following statement (see [16, 76, 9]).

Proposition A. *Suppose that $0 < p < \infty$, $\alpha > -1$ and f is an analytic function in $\mathbb{D}$ which is given by a power series with Hadamard gaps,*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \ \ (z \in \mathbb{D}) \ \textit{ with } n_{k+1} \geq \lambda n_k \ \textit{ for all } k \ \ (\lambda > 1).$$

Then:

(i) $f \in \mathcal{D}^p_\alpha \iff \sum_{k=0}^{\infty} n_k^{p-\alpha-1}|a_k|^p < \infty$, *and*

$$||f - f(0)||^p_{\mathcal{D}^p_\alpha} \asymp \sum_{k=0}^{\infty} n_k^{p-\alpha-1}|a_k|^p.$$

(ii) $f \in H^\infty$ *if and only if* $\sum_{k=0}^{\infty} |a_k| < \infty$, *and*

$$||f||_{H^\infty} \asymp \sum_{k=0}^{\infty} |a_k|.$$

(iii) $f \in \mathcal{B} \iff \sup_n |a_n| < \infty$, *and*

$$\|f\|_{\mathcal{B}} \asymp \sup_n |a_n|.$$

It is also well known that $\mathcal{L} \cap H^p = \mathcal{L} \cap H^2$ for any $p \in (0, \infty)$ but

$$H^\infty \cap \mathcal{L} \subsetneq H^2 \cap \mathcal{L}.$$

Proof of Theorem 1 (i). Assume that $g \in M(\mathcal{B}\cap\mathcal{D}^p_{p-1}, \mathcal{B}\cap\mathcal{D}^q_{q-1})$. Let φ_a be the Möbius transformation that interchanges the origin and a. By the Schwarz-Pick theorem, the family $\{\varphi_a : a \in \mathbb{D}\}$ is uniformly bounded on the Bloch space

$$\|\varphi_a\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1 - |z|^2)|\varphi_a'(z)| + |\varphi_a(0)| = \sup_{z\in\mathbb{D}}(1 - |\varphi_a(z)|^2) + |\varphi_a(0)| < 2. \quad (2.1.2)$$

They are also uniformly bounded in the $\mathcal{D}^p_{p-1}$ spaces for all values of p. This can be proved using [75, Proposition 4.3]

$$\begin{aligned}\|\varphi_a\|^p_{\mathcal{D}^p_{p-1}} &= \int_{\mathbb{D}} |\varphi_a'(z)|^p(1-|z|^2)^{p-1}dA(z) + |\varphi_a(0)|^p \\ &= \int_{\mathbb{D}} \left|\frac{1-|a|^2}{(1-\overline{a}z)^2}\right|^p (1-|z|^2)^{p-1}dA(z) + |a|^p \\ &\lesssim C.\end{aligned} \tag{2.1.3}$$

Consequently,

$$\sup_{a\in\mathbb{D}} \|\varphi_a\|_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} < \infty.$$

Now observe that for any $a, z \in \mathbb{D}$ we have that

$$\begin{aligned}&(1-|z|^2)|\varphi_a'(z)g(z)| = (1-|z|^2)|(\varphi_a\cdot g)'(z) - \varphi_a(z)g'(z)| \\ &\le \|\varphi_a g\|_{\mathcal{B}\cap\mathcal{D}^q_{q-1}} + (1-|z|^2)|\varphi_a(z)g'(z)| \lesssim \|M_g\|_{(\mathcal{B}\cap\mathcal{D}^p_{p-1}\to\mathcal{B}\cap\mathcal{D}^q_{q-1})} + \|g\|_{\mathcal{B}} < \infty.\end{aligned}$$

Since $(1-|a|^2)|\varphi_a'(a)| = 1$, taking $z = a$ we obtain

$$|g(a)| \lesssim \|M_g\|_{(\mathcal{B}\cap\mathcal{D}^p_{p-1}\to\mathcal{B}\cap\mathcal{D}^q_{q-1})} + \|g\|_{\mathcal{B}} < \infty,$$

for any $a \in \mathbb{D}$. Thus, $g \in H^\infty$.

Next consider the test functions, $f_\theta(z) = \log\frac{1}{1-ze^{-i\theta}}$, $\theta \in [0, 2\pi)$. Since

$$\begin{aligned}\|f_\theta\|^p_{\mathcal{D}^p_{p-1}} &= \frac{1}{\pi}\int_0^1 r(1-r)^{p-1}\int_0^{2\pi}\frac{1}{|1-re^{i(t-\theta)}|^p}\,dr\,dt \\ &\lesssim C < \infty,\end{aligned} \tag{2.1.4}$$

and

$$\sup_{\theta\in[0,2\pi)} \|f_\theta\|_{\mathcal{B}} = \sup_{z\in\mathbb{D},\,\theta\in[0,2\pi)} \frac{(1-|z|^2)}{|1-ze^{-i\theta}|} < \infty,$$

we have that $\{f_\theta\}_{\theta\in[0,2\pi)}$ is a uniformly bounded family of functions in $\mathcal{B}\cap\mathcal{D}^p_{p-1}$. Therefore,

$$\begin{aligned}A = \sup_{\theta\in[0,2\pi)} \|gf_\theta\|_{\mathcal{B}} &\le \sup_{\theta\in[0,2\pi)} \|gf_\theta\|_{\mathcal{B}\cap\mathcal{D}^q_{q-1}} \\ &\le \|M_g\|_{(\mathcal{B}\cap\mathcal{D}^p_{p-1}\to\mathcal{B}\cap\mathcal{D}^q_{q-1})} \sup_{\theta\in[0,2\pi)} \|f_\theta\|_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} < \infty,\end{aligned}$$

which implies that

$$\begin{aligned}(1-|z|^2)|g'(z)f_\theta(z)| &= (1-|z|^2)|g'(z)f_\theta(z)+g(z)f'_\theta(z)-g(z)f'_\theta(z)| \\ &\le A+(1-|z|^2)|g(z)f'_\theta(z)| \\ &= A+\|g\|_{H^\infty}\sup_{\theta\in[0,2\pi)}\|f_\theta\|_{\mathcal{B}} \\ &< \infty, \quad \text{for all} \quad z\in\mathbb{D} \text{ and } \theta\in[0,2\pi).\end{aligned}$$

Given $z \in \mathbb{D}$, choose $e^{i\theta} = \frac{z}{|z|}$ to deduce that

$$\sup_{z\in\mathbb{D}} |g'(z)|(1-|z|)\log\frac{1}{1-|z|} < \infty,$$

which together the fact that $g \in H^\infty$, gives $g \in M(\mathcal{B})$.

Suppose now that $g \in M(\mathcal{B})$ and take $f \in \mathcal{B} \cap \mathcal{D}^p_{p-1}$. Then $fg \in \mathcal{B}$. Using Lemma 2 and the closed graph theorem, we obtain

$$\begin{aligned}&\int_{\mathbb{D}} |(fg)'(z)|^q(1-|z|^2)^{q-1}dA(z) \\ &\lesssim \int_{\mathbb{D}} |f'(z)g(z)|^q(1-|z|^2)^{q-1}dA(z) + \int_{\mathbb{D}} |g'(z)f(z)|^q(1-|z|^2)^{q-1}dA(z) \\ &\lesssim \|g\|^q_{H^\infty}\|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} + \int_{\mathbb{D}} |f(z)g'(z)|^q(1-|z|^2)^{q-1}dA(z).\end{aligned} \tag{2.1.5}$$

We shall distinguish two cases to deal with the last integral in (2.1.5). If $1 < q \le 2$, bearing in mind (1.2.2) and the fact that $g \in \mathcal{B}_{\log}$, we see that

$$\begin{aligned}\int_{\mathbb{D}} |f(z)g'(z)|^q(1-|z|^2)^{q-1}dA(z) &\lesssim \int_0^1 \frac{1}{(1-r)^q\log^q\frac{e}{1-r}}M^q_q(r,f)(1-r)^{q-1}\,dr \\ &\lesssim \|f\|^q_{\mathcal{D}^q_{q-1}}\int_0^1 \frac{1}{(1-r)\log^q\frac{e}{1-r}}\,dr \\ &\lesssim \|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-1}}.\end{aligned} \tag{2.1.6}$$

On the other hand, if $2 < q < \infty$, then using that $g \in \mathcal{B}_{\log}$ and the well known fact that

$$M_q(r,f) \le C\|f\|_{\mathcal{B}}\left(\log\frac{1}{1-r}\right)^{1/2}, \quad 0<r<1,$$

(see, e. g., [21]) we get

$$\int_{\mathbb{D}} |f(z)g'(z)|^q (1-|z|^2)^{q-1} dA(z) \lesssim \int_0^1 \frac{1}{(1-r)^q \log^q \frac{e}{1-r}} M_q^q(r,f)(1-r)^{q-1}\, dr$$
$$\lesssim \|f\|_{\mathcal{B}}^q \int_0^1 \frac{1}{(1-r)\log^{q/2}\frac{e}{1-r}}\, dr < \infty. \tag{2.1.7}$$

Joining (2.1.5) (2.1.6) and (2.1.7), we see that in any case we have $fg \in \mathcal{D}^q_{q-1}$ and, hence, $fg \in \mathcal{B} \cap \mathcal{D}^q_{q-1}$. Thus, we have proved that $g \in M(\mathcal{B}\cap\mathcal{D}^p_{p-1}, \mathcal{B}\cap\mathcal{D}^q_{q-1})$ finishing the proof. □

Proof of Theorem 1 (ii). We shall distinguish three cases.

Case 1. $2 < q < \infty$. Assume that $g \in M(\mathcal{B}\cap\mathcal{D}^p_{p-1}, \mathcal{B}\cap\mathcal{D}^q_{q-1})$ and $g \not\equiv 0$. Let f be as in Theorem A with $\beta = \frac{1}{2} - \frac{1}{q}$ and let $\{z_n\}$ its sequence of ordered zeros. Then

$$\prod_{n=1}^N \frac{1}{|z_n|} \neq o\left(\log N\right)^{\frac{1}{2}-\frac{1}{q}}.$$

Observe that the above function f is represented by a power series with Hadamard gaps. By Theorem A, the function f belongs to $\mathcal{D}^p_{p-1}$ and, by Proposition A it belongs to the Bloch space.

Now if $\{w_n\}$ is the sequence of non-zero zeros of gf arranged so that $|w_1| \leq |w_2| \leq |w_3| \ldots$, we have that $|w_n| \leq |z_n|$, for all n, which gives that

$$\prod_{n=1}^N \frac{1}{|w_n|} \geq \prod_{n=1}^N \frac{1}{|z_n|},$$

hence

$$\prod_{n=1}^N \frac{1}{|w_n|} \neq o\left(\log N\right)^{\frac{1}{2}-\frac{1}{q}}.$$

Since $gf \in \mathcal{D}^q_{q-1}$, it follows from (1.2.5) that $g \equiv 0$.

Case 2. $0 < q \leq 2 < p$. The proof is similar to that of case 1. Suppose that $g \not\equiv 0$ and $g \in M(\mathcal{B}\cap\mathcal{D}^p_{p-1}, \mathcal{B}\cap\mathcal{D}^q_{q-1})$. Take $\beta \in \left(0, \frac{1}{2} - \frac{1}{p}\right)$ and take f as in Theorem A with a non-zero sequence of ordered zeros $\{z_n\}_{n=1}^{\infty}$ so that $\{z_n\}$ satisfies

$$\prod_{n=1}^N \frac{1}{|z_n|} \neq o\left(\log N\right)^{\beta}. \tag{2.1.8}$$

Let $\{w_n\}_{n=1}^{\infty}$ be the sequence of ordered non-zero zeros of fg. Since $fg \in \mathcal{D}^q_{q-1}$ and $q \le 2$, it follows that $fg \in H^q$ and, hence, $\{w_n\}_{n=1}^{\infty}$ satisfies the Blaschke condition which is

$$\prod_{k=1}^{N} \frac{1}{|w_k|} = \mathrm{O}(1), \quad \text{as } N \to \infty.$$

This is in contradiction with (2.1.8), because every zero of f is also a zero of fg. Consequently, $g \equiv 0$.

For the the remaining case $0 < p \le 2$ we have a different approach that is based on *random power series* which are defined as follows:

Let $\{c_k\}_{k=1}^{\infty} \in \ell^2$ where $\{r_n(t)\}_{n=0}^{\infty}$ are the Rademacher functions:

$$r_0(t) = \begin{cases} 1, & \text{if } 0 < t < 1/2 \\ -1, & \text{if } 1/2 < t < 1 \\ 0, & \text{if } t = 0, 1/2, 1. \end{cases}$$

$$r_n(t) = r_0(2^n t), \quad n = 1, 2, \dots.$$

Here, of course, we extend the definition of r_0 to $\mathbb{R}$ by periodicity.

We shall consider the random power series

$$\sum_{k=1}^{\infty} c_k r_k(t) z^k. \tag{2.1.9}$$

We refer the reader to [76, Chapter V, Vol. I] or [27, Appendix A] for the properties of these functions. In particular, we shall use the following result of Khinchine.

Proposition B (Khinchine's inequality)**.** *If $\{c_k\}_{k=1}^{\infty} \in \ell^2$ then the random power series $\sum_{k=1}^{\infty} c_k r_k(t)$ converges for almost all values of t . Furthermore, for $0 < p < \infty$ there exist positive constants A_p, B_p such that for every sequence $\{c_k\}_{k=0}^{\infty} \in \ell^2$ we have*

$$A_p \left(\sum_{k=0}^{\infty} |c_k|^2 \right)^{p/2} \le \int_0^1 \left| \sum_{k=0}^{\infty} c_k r_k(t) \right|^p dt \le B_p \left(\sum_{k=0}^{\infty} |c_k|^2 \right)^{p/2}.$$

Case 3. $\mathbf{0 < p \le 2}$. Suppose that $g \not\equiv 0$ and $g \in M(\mathcal{B}\cap\mathcal{D}^p_{p-1}, \mathcal{B}\cap\mathcal{D}^q_{q-1})$. Take $a_n = \frac{1}{n^{1/p+\varepsilon}}$ with $0 < \varepsilon < \frac{1}{q} - \frac{1}{p}$ and $f(z) = \sum_{n=1}^{\infty} a_n z^{2^n}$. Since $\sum_{n=1}^{\infty} a_n^p < \infty$ and $\sum_{n=1}^{\infty} a_n^q = \infty$, then by Proposition A, $f \in \mathcal{B}\cap\mathcal{D}^p_{p-1} \setminus \mathcal{D}^q_{q-1}$.

Let $\{r_k(t)\}$ be the Rademacher functions and let $f_t(z) = \sum_{k=1}^{\infty} r_k(t) a_k z^{2^k}$. By Proposition A (iii)

$$\|f\|_{\mathcal{B}} \asymp \sup_n |a_n| \asymp \|f_t\|_{\mathcal{B}}, \quad t \in [0,1]$$

and

$$\|f_t\|_{H^2}^{2p} = \left(\sum_{k=0}^{\infty} |a_k|^2\right)^p \le \left(\sum_{k=0}^{\infty} |a_k|^p\right)^2 \asymp \|f_t\|_{\mathcal{D}_{p-1}^p}^{2p} \asymp \|f\|_{\mathcal{D}_{p-1}^p}^{2p}, \quad t \in [0,1]. \quad (2.1.10)$$

Then for any $t \in [0,1]$, it follows that

$$\int_{\mathbb{D}} |(gf_t)'(z)|^q (1-|z|^2)^{q-1} dA(z) \lesssim \|f_t\|_{\mathcal{D}_{p-1}^p}^q + \|f_t\|_{\mathcal{B}}^q \asymp \|f\|_{\mathcal{D}_{p-1}^p}^q + \|f\|_{\mathcal{B}}^q < \infty. \quad (2.1.11)$$

So, by Fubini's theorem, Khinchine's inequality and the fact that $g \in \mathcal{D}_{q-1}^q$, we obtain

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{D}} |gf_t'(z)|^q (1-|z|^2)^{q-1} dA(z) dt \\
&\lesssim \int_0^1 \int_{\mathbb{D}} |(gf_t)'(z)|^q (1-|z|^2)^{q-1} dA(z) dt + \int_0^1 \int_{\mathbb{D}} |f_t g'(z)|^q (1-|z|^2)^{q-1} dA(z) dt \\
&\lesssim \|f\|_{\mathcal{B}\cap\mathcal{D}_{p-1}^p}^q + \int_{\mathbb{D}} |g'(z)|^q \int_0^1 |f_t(z)|^q (1-|z|^2)^{q-1} dt dA(z) \\
&\lesssim \|f\|_{\mathcal{B}\cap\mathcal{D}_{p-1}^p}^q + \int_{\mathbb{D}} |g'(z)|^q M_2^q(|z|, f)(1-|z|^2)^{q-1} dA(z) \\
&\lesssim \|f\|_{\mathcal{B}\cap\mathcal{D}_{p-1}^p}^q + \|f\|_{\mathcal{D}_{p-1}^p}^q \int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-1} dA(z) \\
&\lesssim \|f\|_{\mathcal{B}\cap\mathcal{D}_{p-1}^p}^q.
\end{aligned} \quad (2.1.12)$$

On the other hand, since $g \not\equiv 0$, there exists a positive constant C such that $M_q^q(r,g) \ge C$, $1/2 < r < 1$. Using Fubini's theorem, Khinchine's inequality and bearing in mind that f' is also given by a power series with Hadamard gaps (thus $M_2(r, f') \asymp M_q(r, f')$) we have that

$$
\begin{aligned}
&\int_0^1\int_{\mathbb{D}}|gf_t'(z)|^q(1-|z|^2)^{q-1}dA(z)dt\\
&=\int_{\mathbb{D}}|g(z)|^q(1-|z|^2)^{q-1}\left(\int_0^1|f_t'(z)|^qdt\right)dA(z)\\
&\asymp\int_{\mathbb{D}}|g(z)|^q(1-|z|^2)^{q-1}M_2^q(|z|,f')dA(z) \qquad (2.1.13)\\
&\geq C\int_{1/2}^1 M_q^q(r,g)M_q^q(r,f')(1-r^2)^{q-1}dr\\
&\geq C\int_{1/2}^1 M_q^q(r,f')(1-r^2)^{q-1}dr=+\infty.
\end{aligned}
$$

This is in contradiction with (2.4.8). It follows that $g\equiv 0$. □

To deal with the remaining case $0<q\leq 1$, we need a result of its own interest, which is a refinement and extension of a construction of Fournier in [33] and which will also used in further sections. Before we state it, let us say that if $f(z)=\sum_{k=0}^{\infty}a_kz^k$ is analytic in $\mathbb{D}$ and $n<m$, we set $S_{n,m+1}f(z)=\sum_{k=n}^{m}a_kz^k$.

Theorem 2. *Assume that $\{u_k\}_{k=0}^{\infty}\in\ell^2$ and let $\{n_k\}_{k=0}^{\infty}$ be a sequence of positive integers such that $n_{k+1}>2n_k$, for all k. Then, there exists a function $\Psi\in\mathcal{H}ol(\mathbb{D})$ with power series expansion*

$$\Psi(z)=\sum_{n=0}^{\infty}a_nz^n,\quad z\in\mathbb{D},$$

with the following properties:

- *(i)* $\Psi\in H^{\infty}$.
- *(ii)* $a_{n_k}=u_k$, *for all* k.
- *(iii) If we define as* $\Lambda_0=\{n_0\}$ *and* $\Lambda_k=\lfloor n_k-n_{k-1},n_k\rfloor$ *for* $k>0$, *we have that the sets* Λ_k *are pairwise disjoint and satisfy* $\Lambda_k\subset\lfloor n_{k-1}+1,n_k\rfloor$ *for all* $k\geq 1$. *Furthermore,* $a_n=0$ *if* $n\notin\cup_{k=0}^{\infty}\Lambda_k$.
- *(iv) There is an absolute constant* C *such that*

$$\left\|S_{n_k+1,n_{k+1}+1}\Psi\right\|_{H^{\infty}}\leq C|u_k|,\quad \textit{for all } k.$$

Proof. The construction depends on the following equality [33, p. 402]

$$|a+vb|^2+|b-\bar{v}a| = (1+|v|^2)(|a|^2+|b|^2), \quad a,b,v\in\mathbb{C}. \tag{2.1.14}$$

Let us define inductively the following sequences of functions on $\mathbb{T}$

$$\phi_0(\zeta)=u_0\zeta^{n_0}, \quad h_0(\zeta)=1, \quad \zeta\in\mathbb{T}, \tag{2.1.15}$$

and, for $k>0$,

$$\phi_k(\zeta)=\phi_{k-1}(\zeta)+u_k\zeta^{n_k}h_{k-1}(\zeta), \quad h_k(\zeta)=h_{k-1}(\zeta)-\overline{u_k}\zeta^{-n_k}\phi_{k-1}(\zeta), \quad (\zeta\in\mathbb{T}). \tag{2.1.16}$$

Since $n_{k+1}>2n_k$, it is clear that the sets Λ_k, $k=1,2,\dots$, are disjoint and that $\Lambda_k\subset\lfloor n_{k-1}+1,n_k\rfloor$ for all $k\geq 1$.

We claim that that the sequences $\{\phi_k\}$ and $\{h_k\}$ satisfy the following properties

$$\widehat{\phi_k}(n)=0, \quad \text{whenever } k\geq 0 \text{ and } n\notin\bigcup_{j=0}^{k}\Lambda_j. \tag{2.1.17}$$

$$\widehat{h_k}(-n)=0, \quad \text{whenever } k\geq 0 \text{ and } n\geq 1 \text{ and } n\notin\bigcup_{j=1}^{k}\Lambda_j. \tag{2.1.18}$$

$$\widehat{\phi_k}(n)=\widehat{\phi_j}(n), \quad \text{whenever } k\geq j \text{ and } n\leq n_j, \tag{2.1.19}$$

$$\widehat{\phi_k}(n_j)=u_j, \quad \text{whenever } k\geq j\ . \tag{2.1.20}$$

It is clear that (2.1.17) and (2.1.18) hold for $k=0,1$. Arguing by induction, assume that (2.1.17) and (2.1.18) are valid for some value of $k\in\mathbb{N}$. Then,

$$\phi_{k+1}(\zeta)=\phi_k(\zeta)+u_{k+1}\zeta^{n_{k+1}}h_k(\zeta)=\sum_{j=0}^{k}\sum_{n\in\Lambda_j}\widehat{\phi_k}(n)\zeta^n+f_k(\zeta), \tag{2.1.21}$$

where $f_k(\zeta)=u_{k+1}\zeta^{n_{k+1}}h_k(\zeta)$. By the induction hypotheses $\widehat{f_k}(n)=0$ if $n\notin\Lambda_{k+1}$, which gives (2.1.17) for $k+1$. The proof of (2.1.18) is analogous. Now, (2.1.19) follows from (2.1.16), (2.1.17) and the fact that the sets Λ_k are disjoint and (2.1.18). Using again that the sets Λ_k are disjoint, (2.1.19), (2.1.16) and (2.1.15), we deduce (2.1.20).

We have that

$$|\phi_0(\zeta)|^2+|h_0(\zeta)|^2=1+|u_0|^2,$$

so if we assume that $|\phi_k(\zeta)|^2+|h_k(\zeta)|^2=\prod_{j=0}^k(1+|u_j|^2)$, bearing in mind (2.1.14) and (2.1.16), it follows that

$$|\phi_{k+1}(\zeta)|^2+|h_{k+1}(\zeta)|^2=\left(1+|u_{k+1}|^2\right)\left(|\phi_k(\zeta)|^2+|h_k(\zeta)|^2\right)=\prod_{j=0}^{k+1}(1+|u_j|^2),$$

hence we have proved by induction that

$$|\phi_k(\zeta)|^2+|h_k(\zeta)|^2=\prod_{j=0}^{k}(1+|u_j|^2),\quad \zeta\in\mathbb{T},\quad k=0,1,2,\dots.$$

This and the fact that $\{u_k\}_{k=0}^\infty\in\ell^2$ imply that $\{h_k\}_{k=0}^\infty$ and $\{\phi_k\}_{k=0}^\infty$ are uniformly bounded sequences of functions in $L^\infty(\mathbb{T})$. Then, using the Banach-Alaoglu theorem, (2.1.17), (2.1.19) and (2.1.20), we deduce that a subsequence of $\{\phi_k\}$ converges in the weak star topology of $L^\infty(\mathbb{T})$ to a function $\phi\in L^\infty(\mathbb{T})$ with $\hat\phi(n)=0$ for all $n<0$, and $\hat\phi(n_k)=u_k$ for all k. Then if we set $a_n=\hat\phi(n)$ $(n\ge 0)$ it follows that the function Ψ defined by

$$\Psi(z)=\sum_{n=0}^\infty a_nz^n,\quad z\in\mathbb{D},$$

is analytic in $\mathbb{D}$ and satisfies (i), (ii) and (iii).

Finally, we shall prove (iv). Using(2.1.19) and (2.1.21), we see that for any $\zeta\in\mathbb{T}$, we have

$$S_{n_k+1,n_{k+1}+1}\Psi(\zeta)=\sum_{n=n_k+1}^{n_{k+1}}\widehat{\Psi}(n)\zeta^n=\sum_{m=n_k+1}^{n_{k+1}}\left(\lim_{m\to\infty}\widehat{\phi_m}(n)\right)\zeta^n$$
$$=\sum_{n=n_k+1}^{n_{k+1}}\widehat{\phi_{k+1}}(n)\zeta^n=f_k(\zeta)=u_{k+1}\zeta^{n_{k+1}}h_k(\zeta),$$

which, bearing in mind that $\sup_k\|h_k\|_\infty=C<\infty$, implies

$$\left\|S_{n_k+1,n_{k+1}+1}\Psi\right\|_{H^\infty}=|u_{k+1}|\|h_k\|_{L^\infty(\mathbb{T})}\le C|u_{k+1}|.$$

This finishes the proof. □

Next we shall prove the following result.

Proposition 1. *Let* $0 < p \le q \le 1$, $\alpha \in \left(\frac{1}{q}, \infty\right)$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *Then,*

(*i*) *If* $g \in \mathcal{B}_{\log,\alpha} \cap H^\infty$, *then* $g \in M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1})$.

(*ii*) $\left(\mathcal{B}_{\log,\frac{1}{q}} \cap H^\infty\right) \setminus \mathcal{D}^q_{q-1} \neq \{0\}$.

Before presenting a proof of the above result, we observe that the question of obtaining a complete characterization of $M(\mathcal{B}\cap\mathcal{D}^p_{p-1}, \mathcal{B}\cap\mathcal{D}^q_{q-1})$ in the case $0 < p \le q \le 1$ remains open. However, we remark that the inclusion

$$M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) \subset M(\mathcal{B}),$$

is true for any p, q (see the proof of Theorem 1). Using this, the fact that $M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1}) \subset \mathcal{B} \cap \mathcal{D}^q_{q-1}$ and Proposition 1, we see that part (i) of Theorem 1 does not remain true for $0 < q \le 1$.

Corollary 1. *If* $0 < q \le 1$, *then* $M(\mathcal{B}) \setminus \mathcal{D}^q_{q-1} \neq \emptyset$.

Proof of Proposition 1. Part (i) is proved if we argue as in (2.1.22) and (2.1.23). Indeed, suppose that $g \in \mathcal{B}_{\log,\alpha} \cap H^\infty$, $0 < p \le q \le 1$ and take $f \in \mathcal{B} \cap \mathcal{D}^p_{p-1}$. Then $fg \in \mathcal{B}$. Using Lemma 2 and the closed graph theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |(fg)'(z)|^q (1-|z|^2)^{q-1} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)g(z)|^q (1-|z|^2)^{q-1} dA(z) + \int_{\mathbb{D}} |g'(z)f(z)|^q (1-|z|^2)^{q-1} dA(z) \\
&\lesssim \|g\|^q_{H^\infty} \|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} + \int_{\mathbb{D}} |f(z)g'(z)|^q (1-|z|^2)^{q-1} dA(z).
\end{aligned} \tag{2.1.22}$$

Since $\mathcal{D}^p_{p-1} \subset \mathcal{D}^q_{q-1} \subset H^q$, then $M(f,r)^q_q \lesssim \|f\|_{\mathcal{D}^p_{p-1}}$. Thus

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)g'(z)|^q (1-|z|^2)^{q-1} dA(z) &\lesssim \int_0^1 \frac{1}{(1-r)^q \log^{\alpha q} \frac{e}{1-r}} M^q_q(r,f)(1-r)^{q-1}\, dr \\
&\lesssim \|f\|^q_{\mathcal{D}^q_{q-1}} \int_0^1 \frac{1}{(1-r)\log^{\alpha q} \frac{e}{1-r}}\, dr \\
&\lesssim \|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-1}},
\end{aligned} \tag{2.1.23}$$

which completes the proof of (i).

(ii) Assume first that $0<q<1$. Consider the lacunary power series

$$g(z)=\sum_{k=1}^{\infty}\frac{1}{k^{1/q}}z^{2^k}.$$

By Proposition A, $g\in H^\infty\setminus\mathcal{D}^q_{q-1}$. Since $\limsup_{k\to\infty}\frac{1}{k^{1/q}}\left(\log 2^k\right)^{1/q}<\infty$, using [63, p. 245], we conclude that $g\in\mathcal{B}_{\log,\frac{1}{q}}$.

Let us consider now the case $q=1$. The proof in this case is a little bit more involved. Set

$$u_k=\frac{1}{k+1}\quad\text{and}\quad n_k=4^k,\quad k=0,1,2,\dots.$$

Let Ψ be the H^∞-function associated to these sequences via Theorem 2. By [71, Lemma 1.6 (i)],

$$\|\Psi\|_{\mathcal{D}^1_0}\gtrsim\|\{\widehat{\Psi}(4^k\}_{k=0}^{\infty}\|_{\ell^1}=\sum_{k=0}^{\infty}\frac{1}{k+1}=\infty.$$

Finally, we shall see that $\Psi\in\mathcal{B}_{\log}$. Bearing in mind, [59, p. 113], [55, Lemma 3. 1] and Theorem 2 (iv), we deduce

$$\begin{aligned}
M_\infty(r,\Psi')&\le|\widehat{\Psi}(1)|+\sum_{k=0}^{\infty}M_\infty(r,S_{n_k+1,n_{k+1}}\Psi')\\
&\lesssim\|\Psi\|_{H^\infty}+\sum_{k=0}^{\infty}\|S_{n_k+1,n_{k+1}}\Psi'\|_{H^\infty}r^{4^k}\\
&\lesssim\|\Psi\|_{H^\infty}+\sum_{k=0}^{\infty}4^k\|S_{n_k+1,n_{k+1}}\Psi\|_{H^\infty}r^{4^k}\\
&\lesssim\|\Psi\|_{H^\infty}+\sum_{k=0}^{\infty}\frac{4^k}{k+1}r^{4^k}\lesssim\frac{1}{(1-r)\log\frac{2}{1-r}},\quad 0<r<1,
\end{aligned}$$

where in the last inequality we have used that

$$\sum_{k=0}^{\infty}\frac{4^k}{k+1}r^{4^k}\lesssim\frac{1}{(1-r)\log\frac{2}{1-r}},\quad 0<r<1.\tag{2.1.24}$$

Finally, we shall prove (2.1.24). With this aim, take $r_N=1-\frac{1}{4^N}$. Since $f(x)=\frac{3^x}{x+1}$ is increasing on $[0,\infty)$

$$\sum_{k=0}^{N}\frac{4^k}{k+1}r_N^{4^k}\le\sum_{k=0}^{N}\frac{4^k}{k+1}\le\frac{3^N}{N+1}\sum_{k=0}^{N}(4/3)^k\lesssim\frac{4^N}{N+1}.$$

On the other hand, for every $0 < x < 1,\ n \in \mathbb{N}$, the inequality $(1-x)^n \leq 4(nx)^{-2}$ holds,

$$\begin{aligned}\sum_{k=N+1}^{\infty} \frac{4^k}{k+1} r_N^{4^k} &\leq 4^{2N+1} \sum_{n=N+1}^{\infty} \frac{4^{-k}}{k+1} \\ &\leq \frac{4^{2N+1}}{N+1} \sum_{n=N+1}^{\infty} 4^{-k} \lesssim \frac{4^N}{N+1},\end{aligned} \tag{2.1.25}$$

So,

$$\sum_{k=0}^{\infty} \frac{4^k}{k+1} r_N^{4^k} \lesssim \frac{4^N}{N+1} \lesssim \frac{1}{(1-r_N)\log \frac{2}{1-r_N}}.$$

Now, for every $r \in [3/4, 1)$ there $N \in N$ such that $r_N \leq r \leq r_{i+1}$,

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{4^k}{k+1} r^{4^k} \leq \sum_{k=0}^{\infty} \frac{4^k}{k+1} r_{N+1}^{4^k} \lesssim \frac{4^{N+1}}{N+2} \\ &\lesssim \frac{4^N}{N+1} \asymp \frac{1}{(1-r_N)\log \frac{2}{1-r_N}} \lesssim \frac{1}{(1-r)\log \frac{2}{1-r}},\end{aligned}$$

which gives (2.1.24) and finishes the proof. □

Next we provide a sufficient condition, which involves Carleson measures, on a function g to lie in this space of multipliers. It turns out to be also necessary if g is given by a power series with Hadamard gaps.

Theorem 3. *Assume that $0 < p \leq q \leq 1$ and let g be an analytic function in $\mathbb{D}$. Let $\mu_{g,q}$ be the Borel measure in $\mathbb{D}$ defined by $d\mu_{g,q}(z) = |g'(z)|^q(1-|z|^2)^{q-1}\,dA(z)$.*

(a) If $g \in H^\infty \cap \mathcal{B}_{\log}$ and the measure $\mu_{g,q}$ is a Carleson measure, then $g \in M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1})$.

(b) If g is given by a power series with Hadamard gaps, then $g \in M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1})$ if and only if $g \in H^\infty \cap \mathcal{B}_{\log}$ and the measure $\mu_{g,q}$ is a Carleson measure.

Proof. Suppose that $g \in H^\infty \cap \mathcal{B}_{\log}$ and the measure $\mu_{g,q}$ is a Carleson measure. Take $f \in \mathcal{B} \cap \mathcal{D}^p_{p-1}$.

- Using (2.1.1), we see that $g \in M(\mathcal{B})$ and, hence, $fg \in \mathcal{B}$.
- Using [71, Theorem 2.1] we deduce that $g \in M(\mathcal{D}^p_{p-1})$ and, then it follows that $fg \in \mathcal{D}^p_{p-1}$.

Since $\mathcal{D}^p_{p-1} \cap \mathcal{B} \subset \mathcal{D}^q_{q-1} \cap \mathcal{B}$, we have that $fg \in \mathcal{B} \cap \mathcal{D}^q_{q-1}$. Thus, we have proved that $g \in M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1})$. This finishes the proof of part (a).

Suppose now that g is given by a power series with Hadamard gaps and $g \in M(\mathcal{B} \cap \mathcal{D}^p_{p-1}, \mathcal{B} \cap \mathcal{D}^q_{q-1})$. Then $g \in \mathcal{D}^q_{q-1}$. Now, using Theorem 3.2 of [44], we see that this implies that $\mu_{g,q}$ is a Carleson measure. □

2.2 Multipliers from $H^\infty \cap \mathcal{D}^p_{p-1}$ to $H^\infty \cap \mathcal{D}^q_{q-1}$

We turn now our attention to spaces of the type $H^\infty \cap \mathcal{D}^p_{p-1}$. The following theorem gives a complete characterization of the multipliers from $H^\infty \cap \mathcal{D}^p_{p-1}$ to $H^\infty \cap \mathcal{D}^q_{q-1}$ when $0 < p \le q < \infty$.

Theorem 4. *If $0 < p \le q < \infty$, then $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = H^\infty \cap \mathcal{D}^q_{q-1}$.*

Proof. If $g \in M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$ then, since $H^\infty \cap \mathcal{D}^p_{p-1}$ contains the constant functions, it follows trivially that $g \in H^\infty \cap \mathcal{D}^q_{q-1}$.

On the other hand, if $g \in H^\infty \cap \mathcal{D}^q_{q-1}$ and $f \in H^\infty \cap \mathcal{D}^p_{p-1}$, it is clear that $gf \in H^\infty$. We also have

$$\begin{aligned}
&\int_{\mathbb{D}} \left|(g'f + gf')(z)\right|^q (1-|z|^2)^{q-1} dA(z) \\
&\lesssim \int_{\mathbb{D}} |(g'f)(z)|^q (1-|z|^2)^{q-1}\, dA(z) + \int_{\mathbb{D}} |g(z)f'(z)|^q (1-|z|^2)^{q-1} dA(z) \\
&\lesssim \|f\|^q_{H^\infty} \|g\|^q_{\mathcal{D}^q_{q-1}} + \|g\|^q_{H^\infty} \|f\|^{q-p}_{\mathcal{B}} \|f\|^p_{\mathcal{D}^p_{p-1}} < \infty.
\end{aligned}$$

Thus, $gf \in \mathcal{D}^q_{q-1}$ and, hence, $gf \in H^\infty \cap \mathcal{D}^q_{q-1}$. Consequently, we have proved that $g \in M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$. □

Regarding the case $0 < q < p$, let us notice that if $2 \le q < p$ then $H^\infty \cap \mathcal{D}^p_{p-1} = H^\infty \cap \mathcal{D}^q_{q-1} = H^\infty$. Hence we have

$$M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = H^\infty, \quad 2 \le q < p. \tag{2.2.1}$$

When $0 < q < p$ and $0 < q < 2$ the question is more complicated. It is well known (see [40, Theorem 1] and [71]) that, whenever $0 < q < 2$, there exists a function $f \in H^\infty \setminus \mathcal{D}^q_{q-1}$. We improve this result in our next theorem.

Theorem 5. *If* $0 < q < \min\{p, 2\}$, *then there exists a function* $f \in \left(H^\infty \cap \mathcal{D}^p_{p-1}\right) \setminus \left(H^\infty \cap \mathcal{D}^q_{q-1}\right)$.

Proof. Let $\tilde{p} = \min\{p, 1\}$ and $p^\star = \min\{p, 2\}$. We shall split the proof in two cases.
Case 1: $\mathbf{0 < q < 1}$. Take a sequence $\{u_k\}_{k=1}^\infty \in \ell^{\tilde{p}} \setminus \ell^q$ and let f be defined by

$$f(z) = \sum_{k=1}^\infty u_k z^{2^k}, \quad z \in \mathbb{D}.$$

Then, using Proposition A and the fact that $\tilde{p} \leq 1$, we see that $f \in \left(\mathcal{D}^p_{p-1} \cap H^\infty\right) \setminus \mathcal{D}^q_{q-1}$.
Case 2: $\mathbf{1 \leq q < 2}$. Let us consider a sequence $\{u_k\}$ such that $\{u_k\}_{k=1}^\infty \in \ell^{p^\star} \setminus \ell^q$ and let choose $n_k = 4^k$. We claim that the function $\Phi \in H^\infty$ associated to $\{u_k\}$ and $\{n_k\}$ via Theorem 2 satisfies that $\Phi \in H^\infty \cap \mathcal{D}^p_{p-1} \setminus H^\infty \cap \mathcal{D}^q_{q-1}$.

Using [71, Lemma 1.6 (i)] and bearing in mind Theorem 2 (ii), we deduce

$$\|\Phi\|^q_{\mathcal{D}^q_{q-1}} \gtrsim \sum_{k=0}^\infty \left|\widehat{\Phi(n_k)}\right|^q = \|\{u_k\}\|^q_{\ell^q} = \infty,$$

that is, $\Phi \notin \mathcal{D}^q_{q-1}$.

By (1.2.1), if $p \geq 2$ we are done. On the other hand, if $0 < p < 2$ by [43, Theorem 1.1 (ii)], the M. Riesz theorem and Theorem 2 (iv),

$$\begin{aligned} ||\Phi||^q_{\mathcal{D}^p_{p-1}} &\leq \int_0^1 (1-r)^{p-1} M_2^p(r, \Phi')\, dr \\ &\lesssim \sum_{k=0}^\infty \left(\left\|S_{2^k, 2^{k+1}}\Phi\right\|_{H^2}\right)^{p/2} \\ &\lesssim \sum_{k=0}^\infty \left(\left\|S_{4^k+1, 4^{k+1}+1}\Phi\right\|_{H^2}\right)^{p/2} \lesssim ||\{u_k\}||^p_{\ell^p} < \infty, \end{aligned}$$

which finishes the proof. □

Our result on the remaining triangular case is the following.

Theorem 6.

(a) If $0 < q < 1$ and $0 < q < p < \infty$ then $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = \{0\}$.

(b) If $1 \le q < 2 \le p$ then $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1}) = \{0\}$.

Proof. Theorem 6 (a).

Assume that $0 < q < 1$, $0 < q < p$ and that $g \in M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$ and $g \not\equiv 0$. Take

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{2^k}}{k^{\frac{1}{q}}}.$$

Let $\{r_k(t)\}_{k=0}^{\infty}$ be the sequence of Rademacher functions and denote

$$f_t(z) = \textstyle\sum_{k=0}^{\infty} r_k(t) a_k z^{2^k}, \quad 0 \le t \le 1,\ z \in \mathbb{D}.$$

By Proposition A,

$$\|f_t\|_{H^2} \le \|f_t\|_{H^\infty} \le \sum_{n=1}^{\infty} \frac{1}{k^{1/q}} \asymp ||f||_{H^\infty} < \infty, \quad t \in [0,1],$$

and

$$\|f_t\|_{\mathcal{D}^p_{p-1}} \asymp \left(\sum_{k=0}^{\infty} \frac{1}{k^{p/q}} \right)^{1/p} \asymp \|f\|_{\mathcal{D}^p_{p-1}}, \quad t \in [0,1].$$

Then, since $g \in M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$, we see that

$$\int_{\mathbb{D}} |(gf_t)'(z)|^q (1-|z|^2)^{q-1} dA(z) \lesssim \|f_t\|^q_{H^\infty \cap \mathcal{D}^p_{p-1}} \lesssim \|f\|^q_{H^\infty \cap \mathcal{D}^p_{p-1}} < \infty.$$

Integrating gf_t' with respect to t and applying Fubini's theorem and Khinchine's inequality we get

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{D}} |gf_t'(z)|^q (1-|z|^2)^{q-1} dA(z) dt \\
&\lesssim \int_0^1 \int_{\mathbb{D}} |(gf_t)'(z)|^q (1-|z|^2)^{q-1} dA(z) dt + \int_0^1 \int_{\mathbb{D}} |f_t g'(z)|^q (1-|z|^2)^{q-1} dA(z) dt \\
&\lesssim \|f\|^q_{H^\infty \cap \mathcal{D}^p_{p-1}} + \int_{\mathbb{D}} |g'(z)|^q \int_0^1 |f_t(z)|^q (1-|z|^2)^{q-1} dt dA(z) \\
&\lesssim \|f\|^q_{H^\infty \cap \mathcal{D}^p_{p-1}} + \int_{\mathbb{D}} |g'(z)|^q M_2^q(|z|, f)(1-|z|^2)^{q-1} dA(z) \\
&\lesssim \|f\|^q_{H^\infty \cap \mathcal{D}^p_{p-1}} + \|f\|^q_{H^\infty} \int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-1} dA(z) \\
&\lesssim \|f\|^q_{H^\infty \cap \mathcal{D}^p_{p-1}} < \infty.
\end{aligned} \tag{2.2.2}$$

On the other hand, since $g \not\equiv 0$ there exists positive constant C such that

$$M_q^q(r,g) \ge C, \quad 1/2 < r < 1.$$

Then, using again Fubini's Theorem and Khinchine's inequality, we deduce

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{D}} |gf_t'(z)|^q (1-|z|^2)^{q-1} dA(z)dt \\
&= \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} \left(\int_0^1 |f_t'(z)|^q dt \right) dA(z) \\
&\asymp \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} M_2^q(|z|, f') dA(z) \\
&\gtrsim \int_{1/2}^1 M_q^q(r,g) M_q^q(r,f')(1-r^2)^{q-1} dr \\
&\gtrsim \int_{1/2}^1 M_q^q(r,f')(1-r^2)^{q-1} dr = +\infty.
\end{aligned}$$

This is in contradiction with (2.2.2). Therefore $g \equiv 0$.

Assume now that $1 \le q < 2 \le p$. By Theorem B there is a function $f \in H^\infty$ such that

$$\int_0^1 (1-r^2)^{q-1} |f'(re^{i\theta})|^q \, dr = \infty \quad \text{for every } \theta \in B,$$

where B is a subset of $[0, 2\pi]$ whose Lebesgue measure $|B|$ is 2π.

Suppose that $g \in M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$ and $g \not\equiv 0$. Notice that $g \in H^\infty \cap \mathcal{D}^q_{q-1}$. Since

$$\int_{\mathbb{D}} (1-|z|^2)^{q-1} |g'(z) f(z)|^q \, dA(z) \le \|f\|^q_{H^\infty} \|g\|^q_{H^\infty \cap \mathcal{D}^q_{q-1}} < \infty, \tag{2.2.3}$$

it follows that

$$\int_{\mathbb{D}} (1-|z|^2)^{q-1} |g(z) f'(z)|^q \, dA(z) < \infty. \tag{2.2.4}$$

Since $g \in H^\infty$ and $g \not\equiv 0$, there is a set $A = A(g) \subset [0, 2\pi]$ with $|A| > 0$ and such that $\lim_{r\to 1^-} g(re^{i\theta}) \ne 0$ if $\theta \in A$. Then, for every $\theta \in A \cap B$ there is $r_0(\theta) \in (0,1)$ such that $K = \inf_{r_0<r<1} |g(re^{i\theta})| > 0$. Then

$$\int_0^1 (1-r^2)^{q-1} |g(re^{i\theta})|^q |f'(re^{i\theta})|^q \, dr \ge K^q \int_{r_0}^1 (1-r^2)^{q-1} |f'(re^{i\theta})|^q \, dr = \infty,$$

since $|A \cap B| > 0$, this is in contradiction with (2.2.4). Thus g must be identically 0. This finishes the proof. □

The case $1 \leq q < p < 2$ of Theorem 6 remains open. However, if the answer to the following open question were affirmative then it would follow that the space $M(H^\infty \cap \mathcal{D}^p_{p-1}, H^\infty \cap \mathcal{D}^q_{q-1})$ would be trivial also for this range of parameters. (See the proof of Theorem 6 (b)).

Question 1. Suppose that $0 < q < p < 2$. Does there exist a function $f \in H^\infty \cap \mathcal{D}^p_{p-1}$ satisfying (1.2.7)?

2.3 Functions of logarithmic bounded mean oscillation

With the aim of obtaining results concerning multipliers from $\mathcal{D}^p_{p-1} \cap BMOA$ to $\mathcal{D}^q_{q-1} \cap BMOA$, we need to define the space of functions of logarithmic bounded mean oscillation and to study some of their properties. We say that $\psi \in L^1(\mathbb{T})$ has logarithmic bounded mean oscillation or that $\psi \in BMO_{\log}(\mathbb{T})$ if

$$\sup_{I \subset \mathbb{T}} \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_I |\psi(e^{i\theta}) - \psi_I| \, d\theta.$$

The space $BMOA_{\log}(\mathbb{T})$ consists of those $g \in H^1$ such that the function of its boundary values belongs to $BMO_{\log}(\mathbb{T})$. Following the reasoning used in the classical space $BMOA$, it can be proved that $g \in BMOA_{\log}$ if and and only if

$$||g||^2_{BMOA_{\log}} = |g(0)|^2 + \sup_{a \in \mathbb{D}} \frac{\left(\log \frac{2}{1-|a|}\right)^2}{1-|a|} \int_{S(a)} (1-|z|^2)|g'(z)|^2 \, dA(z) < \infty. \quad (2.3.1)$$

The space $BMOA_{\log}$ coincides with that of those $g \in \mathcal{H}ol(\mathbb{D})$ such that T_g : BMOA $\to$ BMOA is bounded [68]. Moreover, it is known that [56]

$$M(BMOA) = H^\infty \cap BMOA_{\log}. \quad (2.3.2)$$

We shall start by proving some embedding relations between $BMOA_{\log}$, $\mathcal{B}_{\log}$ and $BMOA$.

Proposition 2. *If $1>\beta>\frac{1}{2}$, then $BMOA_{\log} \subsetneq \mathcal{B}_{\log} \subsetneq \mathcal{B}_{\log,\beta} \subsetneq BMOA$.*

Proof. First, we prove that $BMOA_{\log} \subset \mathcal{B}_{\log}$. Take $f \in BMOA_{\log}$. Let $a \in \mathbb{D}$ and assume without loss of generality that $|a|>\frac{1}{2}$. Set $a^\star = \frac{3|a|-1}{2} e^{i\arg a}$ so that the disc $D\left(a, \frac{1-|a|}{2}\right)$ of center a and radius $\frac{1-|a|}{2}$ is contained in the Carleson box $S(a^\star)$. This inclusion together with the subharmonicity of $|f'|^2$ and the fact that $(1-|z|) \asymp (1-|a|)$ for $z \in D\left(a, \frac{1-|a|}{2}\right)$ gives

$$\begin{aligned}\left(\log \tfrac{2}{1-|a|}\right)^2 (1-|a|)^2 |f'(a)|^2 &\lesssim \left(\log \tfrac{2}{1-|a|}\right)^2 \int_{D\left(a,\frac{1-|a|}{2}\right)} |f'(z)|^2 \, dA(z) \\ &\asymp \frac{\left(\log \frac{2}{1-|a|}\right)^2}{1-|a|} \int_{D\left(a,\frac{1-|a|}{2}\right)} (1-|z|^2)|f'(z)|^2 \, dA(z) \\ &\asymp \frac{\left(\log \frac{2}{1-|a^\star|}\right)^2}{1-|a^\star|} \int_{D\left(a,\frac{1-|a|}{2}\right)} (1-|z|^2)|f'(z)|^2 \, dA(z) \\ &\lesssim \frac{\left(\log \frac{2}{1-|a^\star|}\right)^2}{1-|a^\star|} \int_{S(a^\star)} (1-|z|^2)|f'(z)|^2 \, dA(z),\end{aligned}$$

so $f \in \mathcal{B}_{\log}$.

Now, let us see that the inclusion is strict. We borrow ideas from [62, Proposition 5.1 (D)]. Assume on the contrary to the assertion that $BMOA_{\log} = \mathcal{B}_{\log}$. By [48, Theorem 1] (see also [1] and [34]) there are $g_1, g_2 \in \mathcal{B}_{\log}$ such that

$$|g_1'(z)| + |g_2'(z)| \gtrsim \frac{1}{(1-|z|)\log \frac{2}{1-|z|}}, \quad z \in \mathbb{D}.$$

Then, for any $a \in \mathbb{D}$

$$\begin{aligned}&\int_{S(a)} \frac{1}{(1-|z|)\log^2 \frac{2}{1-|z|}} \, dA(z) \lesssim \int_{S(a)} \left(|g_1'(z)| + |g_2'(z)|\right)^2 (1-|z|^2) \, dA(z) \\ &\lesssim \int_{S(a)} |g_1'(z)|^2 (1-|z|^2) \, dA(z) + \int_{S(a)} |g_2'(z)|^2 (1-|z|^2) \, dA(z) \\ &\lesssim \frac{(1-|a|)}{\log^2 \frac{2}{1-|a|}},\end{aligned}$$

so bearing in mind that

$$\int_{S(a)} \frac{1}{(1-|z|)\log^2 \frac{2}{1-|z|}} \, dA(z) \asymp \frac{(1-|a|)}{\log \frac{2}{1-|a|}},$$

and letting $|a| \to 1^-$, we obtain a contradiction.

Assume now that $\beta \in \left(\frac{1}{2}, 1\right)$. Then it is clear that $\mathcal{B}_{\log} \subsetneq \mathcal{B}_{\log,\beta}$. Furthermore, $f(z) = \sum_{k=1}^{\infty} \frac{z^{2^k}}{k^\beta} \in \mathcal{B}_{\log,\beta} \setminus \mathcal{B}_{\log}$ (see [63, p. 245]).

The inclusion $\mathcal{B}_{\log,\beta} \subsetneq BMOA$, for $\beta > \frac{1}{2}$, follows easily using the characterization of $BMOA$ in terms of Carleson measures (see [35, p. 669]). Finally, we observe that $f(z) = \log \frac{1}{1-z} \in BMOA \setminus \mathcal{B}_{\log,\beta}$ for any $\beta > 0$. This concludes the proof. □

Next we find a simple sufficient condition for the membership of a function $f \in \mathcal{H}ol(\mathbb{D})$ in the space $BMOA_{\log}$.

Proposition 3. *Let f be an analytic function in $\mathbb{D}$. If*

$$\int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^2 [M_\infty(r,f')]^2 \, dr < \infty \tag{2.3.3}$$

then $f \in BMOA_{\log}$.

Proof. Suppose that f satisfies (2.3.3) Let I be an interval in $\mathbb{T}$ of length h, say $I = \{e^{it} : \theta_0 < t < \theta_0 + h\}$. Then

$$\begin{aligned} \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} (1-|z|^2)|f'(z)|^2 \, dA(z) &\asymp \frac{\left(\log \frac{2}{h}\right)^2}{h} \int_{1-h}^1 \int_{\theta_0}^{\theta_0+h} (1-r)|f'(re^{it})|^2 \, dt \, dr \\ \lesssim \left(\log \tfrac{2}{h}\right)^2 \int_{1-h}^1 (1-r)\,[M_\infty(r,f')]^2 \, dr &\le \int_{1-h}^1 (1-r)\,[M_\infty(r,f')]^2 \left(\log \tfrac{2}{1-r}\right)^2 dr \\ &\le \int_0^1 (1-r)\,[M_\infty(r,f')]^2 \left(\log \tfrac{2}{1-r}\right)^2 dr. \end{aligned}$$

□

Now we turn to the question of finding conditions on the Taylor coefficients of a function $f \in \mathcal{H}ol(\mathbb{D})$ enough to assert that $f \in BMOA_{\log}$. We shall need two lemmas. The first one estimates an integral which may be viewed as a generalization of the classical beta function (compare with Lemma 2 of [28]). We recall that $h : [0,1) \to (0,\infty)$ is essentially decreasing on $[0,1)$ if there exists a constant $C > 0$ such that $h(t) \le Ch(r)$ for all $0 \le r \le t < 1$.

Lemma 3. *Assume that $m \in \mathbb{N}$ and $\alpha > 0$, then*

$$\int_0^1 x^n (1-x)^m \left(\log \frac{e}{1-x}\right)^\alpha dx \asymp \frac{(\log n)^\alpha}{n^{m+1}}, \quad \text{as } n \to \infty, \tag{2.3.4}$$

where the constants involved in the above inequality may depend on m and α but not on n.

Proof. Using [62, Lemma 1.3] for the weight $\omega(s) = (1-s)^m \left(\log \frac{e}{1-s}\right)^\alpha$, we have that

$$\int_0^1 x^n (1-x)^m \left(\log \frac{e}{1-x}\right)^\alpha dx \asymp \int_{1-\frac{1}{n}}^1 (1-x)^m \left(\log \frac{e}{1-x}\right)^\alpha dx.$$

On the other hand, since $h(x) = (1-x)^m \left(\log \frac{e}{1-x}\right)^\alpha$ is essentially decreasing on $[0,1)$

$$\int_{1-\frac{1}{n}}^1 (1-x)^m \left(\log \frac{e}{1-x}\right)^\alpha dx \lesssim \frac{1}{n^m} (\log en)^\alpha \int_{1-\frac{1}{n}}^1 dx = \frac{1}{n^{m+1}} (\log en)^\alpha .$$

Finally, using that $x \mapsto \left(\log \frac{e}{1-x}\right)^\alpha$ is increasing on $[0,1)$

$$\int_{1-\frac{1}{n}}^1 (1-x)^m \left(\log \frac{e}{1-x}\right)^\alpha dx \geq (\log en)^\alpha \int_{1-\frac{1}{n}}^1 (1-x)^m dx = \frac{1}{n^{m+1}} (\log en)^\alpha .$$

This finishes the proof.

□

Lemma 4. *Suppose that $\alpha > 0$ and let g be an analytic function in $\mathbb{D}$, $g(z) = \sum_{n=0}^\infty a_n z^n$ $(z \in \mathbb{D})$. The following two conditions are equivalent:*

(i) $\int_{\mathbb{D}} (1-|z|^2)|g'(z)|^2 \left(\log \frac{2}{1-|z|}\right)^\alpha dA(z) < \infty$.

(ii) $\sum_{n=1}^\infty |a_n|^2 [\log n]^\alpha < \infty$.

Proof. We have

$$\begin{aligned} \int_{\mathbb{D}} (1-|z|^2)|g'(z)|^2 \left(\log \tfrac{2}{1-|z|}\right)^\alpha dA(z) &\asymp \int_0^1 r(1-r) M_2(r,g')^2 \left(\log \tfrac{2}{1-r}\right)^\alpha dr \\ &= \sum_{n=1}^\infty n^2 |a_n|^2 \int_0^1 (1-r) r^{2n-1} \left(\log \tfrac{2}{1-r}\right)^\alpha dr \end{aligned}$$

Using Lemma 3 with $m = 1$ we see that $\int_0^1 (1-r) r^{2n-1} \left(\log \frac{2}{1-|z|}\right)^\alpha dr \asymp \frac{[\log n]^\alpha}{n^2}$. Then it follows that

$$\int_{\mathbb{D}} (1-|z|^2)|g'(z)|^2 \left(\log \frac{2}{1-|z|}\right)^2 dA(z) \asymp \sum_{n=1}^\infty |a_n|^2 [\log(n+1)]^\alpha.$$

□

We close this section by proving a useful result on lacunary series.

Proposition 4. *Let $f \in \mathcal{H}ol(\mathbb{D})$ be given by a lacunary power series, i. ,e., f is of the form*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \quad (z \in \mathbb{D}) \text{ with } n_{k+1} \geq \lambda n_k \text{ for all } k, \text{ for a certain } \lambda > 1.$$

If $\sum_{k=0}^{\infty} |a_k|^2 (\log n_k)^3 < \infty$, then $f \in BMOA_{\log} \cap H^\infty$.

Proof of Proposition 4. Suppose that $\sum_{k=0}^{\infty} |a_k|^2 (\log n_k)^3 < \infty$ and

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \quad (z \in \mathbb{D}) \text{ with } n_{k+1} \geq \lambda n_k \text{ for all } k, \text{ and } \lambda > 1.$$

Using the Cauchy-Schwarz inequality and the fact that $\sum_{k=0}^{\infty} r^{n_k} \lesssim \log \frac{2}{1-r}$ (because the function h given by $h(z) = \sum z^{n_k}$ is a Bloch function), we see that

$$\begin{aligned} [rM_\infty(r, f')]^2 &\leq \left(\sum_{k=0}^{\infty} n_k |a_k| r^{n_k}\right)^2 \\ &\leq \left(\sum_{k=0}^{\infty} n_k^2 |a_k|^2 r^{n_k}\right)\left(\sum_{k=0}^{\infty} r^{n_k}\right) \lesssim \left(\log \frac{2}{1-r}\right) \sum_{k=0}^{\infty} n_k^2 |a_k|^2 r^{n_k}. \end{aligned}$$

Then, using Lemma 3 with $m = 1$ and $\alpha = 3$, we obtain

$$\begin{aligned} \int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^2 [M_\infty(r, f')]^2 \, dr &\lesssim \int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^3 \left(\sum_{k=0}^{\infty} n_k^2 |a_k|^2 r^{n_k}\right) dr \\ = \sum_{k=0}^{\infty} n_k^2 |a_k|^2 \int_0^1 r^{n_k} (1-r)\left(\log \frac{1}{1-r}\right)^3 dr &\lesssim \sum_{k=0}^{\infty} |a_k|^2 (\log n_k)^3 < \infty. \end{aligned}$$

Then Proposition 3 implies that $f \in BMOA_{\log}$.

To see that $f \in H^\infty$ observe that $\lambda^k \lesssim n_k$ and $|a_k|^2 \lesssim (\log n_k)^{-3}$. Then it follows that $|a_k| = \mathrm{O}\left(k^{-3/2}\right)$, as $k \to \infty$ and the result follows. □

Now we turn to study the membership in $BMOA_{\log}$ of certain random power series.

We shall consider random power series analytic in $\mathbb{D}$ of the form

$$\sum_{n=0}^{\infty} \epsilon_n a_n z^n$$

where the ϵ_n's are random signs. More precisely, if $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$), we set

$$f_t(z) = \sum_{n=0}^{\infty} r_n(t) a_n z^n, \quad 0 \leq t < 1, \quad z \in \mathbb{D},$$

where the r_n's are the Rademacher functions. Each function f_t is analytic in $\mathbb{D}$. Littlewood [52] (see also [27, Appendix A]) proved that if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ then

$f_t \in \cap_{0<p<\infty} H^p$ almost surely (a. s.), that is, for almost every t. On the other hand, the condition $\sum_{n=0}^{\infty} |a_n|^2 = \infty$ implies that for almost every t, f_t has a radial limit almost nowhere [27, Appendix A, Theorem A.5].

Paley and Zygmund [57] gave an example of an f with

$$\sum_{n=1}^{\infty} |a_n|^2 \log n < \infty \tag{2.3.5}$$

such that $f_t \notin H^\infty$ for every t.

Anderson, Clunie and Pommerenke [9] used a result of Salem and Zygmund [67] on the behaviour of the maxima of the partial sums of random trigonometric series to prove that (2.3.5) implies that $f_t \in \mathcal{B}$ a. s. and that this condition is the best possible. Later on, Sledd [69] used also the Salem and Zygmund theorem to show that (2.3.5) actually implies that $f_t \in BMOA$ a. s.

Duren proved in [28] the following result.

Theorem C. *If $0 \le \beta \le 1$ and $\sum_{n=1}^{\infty} |a_n|^2 (\log n)^\beta < \infty$, then for almost every $t \in [0,1]$, the function*

$$f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

satisfies

$$\int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^{\beta-1} [M_\infty(r, f_t')]^2 \, dr < \infty. \tag{2.3.6}$$

Using this, Duren gave in [28] a new proof of Sledd's theorem. Next we prove an analogue of Duren's theorem for $\beta = 3$. This will allow us to obtain the analogue of Sledd's theorem for $BMOA_{\log}$.

Theorem 7. *If $\sum_{n=1}^{\infty} |a_n|^2 (\log n)^3 < \infty$ then for almost every $t \in [0,1]$, the function*

$$f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

satisfies

$$\int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^{2} [M_\infty(r, f_t')]^2 \, dr < \infty. \tag{2.3.7}$$

Proof. Set

$$B_n^2 = \sum_{k=1}^{n} k^2 |a_k|^2, \quad n = 1, 2, \dots,$$

and $\psi(r) = (1-r)\sum_{n=1}^{\infty} B_n \sqrt{\log n}\, r^n$ $(0 < r < 1)$. Just as in p. 84 of [28], we have

$$|f_t{}'(z)| \le C\psi(r), \quad |z| = r, \quad 0 < r < 1, \quad \text{almost surely.} \tag{2.3.8}$$

Using Lemma 3, the simple fact that $\frac{\log x}{x^{3/2}}$ decreases as x increases in $[e^{2/3}, \infty)$, and Hilbert's inequality, we deduce

$$\begin{aligned}
&\int_0^1 (1-r)\left(\log\frac{1}{1-r}\right)^2 [\psi(r)]^2\, dr \\
&\asymp \int_0^1 (1-r)^3 \left(\log\frac{1}{1-r}\right)^2 \left[\sum_{n=1}^{\infty} B_n \sqrt{\log n}\, r^n\right]^2 dr \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} B_n\sqrt{\log n} B_j \sqrt{\log j} \int_0^1 r^{n+j}(1-r)^3 \left(\log\frac{1}{1-r}\right)^2 dr \\
&\lesssim \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{B_n\sqrt{\log n} B_j\sqrt{\log j}}{(n+j)^4}[\log(n+j)]^2 \qquad (*) \\
&\le \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{n+j}\frac{B_n[\log n]^{3/2}}{n^{3/2}}\frac{B_j[\log j]^{3/2}}{j^{3/2}} \\
&\lesssim \sum_{n=1}^{\infty} |B_n|^2 \frac{[\log n]^3}{n^3}.
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{n=1}^{\infty} |B_n|^2 \frac{[\log n]^3}{n^3} &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} k^2|a_k|^2 \frac{[\log n]^3}{n^3} \qquad (**) \\
&= \sum_{k=1}^{\infty} k^2 |a_k|^2 \sum_{n=k}^{\infty}\frac{[\log n]^3}{n^3} \asymp \sum_{k=1}^{\infty} |a_k|^2[\log k]^3 < \infty.
\end{aligned}$$

Then (2.3.8), (*) and (**) imply that (2.3.7) holds for almost every t, finishing the proof. □

Now we are ready to prove the analogue of Sledd's theorem. It establishes a sharp condition on the Taylor coefficients a_n of f which implies the almost sure membership of f_t in $BMOA_{\log}$.

Theorem 8. *(i) If $\sum_{n=1}^{\infty} |a_n|^2(\log n)^3 < \infty$ then for almost every $t \in [0,1]$, the function*

$$f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D},$$

belongs to $BMOA_{\log} \cap H^\infty$.

(ii) Furthermore, (i) is sharp in a very strong sense: Given a decreasing sequence of positive numbers $\{\delta_n\}_{n=1}^{\infty}$ with $\delta_n \to 0$, as $n \to \infty$, there exists a sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} a_n^2 \delta_n (\log n)^3 < \infty$ such that, for almost every t the function f_t defined by $f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n$ ($z \in \mathbb{D}$) does not belong to $\mathcal{B}_{\log}$.

Proof. (i) follows from Proposition 3 and Theorem 7.

(ii) is a byproduct of the following result. □

Theorem 9. *Fix $\beta > 1$. Given a decreasing sequence of positive numbers $\{\delta_n\}_{n=1}^{\infty}$ with $\delta_n \to 0$, as $n \to \infty$, there exists a sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$ satisfying that $\sum_{n=1}^{\infty} a_n^2 \delta_n (\log n)^\beta < \infty$ and such that, for almost every t the function f_t defined by $f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n$ ($z \in \mathbb{D}$) does not belong to $\mathcal{B}_{\log, \frac{\beta-1}{2}}$.*

Proof. The proof follows the lines of [9, Theorem 3.7]. Let $\{\delta_n\}_{n=1}^{\infty}$ satisfy the above hypothesis and chose a sequence $\{n_k\}_{k=0}^{\infty}$ such that $n_0 = 1$ and increases so rapidly that $\sum_{k=1}^{\infty} \sqrt{\delta_{n_k}} < \infty$, and $n_k > 2n_{k-1}$. For $n_{k-1} \le n < n_k$ define

$$a_n^2 \log^\beta n = \frac{1}{n_k \sqrt{\delta_{n_{k-1}}}} \tag{2.3.9}$$

Now, we define $f_t(z) = \sum_{n=1}^{\infty} r_n(t) a_n z^n$. Observe that,

$$\begin{aligned}\sum_{n=1}^{\infty}\delta_n a_n^2\log^{\beta}n &= \sum_{k=1}^{\infty}\sum_{j=n_{k-1}}^{n_k-1}\delta_j a_j^2\log^{\beta}j\\ &= \sum_{k=1}^{\infty}\frac{1}{n_k\sqrt{\delta_{n_{k-1}}}}\sum_{j=n_{k-1}}^{n_k-1}\delta_j\\ &\le \sum_{k=1}^{\infty}\frac{\delta_{n_{k-1}}}{n_k\sqrt{\delta_{n_{k-1}}}}\sum_{j=n_{k-1}}^{n_k-1}1 \\ &= \sum_{k=1}^{\infty}(n_k-n_{k-1})\frac{\delta_{n_{k-1}}}{n_k\sqrt{\delta_{n_{k-1}}}}\\ &\le \sum_{k=1}^{\infty}\sqrt{\delta_{n_{k-1}}}<\infty.\end{aligned} \tag{2.3.10}$$

Next, we are going to apply [9, Lemma 3.3] to the Césaro means of $f_t'(z)$. With this target, we write $g_{k,t}(z)=\sum_{m=1}^{n_k}\left(1-\frac{m}{n_k}\right)ma_m r_m(t)z^m$ and $R_k=\sum_{m=0}^{n_k}\left(1-\frac{m}{n_k}\right)^2 m^2a_m^2$. Then,

$$\begin{aligned}R_k=\sum_{m=1}^{n_k}\left(1-\frac{m}{n_k}\right)^2 m^2a_m^2 &\ge \sum_{\frac{3n_k}{4}\ge m>\frac{n_k}{2}}\left(1-\frac{m}{n_k}\right)^2 m^2a_m^2\\ &= \frac{1}{n_k\sqrt{\delta_{n_{k-1}}}}\sum_{\frac{3n_k}{4}\ge m>\frac{n_k}{2}}\left(1-\frac{m}{n_k}\right)^2 m^2\log^{-\beta}m\\ &> \frac{n_k^2}{4\log^{\beta}\frac{3n_k}{4}n_k\sqrt{\delta_{n_{k-1}}}}\sum_{\frac{3n_k}{4}\ge m>\frac{n_k}{2}}\left(1-\frac{m}{n_k}\right)^2\\ &\ge C\frac{n_k^2}{\left(\log^{\beta}n_k\right)\sqrt{\delta_{n_{k-1}}}}.\end{aligned} \tag{2.3.11}$$

Hence

$$\sqrt{R_k\log n_k}>C\frac{\lambda_k n_k}{\log^{\frac{\beta-1}{2}}n_k},\qquad \lambda_k=\frac{1}{\sqrt[4]{\delta_{n_{k-1}}}}\to\infty\quad\text{when } k\to\infty \tag{2.3.12}$$

Now since $x\to\frac{x}{\log^{\beta}x}$ is essentially increasing on $[1,\infty)$

$$\begin{aligned}\sum_{m=n_{j-1}}^{n_j-1}\left(1-\frac{m}{n_k}\right)^4 m^4a_m^4 &= \frac{1}{n_j^2\delta_{n_{j-1}}}\sum_{m=n_{j-1}}^{n_j-1}\left(1-\frac{m}{n_k}\right)^4 m^4\log^{-2\beta} m\\ &\lesssim \frac{n_j^2}{\log^{2\beta} n_j\delta_{n_{j-1}}}\sum_{m=n_{j-1}}^{n_j-1}\left(1-\frac{m}{n_k}\right)^4\\ &\leq \frac{n_j^2(n_j-n_{j-1})}{\log^{2\beta} n_j\delta_{n_{j-1}}},\end{aligned}$$

so

$$\begin{aligned}\sum_{m=1}^{n_k}\left(1-\frac{m}{n_k}\right)^4 m^4a_m^4 &= \sum_{j=1}^{k}\sum_{m=n_{j-1}}^{n_j-1}\left(1-\frac{m}{n_k}\right)^4 m^4a_m^4\\ &\leq \sum_{j=1}^{k}\frac{n_j^2(n_j-n_{j-1})}{\log^{2\beta} n_j\delta_{n_{j-1}}}\\ &\lesssim \frac{n_k^2}{\log^{2\beta} n_k\delta_{n_{k-1}}}\sum_{j=1}^{k}(n_j-n_{j-1})\\ &\leq \frac{n_k^3}{\log^{2\beta} n_k\delta_{n_{k-1}}}\\ &\lesssim \frac{R_k^2}{n_k}\end{aligned}$$

Thus from (2.3.12) and [9, (3.29)] we deduce that

$$\sup_{|z|=1}|g_{k,t}(z)| \gtrsim \frac{\lambda_k n_k}{\log^{\frac{\beta-1}{2}} n_k} \quad \text{a. s.}$$

which together with [76, Vol I. p. 89] gives that

$$4\sup_{|z|=1-\frac{1}{n_k}}|f_t'(z)| \geq \sup_{|z|=1}|g_{k,t}(z)| \gtrsim \frac{\lambda_k n_k}{\log^{\frac{\beta-1}{2}} n_k}, \quad \text{a. s.} \tag{2.3.13}$$

Consequently, bearing in mind that $\lambda_k \to \infty$ as $k \to \infty$, we get that a. s. f_t does not belong to $\mathcal{B}_{\log,\frac{\beta-1}{2}}$. This finishes the proof.

□

2.4 Multipliers from $\mathcal{D}^p_{p-1} \cap BMOA$ to $\mathcal{D}^q_{q-1} \cap BMOA$

In this section we shall prove our results about multipliers from $\mathcal{D}^p_{p-1} \cap BMOA$ to $\mathcal{D}^q_{q-1} \cap BMOA$. Let start with the following result.

Theorem 10. *For any p, q with $0 < p, q < \infty$ we have*

$$M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA) \subset BMOA_{\log} \cap H^\infty = M(BMOA).$$

Proof. The proof uses arguments similar to those in (i) of Theorem 1. By (2.1.3) $\{\varphi_a\}_{a\in\mathbb{D}}$ is uniformly bounded in $\mathcal{D}^\lambda_{\lambda-1}$, for any $\lambda > 0$. Also since $\|\varphi_a\|_{BMOA} \lesssim \|\varphi_a\|_\infty < 1$ for all $\lambda > 0$ the family is uniformly bounded on $\mathcal{D}^\lambda_{\lambda-1} \cap BMOA$. Assume that $g \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$, which implies that $g \in BMOA$.

Then since $BMOA \subset \mathcal{B}$, for any $a, z \in \mathbb{D}$ we have that

$$\begin{aligned}
&(1 - |z|^2)|\varphi_a'(z)g(z)| = (1 - |z|^2)|(\varphi_a \cdot g)'(z) - \varphi_a(z)g'(z)| \\
&\leq \|\varphi_a g\|_{BMOA\cap\mathcal{D}^q_{q-1}} + (1 - |z|^2)|\varphi_a(z)g'(z)| \\
&\lesssim \|M_g\|_{(BMOA\cap\mathcal{D}^p_{p-1}\to BMOA\cap\mathcal{D}^q_{q-1})} + \|g\|_{BMOA} < \infty.
\end{aligned} \tag{2.4.1}$$

Since $(1 - |a|^2)|\varphi_a'(a)| = 1$, taking $z = a$ in (2.4.1) we obtain

$$|g(a)| \lesssim \|M_g\|_{(BMOA\cap\mathcal{D}^p_{p-1}\to BMOA\cap\mathcal{D}^q_{q-1})} + \|g\|_{BMOA} < \infty,$$

for any $a \in \mathbb{D}$. Thus,

$$M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA) \subset H^\infty, \quad 0 < p, q < \infty.$$

Suppose now that $0 < p, q < \infty$ and $g \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$. Let us use the test functions f_a ($a \in \mathbb{D}$) defined by

$$f_a(z) = \log \frac{1}{1 - \bar{a}z}, \quad z \in \mathbb{D}.$$

The family $\{f_a : a \in \mathbb{D}\}$ is also bounded on $\mathcal{D}^\lambda_{\lambda-1} \cap BMOA$ for all $\lambda > 0$.

On the other hand, there exists an absolute constant $C > 0$ such that for any arc $I \subset \partial\mathbb{D}$

$$\frac{1}{C} \log \frac{2}{|I|} \leq |f_a(z)| \leq C \log \frac{2}{|I|}, \quad z \in S(I),$$

where $a = (1 - \frac{|I|}{2\pi})\xi$ with ξ the center of I.

Then we have

$$\begin{aligned}\frac{\log^2\frac{2}{|I|}}{|I|}\int_{S(I)}(1-|z|^2)|g'(z)|^2\,dA(z) &\le \tfrac{C^2}{|I|}\int_{S(I)}(1-|z|^2)|f_a(z)|^2|g'(z)|^2\,dA(z)\\ \lesssim \tfrac{C^2}{|I|}\int_{S(I)}(1-|z|^2)|(f_ag)'(z)|^2\,dA(z) &+ \tfrac{C^2}{|I|}\int_{S(I)}(1-|z|^2)|f_a'(z)|^2|g(z)|^2\,dA(z).\end{aligned}$$

Since $g\in M(\mathcal{D}^p_{p-1}\cap BMOA,\mathcal{D}^q_{q-1}\cap BMOA)$, the family $\{f_ag : a\in\mathbb{D}\}$ is bounded in $BMOA$ and hence $\sup_I \frac{C^2}{|I|}\int_{S(I)}(1-|z|^2)|(f_ag)'(z)|^2\,dA(z)<\infty$. Also, using that $g\in H^\infty$ and that the family $\{f_a : a\in\mathbb{D}\}$ is bounded in $BMOA$, we deduce that $\sup_I \frac{C^2}{|I|}\int_{S(I)}(1-|z|^2)|f_a'(z)|^2|g(z)|^2\,dA(z)<\infty$. Consequently, we have that

$$\sup_I \frac{\log^2\frac{2}{|I|}}{|I|}\int_{S(I)}(1-|z|^2)|g'(z)|^2\,dA(z)<\infty,$$

that is, $g\in BMOA_{\log}$. □

If $\lambda\ge 2$ then $BMOA\subset\mathcal{D}^\lambda_{\lambda-1}$. Hence, we have

$$M(\mathcal{D}^p_{p-1}\cap BMOA,\mathcal{D}^q_{q-1}\cap BMOA)=M(BMOA)=BMOA_{\log}\cap H^\infty, \quad (2.4.2)$$

for $2\le p,q<\infty$. This remains true for other values of p and q.

Theorem 11. *If* $1<q<\infty$ *and* $0<p\le q<\infty$*, then*

$$M(\mathcal{D}^p_{p-1}\cap BMOA,\mathcal{D}^q_{q-1}\cap BMOA)=M(BMOA)=BMOA_{\log}\cap H^\infty.$$

Proof of Theorem 11. Suppose that $1<q<\infty$ and $0<p\le q<\infty$. In view of Theorem 10, we only have to prove that $M(BMOA)\subset M(\mathcal{D}^p_{p-1}\cap BMOA,\mathcal{D}^q_{q-1}\cap BMOA)$.

Take $g\in M(BMOA)$ and $f\in BMOA\cap\mathcal{D}^p_{p-1}$. Then, clearly, $fg\in BMOA$. Using Lemma 2 and the closed graph theorem, we obtain

$$\begin{aligned}&\int_{\mathbb{D}}|(fg)'(z)|^q(1-|z|^2)^{q-1}dA(z)\\ &\lesssim \int_{\mathbb{D}}|f'(z)g(z)|^q(1-|z|^2)^{q-1}dA(z)+\int_{\mathbb{D}}|g'(z)f(z)|^q(1-|z|^2)^{q-1}dA(z) \quad (2.4.3)\\ &\lesssim \|g\|^q_{H^\infty}\|f\|^q_{\mathcal{D}^p_{p-1}\cap BMOA}+\int_{\mathbb{D}}|f(z)g'(z)|^q(1-|z|^2)^{q-1}dA(z).\end{aligned}$$

Now, Proposition 2 implies that $g\in\mathcal{B}_{\log}$. Also, since $BMOA\subset H^q$, we have that $f\in H^q$. Then we see that the last integral in (2.4.3) is finite as in the proof of (2.1.23). Thus, we have proved that $g \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$ finishing the proof. $\square$

To deal with the remaining case, $0 < p \le q \le 1$, we shall use the above results about lacunary power series and random power series. Our main results concerning random power series and multipliers are contained in the following theorem.

Theorem 12. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying*

$$\sum_{n=1}^{\infty} |a_n|^2 (\log n)^3 < \infty. \tag{2.4.4}$$

For $t \in [0,1]$ we set

$$f_t(z) = \sum_{n=0}^{\infty} r_n(t) a_n z^n, \quad z \in \mathbb{D}, \tag{2.4.5}$$

where the r_n's are the Rademacher functions. Then, for almost every $t \in [0,1]$, the function f_t satisfies the following conditions:

(i) $\int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^2 [M_\infty(r, f_t')]^2 \, dr < \infty$.

(ii) $f_t \in BMOA_{\log} \cap H^\infty$.

(iii) $f_t \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$ *whenever* $0 < p \le q$ *and* $q > \frac{1}{2}$.

Furthermore, if $0 < q < \frac{1}{2}$ then there exists a sequence $\{a_n\}$ which satisfies (2.4.4) and such that $f_t \notin \mathcal{D}^q_{q-1}$, for almost every t. Thus, for this sequence $\{a_n\}$ and for almost every t we have:

(a) $f_t \in M(BMOA)$.

(b) If $0 < p \le \lambda$ *and* $\lambda > \frac{1}{2}$ *then* $f_t \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^\lambda_{\lambda-1} \cap BMOA)$.

(c) $f_t \notin M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$ *whenever* $0 < p \le q$.

We remark that Theorem 12 shows that Theorem 11 does not remain true for $q < 1/2$.

To prove Theorem 12, let us first note that (i) follows from Theorem 7 and (ii) from Theorem 8 (i). For (iii) we shall use the following lemma.

Lemma 5. *Suppose that* $0 < q < 2$ *and* $\alpha > 0$. *Let* f *be an analytic function in* $\mathbb{D}$ *of the form* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, *with* $\sum_{n=0}^{\infty} |a_n|^2 [\log n]^\alpha < \infty$. *If* $f \in BMOA_{\log} \cap H^\infty$ *then*

$$f \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA), \quad \text{whenever } 0 < p \leq q \text{ and } \frac{q\alpha}{2-q} > 1.$$

For $0 < q \leq 1$, (iii) of Theorem 12 follows using (ii) and the lemma with $\alpha = 3$, while, for $1 < q < \infty$, it follows from Theorem 11.

Proof. Suppose that f is in the conditions of the lemma and that $0 < p \leq q$ and $\frac{q\alpha}{2-q} > 1$.

Take $h \in \mathcal{D}^p_{p-1} \cap BMOA$. Since $BMOA_{\log} \cap H^\infty = M(BMOA)$, it follows that $fh \in BMOA$.

We have also

$$\begin{aligned} &\int_{\mathbb{D}} (1-|z|)^{q-1} |(fh)'(z)|^q \, dA(z) \\ \lesssim &\int_{\mathbb{D}} (1-|z|)^{q-1} |f(z)|^q |h'(z)|^q \, dA(z) + \int_{\mathbb{D}} (1-|z|)^{q-1} |f'(z)|^q |h(z)|^q \, dA(z) \\ &= I_1 + I_2. \end{aligned}$$

The first summand I_1 is finite because $f \in H^\infty$ and $h \in \mathcal{D}^p_{p-1} \cap BMOA \subset \mathcal{D}^q_{q-1} \cap BMOA$. Let us estimate the second one I_2. Using Hölder's inequality with the exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, we obtain

$$\begin{aligned} I_2 &= \int_{\mathbb{D}} (1-|z|)^{q-1} |f'(z)|^q |h(z)|^q \, dA(z) \\ &= \int_{\mathbb{D}} |f'(z)|^q (1-|z|)^{q/2} \left(\log \tfrac{e}{1-|z|}\right)^{\frac{q\alpha}{2}} \left(\log \tfrac{e}{1-|z|}\right)^{-\frac{q\alpha}{2}} |h(z)|^q (1-|z|)^{\frac{q}{2}-1} \, dA(z) \\ &\leq \left[\int_{\mathbb{D}} |f'(z)|^2 (1-|z|) \left(\log \tfrac{e}{1-|z|}\right)^\alpha dA(z)\right]^{q/2} \left[\int_{\mathbb{D}} \frac{|h(z)|^{\frac{2q}{2-q}}}{\left(\log \frac{e}{1-|z|}\right)^{\frac{q\alpha}{2-q}} (1-|z|)} \, dA(z)\right]^{(2-q)/q}. \end{aligned}$$

Using Lemma 4, it follows that the first integral in the last product is finite. Now, notice that $h \in H^\lambda$ for all $\lambda < \infty$ to deduce

$$\int_{\mathbb{D}} \frac{|h(z)|^{\frac{2q}{2-q}}}{\left(\log \frac{e}{1-|z|}\right)^{\frac{q\alpha}{2-q}} (1-|z|)} \, dA(z) \leq \|h\|^{\frac{2q}{2-q}}_{H^{\frac{2q}{2-q}}} \int_0^1 \frac{1}{\left(\log \frac{e}{1-r}\right)^{\frac{q\alpha}{2-q}} (1-r)} \, dr$$

and this integral is finite because $\frac{q\alpha}{2-q} > 1$. Thus $I_2 < \infty$. Then we have that $fh \in \mathcal{D}^q_{q-1}$ and, hence, $fh \in \mathcal{D}^q_{q-1} \cap BMOA$. Consequently, we have proved that $f \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$. □

To finish the proof of Theorem 12 take $q \in (0, 1/2)$ and let $\{a_n\}$ be defined as follows:

$$a_{2^k} = (k+1)^{-1/q}, \quad k = 0, 1, \dots$$

and $a_n = 0$, if n is not a power of 2. Set

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{k=0}^{\infty} (k+1)^{-1/q} z^{2^k}, \quad z \in \mathbb{D}.$$

It is clear that $\{a_n\}$ satisfies (2.4.4). Furthermore, for almost every t, f_t is given by a lacunary power series, $f_t(z) = \sum_{k=0}^{\infty} r_{2^k}(t) a_{2^k} z^{2^k}$, which does not belong to $\mathcal{D}^q_{q-1}$ because $\sum_{k=0}^{\infty} |a_{2^k}|^q = \infty$.

Now turn to consider multipliers in $M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$ given by power series with Hadamard gaps for the range $0 < p \le q < \infty$.

Theorem 13. *Suppose that $0 < p \le q \le 1$ and let g be an analytic function in $\mathbb{D}$ given by a power series with Hadamard gaps. Then the following conditions are equivalent:*

(a) $g \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$.

(b) $g \in \mathcal{D}^q_{q-1} \cap BMOA_{\log}$.

Proof. Since $\mathcal{D}^p_{p-1} \cap BMOA$ contains the constants functions, it is clear that $M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA) \subset \mathcal{D}^q_{q-1}$ and the inclusion $M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA) \subset BMOA_{\log}$ follows from Theorem 10. Hence, the implication (a) $\Rightarrow$ (b) holds.

Let us prove next the other implication. So take $g \in \mathcal{D}^q_{q-1} \cap BMOA_{\log} \cap \mathcal{L}$,

$$g(z) = \textstyle\sum_{k=0}^{\infty} a_k z^{n_k} \ (z \in \mathbb{D}) \text{ with } n_{k+1} \ge \lambda n_k \text{ for all } k, \text{ for a certain } \lambda > 1.$$

We have $\sum_{k=0}^{\infty} |a_k|^q < \infty$ which, since $q \le 1$, implies that $\sum_{k=0}^{\infty} |a_k| < \infty$. Thus $g \in H^\infty$. Then $g \in BMOA_{\log} \cap H^\infty = M(BMOA)$.

Take $f \in \mathcal{D}^p_{p-1} \cap BMOA$. Since $g \in M(BMOA)$, we have that $gf \in BMOA$. Now,

$$\begin{aligned} &\textstyle\int_{\mathbb{D}} |(gf)'(z)|^q (1-|z|^2)^{q-1}\, dA(z) \\ \lesssim &\textstyle\int_{\mathbb{D}} |g(z)|^q |f'(z)|^q (1-|z|^2)^{q-1}\, dA(z) + \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|^2)^{q-1}\, dA(z) \\ &= I_1 + I_2. \end{aligned}$$

The first summand I_1 is finite because $g \in H^\infty$ and $f \in \mathcal{D}^q_{q-1}$.

Let us estimate the second one. Using Theorem 3. 2 of [44] we see that the measure $\mu_{g,q}$ in $\mathbb{D}$ defined by $d\mu_{g,q}(z) = (1-|z|^2)^{q-1}|g'(z)|^q\, dA(z)$ is a Carleson measure and (see, e. g., [72, Theorem 1] or [71, Theorem 2. 1]) this implies that $\mu_{g,q}$ is a Carleson measure for $\mathcal{D}^q_{q-1}$, that is, $\mathcal{D}^q_{q-1} \subset L^q(d\mu_{g,q})$. Hence $f \in L^q(d\mu_{g,q})$ which is equivalent to saying that $I_2 < \infty$. Hence, $gf \in \mathcal{D}^q_{q-1}$.

So, we have proved that $gf \in \mathcal{D}^q_{q-1} \cap BMOA$ for any $f \in \mathcal{D}^p_{p-1} \cap BMOA$, that is, $g \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$. □

We also obtain the analogue of Theorem 12 for lacunary power series.

Theorem 14. *Let $f \in \mathcal{H}ol(\mathbb{D})$ be given by a lacunary power series, of the form*

$$f(z) = \textstyle\sum_{k=0}^{\infty} a_k z^{n_k} \quad (z \in \mathbb{D}) \text{ with } n_{k+1} \geq \lambda n_k \text{ for all } k, \text{ for a certain } \lambda > 1,$$

and suppose that the sequence of coefficients $\{a_k\}_{k=0}^{\infty}$ satisfies

$$\sum_{k=1}^{\infty} |a_k|^2 (\log n_k)^3 < \infty. \tag{2.4.6}$$

Then the function f satisfies the following conditions:

(i) $f \in M(BMOA)$

(ii) $f \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$ *whenever* $0 < p \leq q$ *and* $q > \frac{1}{2}$.

Furthermore, if $0 < q < \frac{1}{2}$ then there exists a sequence $\{a_k\}$ which satisfies (2.4.6) and such that $f \notin \mathcal{D}^q_{q-1}$. Thus for this sequence $\{a_k\}$ the function f satisfies:

(a) $f \in M(BMOA)$.

(b) If $0 < p \leq \lambda$ *and* $\lambda > 1/2$ *then* $f \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^\lambda_{\lambda-1})$ *whenever* $0 < p \leq \lambda$.

(c) $f \notin M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$ *whenever* $0 < p \leq q$.

Proof. Part (i) follows from Proposition 4. Part (ii) follows from Theorem 11 for $q > 1$ and from Lemma 5 (with $\alpha = 3$) whenever $0 < q \leq 1$.

Now, if $0 < q < \frac{1}{2}$ take

$$a_k = k^{-1/q}, \quad k = 1, 2, \ldots$$

and

$$f(z) = \sum_{k=1}^{\infty} a_k z^{2^k}, \quad z \in \mathbb{D}.$$

Clearly,

$$\sum_{k=1}^{\infty} |a_k|^2 k^3 < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} |a_k|^q = \infty.$$

Then f satisfies conditions (a), (b) and (c) of Theorem 14. □

Finally we prove that whenever $q < p$ then the null function is the only multiplier from $\mathcal{D}^p_{p-1} \cap BMOA$ to $\mathcal{D}^q_{q-1} \cap BMOA$, except in the cases covered by (2.4.2).

Theorem 15. *If* $0 < q < p < \infty$ *and* $q < 2$*, then*

$$M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA) = \{0\}.$$

Proof. Suppose that $0 < q < p < \infty$, $q < 2$ and $g \in M(\mathcal{D}^p_{p-1} \cap BMOA, \mathcal{D}^q_{q-1} \cap BMOA)$ with $g \not\equiv 0$. Take $a_n = \frac{1}{n^\lambda}$ $(n = 1, 2, \dots)$ with

$$\max\left(\frac{1}{2}, \frac{1}{p}\right) < \lambda \leq \frac{1}{q}$$

and set $f(z) = \sum_{n=1}^{\infty} a_n z^{2^n}$ $(z \in \mathbb{D})$. We have that $f \in \mathcal{D}^p_{p-1} \cap BMOA \setminus \mathcal{D}^q_{q-1}$. Then we use the Rademacher functions as in the proof of Case 3 of Theorem 1 (ii).

Let $\{r_k(t)\}$ be the Rademacher functions and $f_t(z) = \sum_{k=1}^{\infty} r_k(t) a_k z^{2^k}$. We have that $f_t \in \mathcal{D}^p_{p-1} \cap BMOA \setminus \mathcal{D}^q_{q-1}$ for all $0 < t \leq 1$. Moreover

$$\|f_t\|_{BMOA} \asymp \|f\|_{H^2}, \quad t \in [0, 1]$$

and

$$\sum_{k=0}^{\infty} |a_k|^p \asymp \|f_t\|^p_{\mathcal{D}^p_{p-1}} \asymp \|f\|^p_{\mathcal{D}^p_{p-1}}, \quad t \in [0, 1]. \tag{2.4.7}$$

Since $g \not\equiv 0$, there exists a positive constant C such that $M_q^q(r, g) \geq C$, $1/2 < r < 1$. Using Fubini's theorem, Khinchine's inequality and bearing in mind that f' is also given by a power series with Hadamard gaps (thus $M_2(r, f') \asymp M_q(r, f')$) we have that

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{D}} |gf_t'(z)|^q (1-|z|^2)^{q-1} dA(z)dt \\
&\lesssim \int_0^1 \int_{\mathbb{D}} |(gf_t)'(z)|^q (1-|z|^2)^{q-1} dA(z)dt + \int_0^1 \int_{\mathbb{D}} |f_t g'(z)|^q (1-|z|^2)^{q-1} dA(z)dt \\
&\lesssim \|f\|^q_{\mathcal{D}^p_{p-1}\cap BMOA} + \int_{\mathbb{D}} |g'(z)|^q \int_0^1 |f_t(z)|^q (1-|z|^2)^{q-1} dt dA(z) \\
&\lesssim \|f\|^q_{\mathcal{D}^p_{p-1}\cap BMOA} + \int_{\mathbb{D}} |g'(z)|^q M_2^q(|z|, f)(1-|z|^2)^{q-1} dA(z) \\
&\lesssim \|f\|^q_{\mathcal{D}^p_{p-1}\cap BMOA} + \|f\|^q_{BMOA} \int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-1} dA(z) \\
&\lesssim \|f\|^q_{\mathcal{D}^p_{p-1}\cap BMOA}.
\end{aligned} \tag{2.4.8}$$

On the other hand, since $g \not\equiv 0$, there exists a positive constant C such that $M_q^q(r, g) \geq C$, $1/2 < r < 1$. Using Fubini's theorem, Khinchine's inequality and bearing in mind that f' is also given by a power series with Hadamard gaps (thus $M_2(r, f') \asymp M_q(r, f')$) we have that

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{D}} |gf_t'(z)|^q (1-|z|^2)^{q-1} dA(z)dt \\
&= \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} \left(\int_0^1 |f_t'(z)|^q dt \right) dA(z) \\
&\asymp \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} M_2^q(|z|, f') dA(z) \\
&\geq C \int_{1/2}^1 M_q^q(r, g) M_q^q(r, f')(1-r^2)^{q-1} dr \\
&\geq C \int_{1/2}^1 M_q^q(r, f')(1-r^2)^{q-1} dr = +\infty.
\end{aligned} \tag{2.4.9}$$

This is in contradiction with (2.4.8). It follows that $g \equiv 0$. □

2.5 Integral operators on Dirichlet subspaces of the Bloch space

We begin studying the integral operator $I_g(f)(z) = \int_0^z g(\xi)f'(\xi)\,d\xi$ on $\mathcal{B}\cap\mathcal{D}^p_{p-1}$ spaces.

Theorem 16. *Let g be an analytic function and $p, q > 0$. Then the following hold:*

(i) If $0 < p \le q < \infty$ then I_g is bounded from $\mathcal{B}\cap\mathcal{D}^p_{p-1}$ to $\mathcal{B}\cap\mathcal{D}^q_{q-1}$ if and only if $g \in H^\infty$.

(ii) If $0 < q < p < \infty$ the operator I_g is bounded from $\mathcal{B}\cap\mathcal{D}^p_{p-1}$ to $\mathcal{B}\cap\mathcal{D}^q_{q-1}$ if and only if $g \equiv 0$.

Proof. Assume that $I_g : \mathcal{B}\cap\mathcal{D}^p_{p-1} \to \mathcal{B}\cap\mathcal{D}^q_{q-1}$ is a bounded operator. Since $\sup_{a\in\mathbb{D}} \|\varphi_a\|_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} < \infty$, for any $z \in \mathbb{D}$

$$\sup_{z,\,a\in\mathbb{D}} (1-|z|^2)|\varphi_a'(z)g(z)| \le \|I_g(\varphi_a)\|_{\mathcal{B}\cap\mathcal{D}^q_{q-1}} \lesssim \|\varphi_a\|_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} \le C < \infty, \tag{2.5.1}$$

in particular taking $z = a$, $\sup_{a\in\mathbb{D}} |g(a)| < C$, that is $g \in H^\infty$.

On the other hand, if $g \in H^\infty$

$$\begin{aligned}\int_{\mathbb{D}} |I_g(f)'(z)|^q(1-|z|^2)^{q-1}\,dA(z) &= \int_{\mathbb{D}} |g(z)f'(z)|^q(1-|z|^2)^{q-1}\,dA(z)\\ &\le \|g\|^q_{H^\infty}\|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-1}},\end{aligned} \tag{2.5.2}$$

where in the last inequality we use that $0 < p \le q < \infty$. Moreover

$$|I_g(f)'(z)|(1-|z|^2) = |g(z)f'(z)|(1-|z|^2) \le \|g\|_{H^\infty}\|f\|_{\mathcal{B}\cap\mathcal{D}^p_{p-1}}. \tag{2.5.3}$$

which finishes the proof of (i).

(ii). Assume that $g \not\equiv 0$ and take $a_n = \frac{1}{n^{1/p+\varepsilon}}$ with $0 < \varepsilon < \frac{1}{q} - \frac{1}{p}$ and $f(z) = \sum_{n=1}^\infty a_n z^{2^n}$. Let $\{r_k(t)\}$ be the Rademacher functions and let $f_t(z) = \sum_{k=1}^\infty r_k(t)a_k z^{2^k}$. Then by Proposition A, $f \in \mathcal{B}\cap\mathcal{D}^p_{p-1}\setminus\mathcal{D}^q_{q-1}$,

$$\|f\|_{\mathcal{B}} \asymp \sup_n |a_n| \asymp \|f_t\|_{\mathcal{B}}, \quad t\in[0,1]$$

and

$$\|f_t\|_{\mathcal{D}^p_{p-1}} \asymp \|f\|_{\mathcal{D}^p_{p-1}}, \quad t\in[0,1]. \tag{2.5.4}$$

Since $g \not\equiv 0$, there exists a positive constant C such that $M_q^q(r,g) \geq C$, $1/2 < r < 1$. Using Fubini's theorem, Khinchine's inequality and bearing in mind that f' is also given by a power series with Hadamard gaps (thus $M_2(r,f') \asymp M_q(r,f')$) we have that

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{D}} |I_g(f_t)'(z)|^q (1-|z|^2)^{q-1} dA(z)dt \\
&= \int_0^1 \int_{\mathbb{D}} |g(z) f_t'(z)|^q (1-|z|^2)^{q-1} dA(z)dt \\
&= \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} \left(\int_0^1 |f_t'(z)|^q dt \right) dA(z) \\
&\asymp \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} M_2^q(|z|, f') dA(z) \\
&\geq C \int_{1/2}^1 M_q^q(r,g) M_q^q(r,f')(1-r^2)^{q-1} dr \\
&\geq C \int_{1/2}^1 M_q^q(r,f')(1-r^2)^{q-1} dr = +\infty,
\end{aligned} \tag{2.5.5}$$

which implies that for some $t \in [0,1]$, $\int_{\mathbb{D}} |I_g(f_t)'(z))|^q (1-|z|^2)^{q-1} dA(z) = \infty$. Consequently g must be the null function.

□

Before presenting our result concerning the integral operator T_g, we need to prove a preliminary result.

Proposition 5. *Assume that* $0 < p, q \leq 2$ *and* $\mathcal{X}$ *is a subspace of* $\mathcal{B}$. *Then*

$$\mathcal{X} \cap \mathcal{D}_{p-1}^p \subset H^q.$$

Proof. If $p \geq q$, the assertion follows from (1.2.2). If $p < q$ then we use that Bloch subspaces of $\mathcal{D}_{p-1}^p$ spaces are nested to assert that

$$\mathcal{X} \cap \mathcal{D}_{p-1}^p \subset \mathcal{X} \cap \mathcal{D}_{q-1}^q \subset \mathcal{D}_{q-1}^q \subset H^q. \tag{2.5.6}$$

This finishes the proof. □

Theorem 17. *Let* g *be an analytic function and* $0 < p, q < \infty$. *The following hold:*

(i) If $2 < q < \infty$, *then* $T_g : \mathcal{B} \cap \mathcal{D}_{p-1}^p \to \mathcal{B} \cap \mathcal{D}_{q-1}^q$ *is bounded if and only if* $g \in \mathcal{B}_{\log}$.

(ii) If $0 < \max\{p, 1\} < q \leq 2$ *then* $T_g : \mathcal{B} \cap \mathcal{D}^p_{p-1} \to \mathcal{B} \cap \mathcal{D}^q_{q-1}$ *is bounded if and only if* $g \in \mathcal{B}_{\log}$.

(iii) If $1 < q \leq p \leq 2$ *then* $T_g : \mathcal{B} \cap \mathcal{D}^p_{p-1} \to \mathcal{B} \cap \mathcal{D}^q_{q-1}$ *is bounded if and only if* $g \in \mathcal{B}_{\log}$.

(iv) If $0 < p \leq q < 1$, $\alpha > \frac{1}{q}$ *and* $g \in \mathcal{B}_{\log,\alpha}$, *then* $T_g : \mathcal{B} \cap \mathcal{D}^p_{p-1} \to \mathcal{B} \cap \mathcal{D}^q_{q-1}$ *is bounded. Moreover, there exists* $g \in \mathcal{B}_{\log,\frac{1}{q}}$ *such that* $T_g(\mathcal{B} \cap \mathcal{D}^p_{p-1}) \not\subset \mathcal{B} \cap \mathcal{D}^q_{q-1}$.

(v) If $0 < q \leq 2 < p$, $q\left(\alpha - \frac{1}{2} + \frac{1}{p}\right) > 1$ *and* $g \in \mathcal{B}_{\log,\alpha}$, *then* $T_g : \mathcal{B} \cap \mathcal{D}^p_{p-1} \to \mathcal{B} \cap \mathcal{D}^q_{q-1}$ *is bounded. Moreover, if* $q\left(\alpha - \frac{1}{2} + \frac{1}{p}\right) = 1$, *there exists* $g \in \mathcal{B}_{\log,\alpha}$ *such that* $T_g(\mathcal{B} \cap \mathcal{D}^p_{p-1}) \not\subset \mathcal{B} \cap \mathcal{D}^q_{q-1}$.

Proof. First we shall prove that $g \in \mathcal{B}_{\log}$ whenever $T_g : \mathcal{B} \cap \mathcal{D}^p_{p-1} \to \mathcal{B} \cap \mathcal{D}^q_{q-1}$, $0 < p, q < \infty$ is bounded. In particular, this gives the necessity part of (i), (ii) and (iii). Set $f_\theta(z) = \log \frac{1}{1-ze^{-i\theta}}$, $\theta \in [0, 2\pi)$. By (2.1.4) $\{f_\theta\}_{\theta\in[0,2\pi)}$ is uniformly bounded in $\mathcal{B} \cap \mathcal{D}^p_{p-1}$. Thus

$$\sup_{z\in\mathbb{D},\,\theta\in[0,2\pi)} (1-|z|^2)|T_g'(f_\theta(z))| = \sup_{z\in\mathbb{D},\,\theta\in[0,2\pi)} (1-|z|^2)|g'(z)f_\theta(z)| \lesssim \sup_{\theta\in[0,2\pi)} \|f_\theta\|_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} < \infty.$$

So for any $z \in \mathbb{D}$, choose $e^{i\theta} = \frac{z}{|z|}$. Then we have that

$$\sup_{z\in\mathbb{D}} |g'(z)|(1-|z|)\log\frac{1}{1-|z|} < \infty,$$

that is $g \in \mathcal{B}_{\log}$.

On the other hand, take $f \in \mathcal{B} \cap \mathcal{D}^p_{p-1}$, $\alpha \geq 1$ and $g \in \mathcal{B}_{\log,\alpha}$. Then

$$\|T_g(f)\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |g'(z)f(z)|(1-|z|) \lesssim \sup_{z\in\mathbb{D}} |g'(z)|(1-|z|)\log\frac{1}{1-|z|} \leq \|g\|_{\mathcal{B}_{\log,\alpha}}$$

Thus, in any case (i)-(v), $T_g(\mathcal{B}\cap\mathcal{D}^p_{p-1}) \subset \mathcal{B}$. Next, we shall prove that $T_g(\mathcal{B}\cap\mathcal{D}^p_{p-1}) \subset \mathcal{D}^p_{p-1}$ for the distinct values of the parameter α.

$$\begin{aligned}
\|T_g(f)\|^q_{\mathcal{D}^q_{q-1}} &= \int_{\mathbb{D}} |f(z)g'(z)|^q(1-|z|^2)^{q-1}dA(z) \\
&\lesssim \int_0^1 \frac{1}{(1-r)^q \log^{\alpha q}\frac{e}{1-r}} M_q^q(r,f)(1-r)^{q-1}\,dr \\
&= \int_0^1 \frac{1}{(1-r)\log^{\alpha q}\frac{e}{1-r}} M_q^q(r,f)\,dr \\
&= A(p,q,\alpha).
\end{aligned} \tag{2.5.7}$$

If $2 < q < \infty$, bearing in mind that $f \in \mathcal{B}$, it follows from [21] that $M_q(f,r) = O\left(\left(\log \frac{e}{1-r}\right)^{\frac{1}{2}}\right)$. Thus

$$
\begin{aligned}
A(p,q,1) &\lesssim \int_0^1 \frac{1}{(1-r)\log^q \frac{e}{1-r}} \left(\log \frac{e}{1-r}\right)^{\frac{q}{2}} dr \\
&= \int_0^1 \frac{1}{(1-r)\log^{\frac{q}{2}} \frac{e}{1-r}} \, dr < \infty,
\end{aligned}
$$

which gives (i).

If p and q are as in (ii) or (iii), then we apply Proposition 5 to get

$$
A(p,q,1) \lesssim \|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} \int_0^1 \frac{1}{(1-r)\log^q \frac{e}{1-r}} \, dr < \infty. \tag{2.5.8}
$$

Now we shall prove (iv). Observe that if $0 < p \le q < 1$ and $\alpha > \frac{1}{q}$, it follows from Proposition 5 that

$$
A(p,q,\alpha) \lesssim \|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-1}} \int_0^1 \frac{1}{(1-r)\log^{\alpha q} \frac{e}{1-r}} \, dr < \infty, \tag{2.5.9}
$$

so $T_g : \mathcal{B}\cap\mathcal{D}^p_{p-1} \to \mathcal{B}\cap\mathcal{D}^q_{q-1}$ is bounded. Furthermore, by Proposition 1, there exists $g \in \mathcal{B}_{\log,1/q} \setminus \mathcal{D}^q_{q-1}$, which implies that $T_g(\mathcal{B}\cap\mathcal{D}^p_{p-1}) \not\subseteq \mathcal{B}\cap\mathcal{D}^q_{q-1}$.

Finally we prove (v). By (1.2.4)

$$
\begin{aligned}
A(p,q,\alpha) &\lesssim \int_0^1 \frac{1}{(1-r)\log^{qa} \frac{e}{1-r}} \left(\log \frac{1}{1-r}\right)^{\frac{q}{2}-\frac{q}{p}} dr \\
&= \int_0^1 \frac{1}{(1-r)\log^{q(\alpha-\frac{1}{2}+\frac{1}{p})}} \, dr < \infty
\end{aligned}
\tag{2.5.10}
$$

because $q\left(\alpha - \frac{1}{2} + \frac{1}{p}\right) > 1$.

Next, assume that $q\left(\alpha - \frac{1}{2} + \frac{1}{p}\right) = 1$ By [1] (see also [48, Theorem 1]) there exist $g_1, g_2 \in \mathcal{B}_{log,\alpha}$ such that

$$
|g_1'(z)| + |g_2'(z)| \gtrsim \frac{1}{(1-|z|)\left(\log \frac{e}{1-|z|}\right)^{\alpha}}, \quad z \in \mathbb{D}.
$$

So,

$$\begin{aligned}\|T_{g_1}(f)\|^q_{\mathcal{D}^q_{q-1}} + \|T_{g_2}(f)\|^q_{\mathcal{D}^q_{q-1}} &= \int_{\mathbb{D}} |f(z)|^q(|g_1'(z)|^q + |g_2'(z)|^q)(1-|z|^2)^{q-1}dA(z)\\ &\geq C\int_{\mathbb{D}} |f(z)|^q(|g_1'(z)| + |g_2'(z)|)^q(1-|z|^2)^{q-1}dA(z)\\ &\geq C\int_0^1 M_q^q(r,f)\frac{1}{(1-r)\left(\log\frac{e}{1-|r|}\right)^{\alpha q}}dr.\end{aligned}$$

Now, let $\frac{1}{p} < \gamma \leq \frac{1}{q}$ and $\Phi : [0,1) \to (1,\infty)$, defined by $\Phi(r) = \left(\log\log\frac{e^{e^2}}{1-r}\right)^{\gamma p-1}$. Then

$$\frac{\Phi'(r)}{\Phi(r)}(1-r) = (\gamma p - 1)\frac{1}{\log\frac{e^{e^2}}{1-r}\log\log\frac{e^{e^2}}{1-r}},$$

which is a decreasing function of r, so applying [61, Lemma 10] we obtain a function $f \in \mathcal{D}^p_{p-1}$ such that for any $r > \frac{1}{2}$

$$M_q(r,f) \gtrsim \left(\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(\frac{\Phi'(r)}{\Phi^2(r)}(1-r)\right)^{\frac{1}{p}} \gtrsim \frac{1}{\left(\log\frac{1}{(1-r)}\right)^{\frac{1}{p}-\frac{1}{2}}\left(\log\log\frac{1}{1-r}\right)^{\gamma}}.$$

An inspection of the proof of [61, Lemma 10] gives us that $f \in \mathcal{L}$, so by Proposition A, $f \in \mathcal{B} \cap \mathcal{D}^p_{p-1}$, and moreover the above estimates gives

$$\begin{aligned}\|T_{g_1}(f)\|^q_{\mathcal{D}^q_{q-1}} + \|T_{g_2}(f)\|^q_{\mathcal{D}^q_{q-1}} &\gtrsim \int_{\frac{1}{2}}^1 M_q^q(r,f)\frac{1}{(1-r)\left(\log\frac{1}{1-|r|}\right)^{\alpha q}}dr\\ &\gtrsim \int_{\frac{1}{2}}^1 \frac{1}{(1-r)\left(\log\frac{1}{(1-r)}\right)^{q(\alpha+\frac{1}{p}-\frac{1}{2})}\left(\log\log\frac{1}{1-r}\right)^{\gamma q}}dr = \infty.\end{aligned}$$

This implies that either $T_{g_1}(f) \notin \mathcal{D}^q_{q-1}$ or $T_{g_2}(f) \notin \mathcal{D}^q_{q-1}$. The proof is complete. □

We turn our attention to Dirichlet subspaces of $BMOA$.

Theorem 18. *Let $g \in \mathcal{H}ol(\mathbb{D})$. Then the following assertions are true:*

(i) If $0 < p \leq q < \infty$, $I_g : BMOA \cap \mathcal{D}^p_{p-1} \to BMOA \cap \mathcal{D}^q_{q-1}$ is bounded if and only if $g \in H^\infty$.

(ii) If $0<q<p<\infty$ *and* $q<2$, $I_g: BMOA\cap\mathcal{D}^p_{p-1}\to BMOA\cap\mathcal{D}^q_{q-1}$ *is bounded if and only if* $g\equiv 0$.

(iii) If $0<q<p<\infty$ *and* $q\geq 2$, $I_g: BMOA\cap\mathcal{D}^p_{p-1}\to BMOA\cap\mathcal{D}^q_{q-1}$ *is bounded if and only if* $g\in H^\infty$.

Proof. The necessity part of (i) can be proved arguing as in part (i) of Theorem 16. Reciprocally, assume that $g\in H^\infty$, then for every $f\in\mathcal{D}^p_{p-1}\cap BMOA$

$$\begin{aligned}&\int_{\mathbb{D}}|I_g'(f)(z)|^q(1-|z|^2)^{q-1}dA(z)+\sup_{I\subset\mathbb{T}}\frac{1}{|I|}\int_{S(I)}|I_g'(f)(z)|^2(1-|z|^2)dA(z)=\\&=\int_{\mathbb{D}}|g(z)f'(z)|^q(1-|z|^2)^{q-1}dA(z)+\sup_{I\subset\mathbb{T}}\frac{1}{|I|}\int_{S(I)}|g(z)f'(z)|^2(1-|z|^2)dA(z)\\&\leq\|g\|_{H^\infty}\|f\|_{BMOA\cap\mathcal{D}^p_{p-1}}.\end{aligned}\tag{2.5.11}$$

(ii). Assume that $g\not\equiv 0$. Take $a_n=\frac{1}{n^\lambda}$ $(n=1,2,\dots)$ with

$$\max\left(\frac{1}{2},\frac{1}{p}\right)<\lambda\leq\frac{1}{q}$$

and set $f_t(z)=\sum_{k=1}^\infty r_k(t)a_kz^{2^k}$, $f(z)=f_1(z)$ where $\{r_k(t)\}$ are, as usual, the Rademacher functions. We have that $f_t\in\mathcal{D}^p_{p-1}\cap BMOA\setminus\mathcal{D}^q_{q-1}$ for all $0<t\leq 1$.

Moreover since f_t are lacunary series

$$\|f_t\|_{BMOA}\asymp\|f_t\|_{H^2}=\|f\|_{H^2},\quad t\in[0,1]$$

and by Proposition A,

$$\|f_t\|_{\mathcal{D}^p_{p-1}}\asymp\|f\|_{\mathcal{D}^p_{p-1}},\quad t\in[0,1].\tag{2.5.12}$$

Since $g\not\equiv 0$, there exists a positive constant C such that $M_q^q(r,g)\geq C$, $1/2<r<1$. Using Fubini's theorem, Khinchine's inequality and bearing in mind that f' is also by a power series with Hadamard gaps (thus $M_2(r,f')\asymp M_q(r,f')$) we have that

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{D}} |I_g'(f_t(z))|^q (1-|z|^2)^{q-1} dA(z)dt \\
&= \int_0^1 \int_{\mathbb{D}} |g f_t'(z)|^q (1-|z|^2)^{q-1} dA(z)dt \\
&= \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} \left(\int_0^1 |f_t'(z)|^q dt \right) dA(z) \\
&\asymp \int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{q-1} M_2^q(|z|, f') dA(z) \\
&\geq C \int_{1/2}^1 M_q^q(r,g) M_q^q(r,f')(1-r^2)^{q-1} dr \\
&\geq C \int_{1/2}^1 M_q^q(r,f')(1-r^2)^{q-1} dr = +\infty.
\end{aligned} \tag{2.5.13}$$

This shows that for some $t \in [0,1]$,

$$\int_{\mathbb{D}} |I_g'(f_t(z))|^q (1-|z|^2)^{q-1} dA(z) = \infty,$$

hence $g \equiv 0$.

If $p \geq q$ and $q \geq 2$ then $\mathcal{D}^p_{p-1} \cap BMOA = \mathcal{D}^q_{q-1} \cap BMOA = BMOA$ and hence (iii) can be reduced to $p = q > 2$ which is covered by (i).

□

Theorem 19. *Let $g \in \mathcal{H}ol(\mathbb{D})$. Then the following hold:*

(i) If $q > 1$ then $T_g : BMOA \cap \mathcal{D}^p_{p-1} \to BMOA \cap \mathcal{D}^q_{q-1}$ is bounded if and only if $g \in BMOA_{log}$.

(ii) If $q \leq 1$ then if $g \in BMOA_{log,\alpha}$, $\alpha \in (\frac{1}{q}, \infty)$, $T_g : BMOA \cap \mathcal{D}^p_{p-1} \to BMOA \cap \mathcal{D}^q_{q-1}$ is bounded.

Proof. Assume first that T_g is bounded and let f_a ($a \in \mathbb{D}$) be the test functions

$$f_a(z) = \log \frac{1}{1-\bar{a}z}, \quad z \in \mathbb{D}.$$

We already showed in the proof of Theorem 10 that the family $\{f_a : a \in \mathbb{D}\}$ is uniformly bounded in $\mathcal{D}^\lambda_{\lambda-1} \cap BMOA$ for all $\lambda > 0$ and there exists an absolute constant $C > 0$ such that for any arc $I \subset \partial\mathbb{D}$

$$\frac{1}{C} \log \frac{2}{|I|} \leq |f_a(z)| \leq C \log \frac{2}{|I|}, \quad z \in S(I),$$

where $a = (1 - \frac{|I|}{2\pi})\xi$ with ξ being the center of I.

So

$$\frac{\log^2 \frac{2}{|I|}}{|I|} \int_{S(I)} (1-|z|^2)|g'(z)|^2\, dA(z) \leq \frac{C^2}{|I|} \int_{S(I)} (1-|z|^2)|f_a(z)|^2|g'(z)|^2\, dA(z)$$
$$= \frac{C^2}{|I|} \int_{S(I)} (1-|z|^2)|T_g(f_a)(z)|^2\, dA(z) \leq C\|f_a\| = C,$$

that is $g \in BMOA_{log}$. Reciprocally, take $f \in \mathcal{D}^p_{p-1} \cap BMOA$. First we observe that if $g \in BMOA_{log}$ then by [68] $\|T_g(f)\|_{BMOA} \leq C\|f\|_{BMOA}$, so we only need to concern ourselves with $\|T_g(f)\|_{\mathcal{D}^q_{q-1}}$.

Let $\alpha > 0$ and $g \in BMOA_{log,\alpha} \subset \mathcal{B}_{log,\alpha}$. Keeping in mind that $BMOA \subset H^q$ for all $q > 0$, we have that

$$\begin{aligned}\int_{\mathbb{D}} |T_g'(f)(z)|^q (1-|z|^2)^{q-1} dA(z) &= \int_{\mathbb{D}} |f(z)g'(z)|^q (1-|z|^2)^{q-1} dA(z) \\ &\lesssim \int_0^1 \frac{1}{(1-r)^q \log^q \frac{e}{1-r}} M_q^q(r,f)(1-r)^{q-1}\, dr \\ &\lesssim \|f\|_{\mathcal{D}^p_{p-1}\cap BMOA} \int_0^1 \frac{1}{(1-r)\log^{\alpha q} \frac{e}{1-r}}\, dr\end{aligned} \tag{2.5.14}$$

if $q > 1$ the last integral converges for $\alpha = 1$ and since we already proved the necessity, (i) holds. If $q \leq 1$, and $\alpha \in (\frac{1}{q}, \infty)$, the integral again converges hence (ii) holds. □

Regarding the study of integration operators on $H^\infty \cap \mathcal{D}^p_{p-1}$, I would like to comment that our results are not satisfactory. The problem ends up with that of describing those analytic functions g such that $T_g : H^\infty \to H^\infty$ is bounded. This is something that does not follows with our techniques.

Chapter 3

A generalized Hilbert matrix acting on Hardy spaces

In this chapter we shall study a new class of integral operators associated with certain Hankel matrices acting on Hardy spaces. Most of our results concerning this topic are included in [20].

If μ is a finite positive Borel measure on $[0,1)$ and $n = 0,1,2,\dots$, we let μ_n denote the moment of order n of μ, that is,

$$\mu_n = \int_{[0,1)} t^n \, d\mu(t),$$

and we define $\mathcal{H}_\mu$ to be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathcal{H}_\mu$ can be viewed as an operator on spaces of analytic functions by its action on the Taylor coefficients:

$$a_n \mapsto \sum_{k=0}^{\infty} \mu_{n,k} a_k, \quad n = 0,1,2,\cdots .$$

To be precise, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbb{D})$ we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, \tag{3.0.1}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = \left((n+k+1)^{-1}\right)_{n,k\geq 0}$, which induces the classical Hilbert operator $\mathcal{H}$, a prototype of a Hankel operator which has attracted a considerable amount of attention during the last years. Indeed, the study of the boundedness, the operator norm and the spectrum of $\mathcal{H}$ on Hardy and weighted Bergman spaces [5, 23, 24, 37, 63] links $\mathcal{H}$ up to weighted composition operators, the Szegö projection, Legendre functions and the theory of Muckenhoupt weights.

Hardy's inequality [27, page 48] guarantees that $\mathcal{H}(f)$ is a well defined analytic function in $\mathbb{D}$ for every $f \in H^1$. However, the resulting Hilbert operator $\mathcal{H}$ is bounded from H^p to H^p if and only if $1 < p < \infty$ [23]. In a recent paper [51] Lanucha, Nowak, and Pavlovic have considered the question of finding subspaces of H^1 which are mapped by $\mathcal{H}$ into H^1.

Galanopoulos and Peláez [38] have described the measures μ so that the generalized Hilbert operator $\mathcal{H}_\mu$ becomes well defined and bounded on H^1. Carleson measures play a basic role in the work. Indeed, Galanopoulos and Peláez proved in [38] that if μ is a Carleson measure then the operator $\mathcal{H}_\mu$ is well defined in H^1, obtaining en route the following integral representation

$$\mathcal{H}_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t), \quad z \in \mathbb{D}, \quad \text{for all } f \in H^1. \tag{3.0.2}$$

For simplicity, we shall write throughout the chapter

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t), \tag{3.0.3}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. It was also proved in [38] that if $I_\mu(f)$ defines an analytic function in $\mathbb{D}$ for all $f \in H^1$, then μ has to be a Carleson measure. This condition does not ensure the boundedness of $\mathcal{H}_\mu$ on H^1, as the classical Hilbert operator $\mathcal{H}$ shows.

Let μ be a positive Borel measure in $\mathbb{D}$, $0 \le \alpha < \infty$, and $0 < s < \infty$. Following [73], we say that μ is an α-logarithmic s-Carleson measure, respectively, a vanishing α-logarithmic s-Carleson measure, if

$$\sup_{a\in\mathbb{D}} \frac{\mu(S(a))\left(\log\frac{2}{1-|a|^2}\right)^\alpha}{(1-|a|^2)^s} < \infty, \quad \text{respectively,} \quad \lim_{|a|\to 1^-} \frac{\mu(S(a))\left(\log\frac{2}{1-|a|^2}\right)^\alpha}{(1-|a|^2)^s} = 0.$$

Theorem 1. 2 of [38] asserts that if μ is a Carleson measure on $[0,1)$, then $\mathcal{H}_\mu$ is a bounded (respectively, compact) operator from H^1 into H^1 if and only if μ is a 1-logarithmic 1-Carleson measure (respectively, a vanishing 1-logarithmic 1-Carleson measure).

It is also known that $\mathcal{H}_\mu$ is bounded from H^2 into itself if and only if μ is a Carleson measure (see [64, p. 42, Theorem 7.2]).

Our main aim in this chapter is to study the generalized Hilbert matrix $\mathcal{H}_\mu$ acting on H^p spaces ($0 < p < \infty$). Namely, for any given p, q with $0 < p, q < \infty$, we wish to characterize those for which $\mathcal{H}_\mu$ is a bounded (compact) operator from H^p into H^q and to describe those measures μ such that $\mathcal{H}_\mu$ belongs to the Schatten class $\mathcal{S}_p(H^2)$. A key tool will be a description of those positive Borel measures μ on $[0, 1)$ for which $\mathcal{H}_\mu$ is well defined in H^p and $\mathcal{H}_\mu(f) = I_\mu(f)$. Let us start with the case $p \leq 1$.

Theorem 20. *Suppose that $0 < p \leq 1$ and let μ be a positive Borel measure on $[0, 1)$. Then the following two conditions are equivalent:*

(i) μ is an $\frac{1}{p}$-Carleson measure.

(ii) $I_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ for any $f \in H^p$.

Furthermore, if (i) and (ii) hold and $f \in H^p$, then $\mathcal{H}_\mu(f)$ is also a well defined analytic function in $\mathbb{D}$, and $\mathcal{H}_\mu(f) = I_\mu(f)$, for all $f \in H^p$.

We remark that for $p = 1$, this reduces to [38, Proposition 1. 1].

For $0 < q < 1$, we let B_q denote the space consisting of those $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$\int_0^1 (1 - r)^{\frac{1}{q} - 2} M_1(r, f)\, dr < \infty.$$

The Banach space B_q is the "containing Banach space" of H^q, that is, H^q is a dense subspace of B_q, and the two spaces have the same continuous linear functionals [26]. Next we shall show that if μ is an $1/p$-Carleson measure then $\mathcal{H}_\mu$ actually applies H^p into B_q for all $q < 1$. We shall also give a characterization of those μ for which $\mathcal{H}_\mu$ maps H^p into H^q ($q \geq 1$). Before stating these results precisely, let us mention that all over the chapter we shall use the notation that for any given $\alpha > 1$, α' will denote the conjugate exponent of α, that is, $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, or $\alpha' = \frac{\alpha}{\alpha - 1}$.

Theorem 21. *Suppose that $0 < p \leq 1$ and let μ be a positive Borel measure on $[0, 1)$ which is an $\frac{1}{p}$-Carleson measure.*

(i) If $0 < q < 1$, then $\mathcal{H}_\mu$ is a bounded operator from H^p into B_q, the containing Banach space of H^q.

(ii) $\mathcal{H}_\mu$ *is a bounded operator from* H^p *into* H^1 *if and only if* μ *is an* 1*-logarithmic* $\frac{1}{p}$*-Carleson measure.*

(iii) *If* $q > 1$ *then* $\mathcal{H}_\mu$ *is a bounded operator from* H^p *into* H^q *if and only if* μ *is an* $\frac{1}{p} + \frac{1}{q'}$*-Carleson measure.*

Let us state next our results for $p > 1$.

Theorem 22. *Suppose that* $1 < p < \infty$ *and let* μ *be a positive Borel measure on* $[0, 1)$*. Then:*

(i) $I_\mu(f)$ *is a well defined analytic function in* $\mathbb{D}$ *for any* $f \in H^p$ *if and only if* μ *is an* 1*-Carleson measure for* H^p*, or, equivalently, if and only if*

$$\int_0^1 \left(\int_0^{1-s} \frac{d\mu(t)}{1-t} \right)^{p'} ds < \infty. \tag{3.0.4}$$

(ii) If μ *satisfies (3.0.4) then* $\mathcal{H}_\mu(f)$ *is also a well defined analytic function in* $\mathbb{D}$*, whenever* $f \in H^p$*, and*

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \textit{for every } f \in H^p.$$

Theorem 23. *Suppose that* $1 < p < \infty$ *and let* μ *be a positive Borel measure on* $[0, 1)$ *which satisfies (3.0.4).*

(i) *If* $1 < p \le q < \infty$*, then* $\mathcal{H}_\mu$ *is a bounded operator from* H^p *to* H^q *if and only if* μ *is an* $\frac{1}{p} + \frac{1}{q'}$*-Carleson measure.*

(ii) *If* $1 < q < p$*, then* $\mathcal{H}_\mu$ *is a bounded operator from* H^p *to* H^q *if and only if the function defined by* $s \mapsto \int_0^{1-s} \frac{d\mu(t)}{1-t}$ $(s \in [0, 1))$ *belongs to* $L^{\left(\frac{pq'}{p+q'}\right)'}([0, 1))$*.*

(iii) $\mathcal{H}_\mu$ *is a bounded operator from* H^p *to* H^1 *if and only if the function defined by* $s \mapsto \int_0^{1-s} \frac{\log \frac{1}{1-t} d\mu(t)}{1-t}$ $(s \in [0, 1))$ *belongs to* $L^{p'}([0, 1))$*.*

(iv) *If* $0 < q < 1$*, then* $\mathcal{H}_\mu$ *is a bounded operator from* H^p *into* B_q*.*

Let us remark that both if $0 < p \le 1$ and μ is an $1/p$-Carleson measure, or if $1 < p < \infty$ and μ satisfies (3.0.4), we have that μ is an 1-Carleson measure for H^p. By the closed graph theorem it follows that, for any $q > 0$,

$$\mathcal{H}_\mu(H^p) \subset H^q \;\Leftrightarrow\; \mathcal{H}_\mu \text{ is a bounded operator form } H^p \text{ into } H^q.$$

Substitutes of Theorem 21 and Theorem 23 regarding compactness will be stated and proved in Section 3.4.

Finally, we address the question of describing those measures μ such that $\mathcal{H}_\mu$ belongs to the Schatten class $\mathcal{S}_p(H^2)$, $(1 < p < \infty)$. Given a separable Hilbert space X and $0 < p < \infty$, let $\mathcal{S}_p(X)$ denote the Schatten p-class of operators on X. The class $\mathcal{S}_p(X)$ consists of those compact operators T on X whose sequence of singular numbers $\{\lambda_n\}$ belongs to ℓ^p, the space of p-summable sequences. It is well known that, if λ_n are the singular numbers of an operator T, then

$$\lambda_n = \lambda_n(T) = \inf\{\|T - K\| : \operatorname{rank} K \leq n\}.$$

Thus finite rank operators belong to every $\mathcal{S}_p(X)$, and the membership of an operator in $\mathcal{S}_p(X)$ measures in some sense the size of the operator. In the case when $1 \leq p < \infty$, $\mathcal{S}_p(X)$ is a Banach space with the norm

$$\|T\|_p = \left(\sum_n |\lambda_n|^p\right)^{1/p},$$

while for $0 < p < 1$ we have the following inequality $\|T + S\|_p^p \leq \|T\|_p^p + \|S\|_p^p$. We refer to [75] for more information about $\mathcal{S}_p(X)$.

Galanopoulos and Peláez [38, Theorem 1. 6] found a characterization of those μ for which $\mathcal{H}_\mu$ is a Hilbert-Schmidt operator on H^2 improving a result of [66]. In [64, p. 239, Corollary 2. 2] it is proved that, for $1 < p < \infty$, $\mathcal{H}_\mu \in \mathcal{S}_p(H^2)$ if and only if $h_\mu(z) = \sum_{n=1}^\infty \mu_{n+1} z^n$ belongs to the Besov space B^p (see [75, Chapter 5]) of those analytic functions g in $\mathbb{D}$ such that

$$||g||_{B^p}^p = |g(0)|^p + \int_{\mathbb{D}} |g'(z)|^p (1 - |z|^2)^{p-2}\, dA(z) < \infty.$$

We simplify this result describing the membership of $\mathcal{H}_\mu$ in the Schatten class $\mathcal{S}_p(H^2)$ in terms of the moments μ_n.

Theorem 24. *Assume that* $1 < p < \infty$ *and let* μ *be a positive Borel measure on* $[0, 1)$. *Then,* $\mathcal{H}_\mu \in \mathcal{S}_p(H^2)$ *if and only if* $\sum_{n=0}^\infty (n + 1)^{p-1} \mu_n^p < \infty$.

3.1 Preliminary results

In this section we shall collect a number of results which will be needed in our work. We start by obtaining a characterization of s-Carleson measures in terms of the moments.

Proposition 6. *Let μ be a positive Borel measure on $[0,1)$ and $s>0$. Then μ is an s-Carleson measure if and only if the sequence of moments $\{\mu_n\}_{n=0}^{\infty}$ satisfies*

$$\sup_{n\geq 0}(1+n)^s\,\mu_n<\infty. \tag{3.1.1}$$

The proof is simple and will be omitted.

The following result, which may be of independent interest, asserts that for any function $f\in H^p$ $(0<p<\infty)$ we can find another one F with the same H^p-norm and which is non-negative and bigger than $|f|$ in the radius $(0,1)$.

Proposition 7. *Suppose that $0<p<\infty$ and $f\in H^p$, $f\not\equiv 0$. Then there exists a function $F\in H^p$ with $\|F\|_{H^p}=\|f\|_{H^p}$ and satisfying the following properties:*

(i) $F(r)>0$, for all $r\in(0,1)$.

(ii) $|f(r)|\leq F(r)$, for all $r\in(0,1)$.

(iii) F has no zeros in $\mathbb{D}$.

Proof. Let us consider first the case $p=2$. So, take $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H^2$, $f\not\equiv 0$. Set $G(z)=\sum_{n=0}^{\infty}|a_n|z^n$ $(z\in\mathbb{D})$. Then $G\in H^2$ and $\|G\|_{H^2}=\|f\|_{H^2}$. Furthermore, we have:

$$0\leq|f(r)|\leq G(r)\ \text{ and } G(r)>0,\quad \text{for all } r\in(0,1), \tag{3.1.2}$$

and

$$G(\overline{z})=\overline{G(z)},\quad z\in\mathbb{D}. \tag{3.1.3}$$

By (3.1.2) and (3.1.3) we see that the sequence $\{z_n\}$ of the zeros of G with $z_n\neq 0$ (which is a Blaschke sequence) can be written in the form $\{z_n\}=\{\alpha_n\}\cup\{\overline{\alpha_n}\}\cup\{\beta_n\}$ where $\mathrm{Im}(\alpha_n)>0$ and $-1<\beta_n<0$. Then the Blaschke product B with the same zeros that G is

$$B(z)=z^m\prod\left(\frac{\alpha_n-z}{1-\overline{\alpha_n}z}\frac{\overline{\alpha_n}-z}{1-\alpha_nz}\right)\prod\frac{z-\beta_n}{1-\beta_nz},$$

where m is the order of 0 as zero of G (maybe 0). Using the Riesz factorization theorem [27, Theorem 2. 5], we can factor G in the form $G = B \cdot F$ where F is an H^2-function with no zeros and with $\|f\|_{H^2} = \|G\|_{H^2} = \|F\|_{H^2}$. Notice that $B(r) > 0$, for all $r \in (0,1)$. This together with (3.1.2) gives that $F(r) > 0$, for all $r \in (0,1)$. Finally since $|B(z)| \leq 1$, for all z, we have that $G(r) \leq F(r)$ $(r \in (0,1))$ and then (3.1.2) implies $|f(r)| \leq F(r)$ $(r \in (0,1))$. This finishes the proof in the case $p = 2$.

If $0 < p < \infty$ and $f \in H^p$, $f \not\equiv 0$, write f in the form $f = B \cdot g$ where B is a Blaschke product and g is an H^p-function without zeros and with $\|g\|_{H^p} = \|f\|_{H^p}$. Now $g^{p/2} \in H^2$. By the previous case, we have a function $G \in H^2$ without zeros, which take positive values in the radius $(0,1)$, and satisfying $\|G\|_{H^2} = \|g^{p/2}\|_{H^2}$ and $|g(r)|^{p/2} \leq G(r)$, for all $r \in (0,1)$. It is clear that the function $F = G^{2/p}$ satisfies the conclusion of Proposition 7. □

We shall also use the following description of α-logarithmic s-Carleson measures (see [73, Theorem 2]).

Lemma B. *Suppose that $0 \leq \alpha < \infty$ and $0 < s < \infty$ and μ is a positive Borel measure in $\mathbb{D}$. Then μ is an α-logarithmic s-Carleson measure if and only if*

$$K_{\alpha,s}(\mu) \stackrel{def}{=} \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^{\alpha} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{s} d\mu(z) < \infty. \tag{3.1.4}$$

When $\alpha = 0$, the constant $K_{0,s}(\mu)$ will be simply written as $K_s(\mu)$. We remark that, if $s \geq 1$, then $K_s(\mu)$ is equivalent to the norm of the embedding $i : H^p \to L^{ps}(d\mu)$ for any $p \in (0,\infty)$.

Next we recall the following useful characterization of q-Carleson measures for H^p in the case $0 < q < p < \infty$ (see [14, 54, 70]).

Take α with $0 < \alpha < \frac{\pi}{2}$. Given $s \in \mathbb{R}$, we let $\Gamma_\alpha(e^{is})$ denote the Stolz angle with vertex e^{is} and semi-aperture α, that is, the interior of the convex hull of $\{e^{is}\} \cup \{|z| < \sin\alpha\}$. If μ is a positive Borel measure in $\mathbb{D}$, we define "the α-balagaye" $\tilde{\mu}_\alpha$ of μ as follows:

$$\tilde{\mu}_\alpha(e^{is}) = \int_{\Gamma_\alpha(e^{is})} \frac{d\mu(z)}{1-|z|}, \quad s \in \mathbb{R}.$$

Theorem D. *Let μ be a positive Borel measure on $\mathbb{D}$ and $0 < q < p < \infty$. Then μ is a q-Carleson measure for H^p if and only if $\tilde{\mu}_\alpha \in L^{\frac{p}{p-q}}(\partial\mathbb{D})$ for some (equivalently, for all) $\alpha \in (0, \frac{\pi}{2})$.*

A simple geometric argument shows that if the measure μ is supported in $[0,1)$ then $\Gamma_\alpha(e^{is}) \cap [0,1) = [0, s_\alpha)$, where

$$1 - s_\alpha \sim (\tan\alpha)\, s, \quad \text{as } s \to 0.$$

In particular, this implies the following.

Theorem E. *Let μ be a positive Borel measure on $\mathbb{D}$ supported in $[0,1)$, $0 < q < p < \infty$. Then μ is a q-Carleson measure for H^p if and only if*

$$\int_0^1 \left(\int_0^{1-s} \frac{d\mu(t)}{1-t} \right)^{\frac{p}{p-q}} ds \, < \, \infty. \tag{3.1.5}$$

3.2 Proofs of the main results. Case $p \leq 1$.

Proof of Theorem 20.

(i) $\Rightarrow$ (ii). Suppose that μ is an $1/p$-Carleson measure. Using [27, Theorem 9. 4], we see that there exists a positive constant C such that

$$\int_{[0,1)} |f(t)|\, d\mu(t) \, \leq \, C\|f\|_{H^p}, \quad \text{for all } f \in H^p. \tag{3.2.1}$$

Take $f \in H^p$. Using (3.2.1) we obtain that

$$\sum_{n=0}^{\infty} \Big(\int_{[0,1)} t^n |f(t)|\, d\mu(t) \Big) |z|^n \, \leq \, \frac{C\,\|f\|_{H^p}}{1-|z|}, \quad z \in \mathbb{D}.$$

This implies that, for every $z \in \mathbb{D}$, the integral

$$\int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t) \, = \, \int_{[0,1)} f(t) \left(\sum_{n=0}^{\infty} t^n z^n \right) d\mu(t)$$

converges and that

$$I_\mu(f)(z) \, = \, \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t) \, = \, \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t)\, d\mu(t) \right) z^n, \quad z \in \mathbb{D}. \tag{3.2.2}$$

Thus $I_\mu(f)$ is a well defined analytic function in $\mathbb{D}$.

(ii) $\Rightarrow$ (i). We claim that

$$\int_{[0,1)} |f(t)|\, d\mu(t) \,<\, \infty, \quad \text{for all } f \in H^p. \tag{3.2.3}$$

Indeed, take $f \in H^p$. Let F be the function associated to f by Proposition 7. Since $F \in H^p$, we have that the integral $\int_{[0,1)} \frac{F(t)}{1-tz}\, d\mu(t)$ converges for all $z \in \mathbb{D}$. Taking $z = 0$ and bearing in mind that $0 \le |f(t)| \le F(t)$ $(t \in (0,1))$, we obtain that

$$\int_{[0,1)} |f(t)|\, d\mu(t) \,\le\, \int_{[0,1)} F(t)\, d\mu(t) \,<\infty.$$

Thus (3.2.3) holds.

For any $\beta \in [0,1)$ and $f \in H^p$ define

$$T_\beta(f) = f \cdot \chi_{\{0 \le |z| < \beta\}}.$$

By (3.2.3), T_β is a linear operator from H^p into $L^1(d\mu)$ and by the lemma in [27, Section 3. 2],

$$\|T_\beta(f)\|_{L^1(d\mu)} \,=\, \int_{[0,\beta)} |f(t)|\, d\mu(t) \,\le\, \left[\sup_{|z|\le\beta} |f(z)|\right] \cdot \mu([0,\beta)) \le C_\beta \|f\|_{H^p}, \quad f \in H^p.$$

Thus, for every $\beta \in [0,1)$, T_β is a bounded linear operator from H^p into $L^1(d\mu)$. Furthermore, (3.2.3) also implies that

$$\sup_{0\le\beta<1} \|T_\beta(f)\|_{L^1(d\mu)} \,\le\, \int_{[0,1)} |f(t)|\, d\mu(t) = C_f < \infty, \quad \text{for all } f \in H^p.$$

Then, by the principle of uniform boundedness, we deduce that $\sup_{\beta\in[0,1)} \|T_\beta\| < \infty$ which implies that the identity operator is bounded from H^p into $L^1(d\mu)$. Using again [27, Theorem 9. 4] we obtain that μ is an $1/p$-Carleson measure.

Assume now (i) (and (ii)), that is, assume that μ is $1/p$-Carleson measure. Take $f \in H^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$. By Proposition 6 and [27, Theorem 6. 4] we have that there exists $C > 0$ such that

$$|\mu_{n,k}| \,=\, |\mu_{n+k}| \,\le\, \frac{C}{(k+1)^{1/p}} \;\text{ and }\; |a_k| \le C(k+1)^{(1-p)/p}, \quad \text{for all } n, k.$$

Then it follows that, for every n,

$$\begin{aligned}\sum_{k=0}^{\infty} |\mu_{n,k}||a_k| &\le C \sum_{k=0}^{\infty} \frac{|a_k|}{(k+1)^{1/p}} \,=\, C \sum_{k=0}^{\infty} \frac{|a_k|^p\, |a_k|^{1-p}}{(k+1)^{1/p}} \\ &\le C \sum_{k=0}^{\infty} \frac{|a_k|^p\, (k+1)^{(1-p)^2/p}}{(k+1)^{1/p}} \,=\, C \sum_{k=0}^{\infty} (k+1)^{p-2} |a_k|^p\end{aligned}$$

and then by a well known result of Hardy and Littlewood ([27, Theorem 6. 2]) we deduce that

$$\sum_{k=0}^{\infty} |\mu_{n,k}||a_k| \le C \|f\|_{H^p}^p, \quad \text{for all } n.$$

This implies that $\mathcal{H}_\mu$ is a well defined analytic function in $\mathbb{D}$ and that

$$\int_{[0,1)} t^n f(t)\, d\mu(t) = \sum_{k=0}^{\infty} \mu_{n,k} a_k, \quad \text{for all } n,$$

bearing in mind (3.2.2), this gives that $\mathcal{H}_\mu(f) = I_\mu(f)$. $\square$

Proof of Theorem 21.

Since μ is an $1/p$-Carleson measure, there exists $C > 0$ such that (3.2.1) holds. This implies that

$$\int_0^{2\pi} \int_{[0,1)} \left| \frac{f(t)\, g(e^{i\theta})}{1 - re^{i\theta} t} \right| d\mu(t)\, d\theta < \infty, \quad 0 \le r < 1, \ f \in H^p, \ g \in H^1. \tag{3.2.4}$$

Using Theorem 20, (3.2.4) and Fubini's theorem, and Cauchy's integral representation of H^1-functions [27, Theorem 3. 6], we obtain

$$\int_0^{2\pi} \mathcal{H}_\mu(f)(re^{i\theta})\, \overline{g(e^{i\theta})}\, d\theta = \int_0^{2\pi} \left(\int_{[0,1)} \frac{f(t)\, d\mu(t)}{1 - re^{i\theta} t} \right) \overline{g(e^{i\theta})}\, d\theta$$

$$= \int_{[0,1)} f(t) \int_0^{2\pi} \frac{\overline{g(e^{i\theta})}}{1 - re^{i\theta} t}\, d\theta\, d\mu(t) = \int_{[0,1)} f(t) \overline{g(rt)} d\mu(t), \ 0 \le r < 1, \ f \in H^p, \ g \in H^1. \tag{3.2.5}$$

(i) Take $q \in (0,1)$. Bearing in mind (3.2.5) and (3.2.1) we deduce that

$$\left| \int_0^{2\pi} \mathcal{H}_\mu(f)(re^{i\theta})\, \overline{g(e^{i\theta})}\, d\theta \right| \le C \|f\|_{H^p} \|g\|_{H^\infty}, \ 0 \le r < 1, \ f \in H^p, \ g \in H^\infty. \tag{3.2.6}$$

Now we recall [26, Theorem 10] that B_q can be identified with the dual of a certain subspace X of H^∞ under the pairing

$$< f, g >= \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, \overline{g(e^{i\theta})}\, d\theta, \quad f \in B_q, \quad f \in X.$$

This together with (3.2.6) gives that $\mathcal{H}_\mu$ is a bounded operator from H^p into B_q.

(ii) We shall use Fefferman's duality theorem [30, 31], which says that $(H^1)^\star \cong BMOA$ and $(VMOA)^\star \cong H^1$, under the Cauchy pairing

$$\langle f, g\rangle = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta, \quad f\in H^1, \quad g\in BMOA \text{ (resp. } VMOA), \tag{3.2.7}$$

We mention [12, 39, 41], as general references for the spaces $BMOA$ and $VMOA$. In particular, Fefferman's duality theorem can be found in [41, Section 7].

Using the duality theorem and (3.2.4) it follows that $\mathcal{H}_\mu$ is a bounded operator from H^p into H^1 if and only if there exists a positive constant C such that

$$\left|\int_{[0,1)} f(t)\overline{g(rt)}\,d\mu(t)\right| \le C\|f\|_{H^p}\|g\|_{BMOA}, \quad 0<r<1, \;\; f\in H^p, \;\; g\in VMOA. \tag{3.2.8}$$

Suppose that $\mathcal{H}_\mu$ is a bounded operator from H^p to H^1. For $0<a,b<1$, let the functions g_a and f_b be defined by

$$g_a(z) = \log\frac{2}{1-az}, \qquad f_b(z) = \left(\frac{1-b^2}{(1-bz)^2}\right)^{1/p}, \quad z\in\mathbb{D}. \tag{3.2.9}$$

A calculation shows that $\{g_a\}\subset VMOA$, $\{f_b\}\subset H^p$, and

$$\sup_{a\in[0,1)} ||g_a||_{BMOA} < \infty \quad \text{and} \quad \sup_{b\in[0,1)} ||f_b||_{H^p} < \infty. \tag{3.2.10}$$

Next, taking $a=b\in[0,1)$ and $r\in[a,1)$, we obtain

$$\begin{aligned}\left|\int_0^1 f_a(t)\overline{g_a(rt)}\,d\mu(t)\right| &\ge \int_a^1 \left(\frac{1-a^2}{(1-at)^2}\right)^{1/p}\log\frac{2}{1-rat}\,d\mu(t),\\ &\ge C\frac{\log\frac{2}{1-a^2}}{(1-a^2)^{1/p}}\mu\left([a,1)\right),\end{aligned}$$

which, bearing in mind (3.2.8) and (3.2.10), implies that μ is an 1-logarithmic $\frac{1}{p}$-Carleson measure.

Reciprocally, suppose that μ is an 1-logarithmic $\frac{1}{p}$-Carleson measure. Let us see that $\mathcal{H}_\mu$ is a bounded operator from H^p to H^1. Using (3.2.8), it is enough to prove that there exists $C>0$ such that

$$\int_0^1 |f(t)||g(rt)|\,d\mu(t) \le C||f||_{H^p}||g||_{BMOA}, \quad \text{for all } r\in(0,1),\ f\in H^p, \text{ and } g\in VMOA.$$

By [27, Theorem 9.4], this is equivalent to saying that, for every $r \in (0,1)$ and every $g \in VMOA$, the measure $|g(rz)|\,d\mu(z)$ is an $1/p$-Carleson measure with constant bounded by $C\|g\|_{BMOA}$. Using Lemma B this can be written as

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{1/p} |g(rz)|\,d\mu(z) \leq C||g||_{BMOA}, \quad 0<r<1, \;\; g \in VMOA. \tag{3.2.11}$$

So take $r \in (0,1)$, $a \in \mathbb{D}$ and $g \in VMOA$. We have

$$\begin{aligned}
&\int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{1/p} |g(rz)|\,d\mu(z)\\
&\leq |g(ra)| \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{1/p} d\mu(z) + \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{1/p} |g(rz)-g(ra)|\,d\mu(z)\\
&= I_1(r,a) + I_2(r,a).
\end{aligned}$$

Bearing in mind that any function g in the Bloch space $\mathcal{B}$ (see [9]) satisfies the growth condition

$$|g(z)| \leq 2\|g\|_{\mathcal{B}} \log\frac{2}{1-|z|}, \quad \text{for all } z \in \mathbb{D}, \tag{3.2.12}$$

and $BMOA \subset \mathcal{B}$ [41, Theorem 5. 1]), by Lemma B we have that

$$\begin{aligned}
I_1(r,a) &\leq C||g||_{BMOA} \log\frac{2}{1-|a|} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{1/p} d\mu(z)\\
&\leq CK_{1,\frac{1}{p}}(\mu)||g||_{BMOA} < \infty.
\end{aligned} \tag{3.2.13}$$

Now, since μ is an $1/p$-Carleson measure, Lemma B yields

$$\begin{aligned}
I_2(r,a) &\leq CK_{\frac{1}{p}}(\mu) \left\| \left(\frac{1-|a|^2}{(1-\bar{a}\,z)^2}\right)^{1/p} [g(rz)-g(ra)] \right\|_{H^p}\\
&= CK_{\frac{1}{p}}(\mu) \left(\int_0^{2\pi} \frac{1-|a|^2}{|1-\bar{a}e^{i\theta}|^2} |g_r(e^{i\theta}) - g_r(a)|^p\,d\theta \right)^{1/p}\\
&\leq CK_{\frac{1}{p}}(\mu) \int_0^{2\pi} \frac{1-|a|^2}{|1-\bar{a}e^{i\theta}|^2} |g_r(e^{i\theta}) - g_r(a)|\,d\theta,
\end{aligned}$$

where, $g_r(z) = g(rz)$ $(z \in \mathbb{D})$. Now, using the conformal invariance of $BMOA$ ([41, Theorem 3. 1])) and the fact that $BMOA$ is closed under subordination [41, Theorem 10. 3], we obtain that

$$\int_0^{2\pi} \frac{1-|a|^2}{|1-\bar{a}e^{i\theta}|^2}|g_r(e^{i\theta}) - g_r(a)|\,d\theta \leq C\|g\|_{BMOA}$$

and then it follows that $I_2(r,a) \leq CK_{\frac{1}{p}}(\mu)\|g\|_{BMOA}$. This and (3.2.13) give (3.2.11), finishing the proof of part (ii).

(iii) Using (3.2.5), the duality theorem for H^q [27, Section 7. 2] and arguing as in the proof of part (ii), we can assert that $\mathcal{H}_\mu$ is a bounded operator from H^p to H^q if and only if there exists a positive constant C such that

$$\left|\int_{[0,1)} f(t)\,\overline{g(t)}\,d\mu(t)\right| \leq C\,\|f\|_{H^p}\,\|g\|_{H^{q'}}, \quad f \in H^p, \quad g \in H^{q'}. \tag{3.2.14}$$

Now, by Proposition 7, it follows that (3.2.14) is equivalent to

$$\int_{[0,1)} |f(t)|\,|g(t)|\,d\mu(t) \leq C\,\|f\|_{H^p}\,\|g\|_{H^{q'}}, \quad f \in H^p, \quad g \in H^{q'}, \tag{3.2.15}$$

and, by Lemma B, this is the same as saying the following:

For every $g \in H^{q'}$, the measure μ_g supported on $[0,1)$ and defined by $d\mu_g(z) = |g(z)|\,d\mu(z)$ is a $1/p$-Carleson measure with $K_{\frac{1}{p}}(\mu_g) \leq C\,\|g\|_{H^{q'}}$, that is,

$$\sup_{a\in\mathbb{D}} \int_{[0,1)} \left(\frac{1-|a|^2}{|1-\overline{a}t|^2}\right)^{1/p} |g(t)|\,d\mu(t) \leq C\|g\|_{H^{q'}}, \quad g \in H^{q'}. \tag{3.2.16}$$

Suppose that $\mathcal{H}_\mu$ is a bounded operator from H^p to H^q. Then (3.2.16) holds. For $a \in \mathbb{D}$, take

$$g_a(z) = \left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)^{1/q'}, \quad z \in \mathbb{D}.$$

Since $\sup_{a\in\mathbb{D}} \|g_a\|_{H^{q'}} < \infty$, (3.2.16) implies that

$$\sup_{a\in\mathbb{D}} \int_{[0,1)} \left(\frac{1-|a|^2}{|1-\overline{a}t|^2}\right)^{\frac{1}{p}+\frac{1}{q'}} d\mu(t) < \infty,$$

that is, μ is a $\frac{1}{p}+\frac{1}{q'}$-Carleson measure, by Lemma B.

Suppose now that μ is an $\frac{1}{p}+\frac{1}{q'}$-Carleson measure. Set $s = 1+\frac{p}{q'}$. The conjugate exponent of s is $s' = 1+\frac{q'}{p}$ and $\frac{1}{p}+\frac{1}{q'} = \frac{s}{p} = \frac{s'}{q'}$. Then, by [27, Theorem 9.4], H^p

is continuously embedded in $L^s(d\mu)$ and $H^{q'}$ is continuously embedded in $L^{s'}(d\mu)$, that is,

$$\left(\int_{[0,1)} |f(t)|^s \, d\mu(s)\right)^{1/s} \leq C \|f\|_{H^p}, \quad f \in H^p, \tag{3.2.17}$$

and

$$\left(\int_{[0,1)} |g(t)|^{s'} \, d\mu(s)\right)^{1/s'} \leq C \|g\|_{H^{q'}}, \quad g \in H^{q'}. \tag{3.2.18}$$

Using Hölder's inequality with exponents s and s', (3.2.17) and (3.2.18), we obtain

$$\begin{aligned}\int_{[0,1)} |f(t)|\,|g(t)| \, d\mu(t) \leq & \left(\int_{[0,1)} |f(t)|^s \, d\mu(s)\right)^{1/s} \left(\int_{[0,1)} |g(t)|^{s'} \, d\mu(s)\right)^{1/s'} \\ \leq & C \|f\|_{H^p} \|g\|_{H^{q'}}, \quad f \in H^p, \ g \in H^{q'}.\end{aligned}$$

Hence, (3.2.15) holds and then it follows that $\mathcal{H}_\mu$ is a bounded operator from H^p to H^q. □

3.3 Proofs of the main results. Case $p > 1$.

Proof of Theorem 22 (i). Since μ is an 1-Carleson measure for H^p, (3.2.1) holds for a certain $C > 0$. Then the argument used in the proof of the implication (i) ⇒ (ii) in Theorem 20 gives that, for every $f \in H^p$, $I_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ and

$$I_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t) \, d\mu(t)\right) z^n, \quad z \in \mathbb{D}. \tag{3.3.1}$$

The reverse implication can be proved just as (ii) ⇒ (i) in Theorem 20.

The fact that μ being an 1-Carleson measure for H^p is equivalent to (3.0.4) follows from Theorem E.

(ii). Take $f \in H^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$. Set

$$S_n(f)(z) = \sum_{k=0}^{n} a_k z^k, \quad R_n(f)(z) = \sum_{k=n+1}^{\infty} a_k z^k, \quad z \in \mathbb{D}, \quad n = 0, 1, 2, \ldots.$$

Whenever $0 \leq N < M$ and $n \geq 0$, we have

$$\begin{aligned}\left|\sum_{k=N+1}^{M} \mu_{n,k} a_k\right| &= \left|\int_{[0,1)} t^n \left(\sum_{k=N+1}^{M} a_k\, t^k\right) d\mu(t)\right| \\ &= \left|\int_{[0,1)} t^n \left[S_M(f)(t) - S_N(f)(t)\right] d\mu(t)\right| \\ &\leq \int_{[0,1)} |S_M(f)(t) - S_N(f)(t)|\, d\mu(t).\end{aligned}$$

Using this, the fact that μ is an 1-Carleson measure for H^p, and the Riesz projection theorem, we deduce that

$$\sum_{k=N+1}^{M} \mu_{n,k} a_k \to 0, \quad \text{as } N, M \to \infty$$

for all n. This gives that the series $\sum_{k=0}^{\infty} \mu_{n,k} a_k$ converges for all n.

For $n, N \geq 0$, we have

$$\begin{aligned}\left|\int_{[0,1)} t^n f(t)\, d\mu(t) - \sum_{k=0}^{N} \mu_{n,k}\, a_k\right| &= \left|\int_{[0,1)} t^n f(t)\, d\mu(t) - \int_{[0,1)} t^n \left(\sum_{k=0}^{N} a_k\, t^t\right) d\mu(t)\right| \\ &= \left|\int_{[0,1)} t^n R_N(f)(t)\, d\mu(t)\right| \\ &\leq C\|R_N(f)\|_{H^p}.\end{aligned}$$

Since $1 < p < \infty$, $\|R_N(f)\|_{H^p} \to 0$, as $N \to \infty$, and then it follows that

$$\sum_{k=0}^{\infty} \mu_{n,k} a_k = \int_{[0,1)} t^n f(t)\, d\mu(t), \quad \text{for all } n,$$

which together with (3.2.1) implies that $\mathcal{H}_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ and, by (3.3.1), $\mathcal{H}_\mu(f) = I_\mu(f)$. □

Let us turn to prove Theorem 23. In view of Theorem 22, $\mathcal{H}_\mu$ coincides with I_μ on H^p. This fact will be used repeatedly in the following.

Recall that (3.0.4) implies that μ is an 1-Carleson measure for H^p, that is, we have

$$\int_{[0,1)} |f(t)|\, d\mu(t) \leq C\|f\|_{H^p}, \quad f \in H^p.$$

Then arguing as in the proof of Theorem 21, we obtain

$$\int_0^{2\pi} \mathcal{H}_\mu(f)(re^{i\theta})\,\overline{g(e^{i\theta})}\,d\theta = \int_0^{2\pi}\left(\int_{[0,1)} \frac{f(t)\,d\mu(t)}{1-re^{i\theta}t}\right)\overline{g(e^{i\theta})}\,d\theta$$
$$= \int_{[0,1)} f(t)\int_0^{2\pi} \frac{\overline{g(e^{i\theta})}}{1-re^{i\theta}t}\,d\theta\,d\mu(t) = \int_{[0,1)} f(t)\overline{g(rt)}d\mu(t),\ 0\le r<1,\ f\in H^p,\ g\in H^1. \tag{3.3.2}$$

Once (3.3.2) is established, (i) can be proved with the argument used in the proof of part (iii) of Theorem 21.

Part (ii) of Theorem 23 is a byproduct of the following result.

Proposition 8. *Assume that $1<q<p<\infty$ and let μ be a positive Borel measure on $[0,1)$ satisfying (3.0.4). Then, the following conditions are equivalent:*

(a) $\mathcal{H}_\mu$ is a bounded operator from H^p to H^q.

(b) $\mathcal{H}_\mu$ is a bounded operator from $H^{\frac{2pq'}{p+q'}}$ to $H^{\left(\frac{2pq'}{p+q'}\right)'}$.

(c) $H^{\frac{2pq'}{p+q'}}$ is continuously contained in $L^2(\mu)$.

(d) The function defined by $s\mapsto \int_0^{1-s}\frac{d\mu(t)}{1-t}$ $(s\in[0,1))$ belongs to $L^{\left(\frac{pq'}{p+q'}\right)'}([0,1))$.

Proof. (a) $\Rightarrow$ (b). Using duality as above, we see that (a) is equivalent to

$$\left|\int_{[0,1)} f(t)\,\overline{g(t)}\,d\mu(t)\right| \lesssim \|f\|_{H^p}\,\|g\|_{H^{q'}},\quad f\in H^p,\quad g\in H^{q'}. \tag{3.3.3}$$

Take $f\in H^p$ and $g\in H^{q'}$, and let $F\in H^p$ and $G\in H^{q'}$ be the functions associated to f and g by Proposition 7, respectively. Using (3.3.3) and Hölder's inequality, we obtain

$$\begin{aligned}\int_0^1 |f(t)||g(t)|\,d\mu(t) \lesssim \int_0^1 F(t)\,G(t)\,d\mu(t) &= \left|\int_0^1 F(t)\,\overline{G(t)}\,d\mu(t)\right| \\ &\lesssim ||F||_{H^p}||G||_{H^{q'}} \lesssim ||f||_{H^p}||g||_{H^{q'}}.\end{aligned} \tag{3.3.4}$$

Take now $\phi\in H^{\frac{2pq'}{p+q'}}$. By the outer-inner factorization [27, Chapter 2], $\phi=\Phi\cdot I$ where I is an inner function and $\Phi\in H^{\frac{2pq'}{p+q'}}$ is free from zeros and $||\Phi||_{H^{\frac{2pq'}{p+q'}}} = ||\phi||_{H^{\frac{2pq'}{p+q'}}}$.

Now let us consider the analytic functions $f = \Phi^{\frac{2q'}{p+q'}}$ and $g = \Phi^{\frac{2p}{p+q'}}$. We have

$$f = \Phi^{\frac{2q'}{p+q'}} \in H^p, \quad \text{with } ||f||_{H^p} = ||\Phi||^{\frac{2q'}{p+q'}}_{H^{\frac{2pq'}{p+q'}}}$$

and

$$g = \Phi^{\frac{2p}{p+q'}} \in H^{q'}, \quad \text{with } ||g||_{H^{q'}} = ||\Phi||^{\frac{2p}{p+q'}}_{H^{\frac{2pq'}{p+q'}}}.$$

Bearing in mind (3.3.4), it follows that

$$\begin{aligned}\int_0^1 |\phi(t)|^2\, d\mu(t) &\le \int_0^1 |\Phi(t)|^2\, d\mu(t)\\ &= \int_0^1 |f(t)||g(t)|\, d\mu(t)\\ &\lesssim ||f||_{H^p}||g||_{H^{q'}} = ||\Phi||^2_{H^{\frac{2pq'}{p+q'}}} = ||\phi||^2_{H^{\frac{2pq'}{p+q'}}},\end{aligned}$$

which gives (b).

(b) $\Rightarrow$ (c). Since $p > q > 1$, $\frac{pq'}{p+q'} > 1$, by duality, as above, (b) is equivalent to

$$\left|\int_0^1 f(t)\overline{g(t)}\, d\mu(t)\right| \le C\,||f||_{H^{\frac{2pq'}{p+q'}}}||g||_{H^{\frac{2pq'}{p+q'}}}, \quad f, g \in H^{\frac{2pq'}{p+q'}}.$$

Taking $f = g$ we obtain

$$\int_0^1 |f(t)|^2\, d\mu(t) \le C\,||f||^2_{H^{\frac{2pq'}{p+q'}}}.$$

This is (c).

Theorem E gives that (c) and (d) are equivalent.

(d) $\Rightarrow$ (a). Using again Theorem E we have that H^p is continuously contained in $L^{\frac{p+q'}{q'}}(d\mu)$ and $H^{q'}$ is continuously contained in $L^{\frac{p+q'}{p}}(d\mu)$, which together with Hölder's inequality gives

$$\begin{aligned}\int_0^1 |f(t)||g(t)|\, d\mu(t) &\le \left(\int_0^1 |f(t)|^{\frac{p+q'}{q'}}\, d\mu(t)\right)^{\frac{q'}{p+q'}} \left(\int_0^1 |g(t)|^{\frac{p+q'}{p}}\, d\mu(t)\right)^{\frac{p}{p+q'}}\\ &\le C||f||_{H^p}||g||_{H^{q'}}, \quad f \in H^p, \quad g \in H^q,\end{aligned}$$

and this is equivalent to (a). □

Proof of Theorem 23 (iii). Just as in the proof of Theorem 21 (ii), $\mathcal{H}_\mu$ is a bounded operator from H^p into H^1 if and only there exists a positive constant C such that

$$\left|\int_{[0,1)} f(t)\overline{g(rt)}\,d\mu(t)\right| \le C\|f\|_{H^p}\|g\|_{BMOA}, \quad 0<r<1,\; f\in H^p,\; g\in VMOA. \tag{3.3.5}$$

Let ν be the measure on $[0,1)$ defined by $d\nu(t) = \log\frac{1}{1-t}\,d\mu(t)$, by Theorem E it follows that the function $s \mapsto \int_0^{1-s} \frac{\log\frac{1}{1-t}d\mu(t)}{1-t}$ $(s\in[0,1))$ belongs to $L^{p'}([0,1))$ if and only if the measure ν is an 1-Carleson measure for H^p.

Consequently, we have to prove that

$$(3.3.5) \Leftrightarrow \nu \text{ is an 1-Carleson measure for } H^p.$$

Suppose that (3.3.5) holds. For $0<\rho<1$, let g_ρ be the function defined by $g_\rho(z) = \log\frac{1}{1-\rho z}$ $(z\in\mathbb{D})$, then

$$g_\rho \in VMOA, \text{ for all } \rho\in(0,1), \quad \text{and} \quad \sup_{0<\rho<1}\|g_\rho\|_{BMOA} = A < \infty\,.$$

On the other hand, if $f\in H^p$, $0<r<1$, and F is the function associated to f by Proposition 7, it follows that

$$\int_{[0,1)} |f(t)|\,\log\frac{1}{1-\rho rt}\,d\mu(t) \le \int_{[0,1)} F(t)\,\overline{g_\rho(rt)}\,d\mu(t) \le C\,A\,\|F\|_{H^p} = C\,A\,\|f\|_{H^p},$$

for every $\rho\in(0,1)$. Letting r and ρ tend to 1, we obtain

$$\int_{[0,1)} |f(t)|\,\log\frac{1}{1-t}\,d\mu(t) \le C\,A\,\|f\|_{H^p}.$$

Thus ν is an 1-Carleson measure for H^p.

Conversely, assume that ν is an 1-Carleson measure for H^p. Take $r\in(0,1)$, $f\in H^p$, and $g\in VMOA$. Using (3.2.12), we obtain

$$\left|\int_{[0,1)} f(t)\overline{g(rt)}\,d\mu(t)\right| \le C\,\|g\|_{BMOA}\int_{[0,1)} |f(t)|\log\frac{2}{1-t}\,d\mu(t) \le C\,\|g\|_{BMOA}\,\|f\|_{H^p}.$$

□

Proof of Theorem 23 (iv). Assume that the function defined by $s\mapsto \int_0^{1-s}\frac{d\mu(t)}{1-t}$ $(s\in[0,1))$ belongs to $L^{p'}([0,1)$. By Theorem E, this implies that H^p is continuously contained in $L^1(d\mu)$. From now on, the proof is analogous to the proof of Theorem 21 (i). □

3.4 Compactness.

The next theorem gathers our main results concerning the study of the compactness of $\mathcal{H}_\mu$ on Hardy spaces.

Theorem 25. *Let μ be a positive Borel measure on $[0,1)$.*

(i) If $0 < p \le 1$ and μ is a $1/p$-Carleson measure, then $\mathcal{H}_\mu$ is a compact operator from H^p to H^1 if and only if μ is a vanishing 1-logarithmic $1/p$-Carleson measure.

(ii) If $0 < p \le 1 < q$ and μ is a $1/p$-Carleson measure, then $\mathcal{H}_\mu$ is a compact operator from H^p to H^q if and only if μ is a vanishing $\frac{1}{p}+\frac{1}{q'}$-Carleson measure.

(iii) If $1 < p < q$ and μ satisfies (3.0.4), then $\mathcal{H}_\mu$ is a compact operator from H^p to H^q if and only if μ is a vanishing $\frac{1}{p}+\frac{1}{q'}$-Carleson measure.

(iv) If $1 < p < \infty$, μ satisfies (3.0.4) and $1 \le q < p$, then $\mathcal{H}_\mu$ is a compact operator from H^p to H^q if and only if it is a bounded operator from H^p to H^q.

The following lemma will be used in the proof of cases (i), (ii) and (iii).

Lemma 6. *Suppose that $0 < p < \infty$ and let μ be a positive Borel measure on $[0,1)$ which is an 1-logarithmic $1/p$-Carleson measure. Let f_b, $(0 \le b < 1)$, be defined as in (3.2.9). Then*

$$\lim_{b\to 1^-} \int_{[0,1)} f_b(t)\, d\mu(t) \;=\; 0. \tag{3.4.1}$$

Proof. For $0 \le t < 1$, set $F(t) = \mu([0,t)) - \mu([0,1)) = -\mu([t,1))$. Integrating by parts and using the fact that μ is an 1-logarithmic $1/p$-Carleson measure, we obtain

$$\int_{[0,1)} f_b(t)\, d\mu(t) \;=\; f_b(0)\,\mu([0,1)) + \int_0^1 f_b'(t)\,\mu([t,1))\; dt. \tag{3.4.2}$$

Using that μ is an 1-logarithmic $1/p$-Carleson measure and the fact that $bt < b$ and

$bt < t$ $(0 < b, t < 1)$, we deduce

$$\begin{aligned}
&\int_0^1 f_b'(t)\,\mu([t,1))\,dt \le C\int_0^1 \frac{(1-b)^{1/p}(1-t)^{1/p}}{(1-bt)^{\frac{2}{p}+1}\log\frac{e}{1-t}}\,dt\\
=&C\int_0^b \frac{(1-b)^{1/p}(1-t)^{1/p}}{(1-bt)^{\frac{2}{p}+1}\log\frac{e}{1-t}}\,dt + C\int_b^1 \frac{(1-b)^{1/p}(1-t)^{1/p}}{(1-bt)^{\frac{2}{p}+1}\log\frac{e}{1-t}}\,dt\\
\le&C\,(1-b)^{1/p}\int_0^b \frac{dt}{(1-t)^{\frac{1}{p}+1}\log\frac{e}{1-t}} + \frac{C}{(1-b)^{\frac{1}{p}+1}}\int_b^1 \frac{(1-t)^{1/p}}{\log\frac{e}{1-t}}\,dt\\
=&I(b) + II(b).
\end{aligned}$$

Now, it is a simple calculus exercise to show that $I(b)$ and $II(b)$ tend to 0, as $b \to 1$. Using this, the fact that $\lim_{b\to 1} f_b(0) \to 0$, and (3.4.2), we deduce (3.4.1). □

Proof of Theorem 25 (i). Suppose that $\mathcal{H}_\mu$ is a compact operator from H^p to H^1. Let f_b, $(0 \le b < 1)$, be defined as in (3.2.9). Let $\{b_n\} \subset (0,1)$ be any sequence with $b_n \to 1$ and such that the sequence $\{\mathcal{H}_\mu(f_{b_n})\}$ converges in H^1 (such a sequence exists because $\sup_{0<b<1}\|f_b\|_{H^p} < \infty$ and $\mathcal{H}_\mu$ is compact) and let g be the limit (in H^1) of $\{\mathcal{H}_\mu(f_{b_n})\}$. Then $\mathcal{H}_\mu(f_{b_n}) \to g$, uniformly on compact subsets of $\mathbb{D}$. Now, by Theorem 20, we have

$$0 \le H_\mu(f_{b_n})(r) = \int_{[0,1)} \frac{f_{b_n}(t)}{1-rt}\,d\mu(t) \le \frac{1}{1-r}\int_{[0,1)} f_{b_n}(t)\,d\mu(t), \quad 0 < r < 1.$$

Since $\mathcal{H}_\mu$ is continuous from H^p to H^1, μ is an 1-logarithmic $1/p$-Carleson measure. Then, by Lemma 6, it follows that $g(r) = 0$ for all $r \in (0,1)$. Hence, $g \equiv 0$. In this way we have proved that

$$\mathcal{H}_\mu(f_b) \to 0, \quad \text{as } b \to 1, \text{ in } H^1.$$

Arguing as in proof of the boundedness (Theorem 21 (ii)), this yields

$$\lim_{b\to 1^-} \frac{\mu([b,1))\log\frac{e}{1-b}}{(1-b)^{1/p}} = 0,$$

which is equivalent to saying that μ is a vanishing 1-logarithmic $1/p$-Carleson measure.

Suppose now that μ is a vanishing 1-logarithmic $1/p$-Carleson measure. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in H^p with $\sup\|f_n\|_{H^p} < \infty$ and such that $f_n \to 0$, uniformly on compact subsets of $\mathbb{D}$. For $0 < r < 1$, let us write

$$d\mu_r(t) = \chi_{r<|z|<1}(t)\,d\mu(t).$$

Since μ is a vanishing 1-logarithmic $1/p$-Carleson measure, $\lim_{r\to 1} K_{1,\frac{1}{p}}(\mu_r) = 0$. This together with the fact that $f_n \to 0$, uniformly on compact subsets of $\mathbb{D}$ gives that

$$\lim_{n\to\infty} \int_0^1 |f_n(t)|\,|g(t)|\,d\mu(t) = 0 \quad \text{for all } g \in VMOA.$$

Using the duality relation $(VMOA)^\star \cong H^1$ as in the proof of Theorem 21, this implies that $\mathcal{H}_\mu(f_n) \to 0$, in H^1. So, $\mathcal{H}_\mu : H^p \to H^1$ is compact.

Parts (ii) and (iii) can be proved similarly to the preceding one. We shall omit the details. Let us simply remark, for the necessity, that if μ is a vanishing $\frac{1}{p} + \frac{1}{q'}$-Carleson measure, then it is an 1-logarithmic $1/p$-Carleson measure and then we can use Lemma 6. □

Proof of Theorem 25 (iv). Suppose that $1 < p < \infty$ and $0 < q < p$. Looking at the proof of parts (ii) and (iii) of Theorem 23, we see $\mathcal{H}_\mu$ applies H^p into H^q if and only if:

- $H^{\frac{2pq'}{p+q'}}$ is continuously embedded in $L^2(d\mu)$, in the case $1 < q < p$.
- H^p is continuously embedded in $L^1(d\nu)$, where $d\nu(t) = \log\frac{1}{1-t}\,d\mu(t)$, in the case $q = 1 < p < \infty$.

Now, using the results in Section 3 of [14], we see that when any of these embeddings exists as a continuous operator, then it is compact. Then arguments similar to those used in the boundedness case can be used to yield the compactness. We omit the details. □

3.5 Schatten classes.

Before presenting the proof of Theorem 24 let us recall some definitions which connect the operator $\mathcal{H}_\mu$ with the classical theory of Hankel operators.

Given $\varphi(\xi) \sim \sum_{n=-\infty}^{+\infty} \widehat{\varphi}(n)\xi^n \in L^2(\mathbb{T})$, the associated (big) Hankel operator $H_\varphi : H^2 \to H^2_-$ (see [64, p. 6]) is formally defined as

$$H_\varphi(f) = I - P(\varphi f)$$

where P is the Riesz projection.

Moreover, if μ is a classical Carleson measure, Nehari's theorem implies that (see [64, p. 3 and p. 42]) there exists $\varphi_\mu \in L^\infty(\mathbb{T})$ with $\mu_{n+1} = \widehat{\varphi_\mu}(-n)$, so

$$\mathcal{H}_\mu(f)(z) - \mathcal{H}_\mu(f)(0) = H_{\varphi_\mu}(f)(\bar{z}).$$

In particular, $\mathcal{H}_\mu$ is bounded on H^2 if and only if $H_{\varphi_\mu} : H^2 \to H^2_-$ is bounded, that is, if and only if μ is a Carleson measure. Finally let us observe that,

$$(I-P)(\varphi_\mu)(\bar{z}) = \sum_{n=1}^{\infty} \mu_{n+1} z^n$$

Proof of Theorem 24. It follows from the above observation [64, p. 240, Corollary 2. 2] and [64, Appendix 2.6] that $\mathcal{H}_\mu \in \mathcal{S}_p(H^2)$ if and only if $h_\mu(z) = \sum_{n=1}^\infty \mu_{n+1} z^n \in B^p$. Bearing in mind [55, Theorem 2.1] (see also [59, p. 120, 7.5.8]), [59, p. 120, 7.3.5], the fact that $\{\mu_n\}$ decreases to zero and [51, Lemma 3.4], we deduce that

$$\begin{aligned} ||h_\mu||^p_{B^p} &\asymp \sum_{n=0}^{\infty} 2^{-n(p-1)} \left\| \sum_{k=2^n}^{2^{n+1}-1} k\mu_{k+1} z^{k-1} \right\|^p_{H^p} \\ &\asymp \sum_{n=0}^{\infty} 2^{n} \left\| \sum_{k=2^n}^{2^{n+1}-1} \mu_{k+1} z^{k-1} \right\|^p_{H^p} \\ &\lesssim \sum_{n=0}^{\infty} 2^{n} \mu_{2^n}^p \left\| \sum_{k=2^n}^{2^{n+1}-1} z^{k-1} \right\|^p_{H^p}. \end{aligned}$$

We claim that

$$\left\| \sum_{k=2^n}^{2^{n+1}-1} z^{k-1} \right\|^p_{H^p} \asymp 2^{n(p-1)}. \tag{3.5.1}$$

Then, using again that $\{\mu_n\}$ is decreasing and the results of [60], we obtain

$$||h_\mu||^p_{B^p} \lesssim \sum_{n=0}^{\infty} 2^{np} \mu_{2^n}^p \lesssim \sum_{n=0}^{\infty} 2^{n(p-1)} \sum_{k=2^n}^{2^{n+1}-1} \mu_k^p \asymp \sum_{k=0}^{\infty} (k+1)^{p-1} \mu_k^p.$$

An analogous reasoning using the left hand inequality in [51, Lemma 3.4] proves that

$$||h_\mu||^p_{B^p} \gtrsim \sum_{n=0}^{\infty} (k+1)^{p-1} \mu_k^p.$$

Finally, we shall prove (3.5.1). By [55, Lemma 3.1] and the M. Riesz projection theorem, it follows that

$$\left\|\sum_{k=2^n}^{2^{n+1}-1} z^{k-1}\right\|_{H^p}^p \lesssim M_p^p\left(1-\frac{1}{2^n}, \sum_{k=2^n}^{2^{n+1}-1} z^{k-1}\right) \lesssim M_p^p\left(1-\frac{1}{2^n}, \frac{1}{1-z}\right) \asymp 2^{n(p-1)}. \tag{3.5.2}$$

On the other hand, using [55, Lemma 3.1], we obtain

$$M_\infty\left(1-\frac{1}{2^n}, \sum_{k=2^n}^{2^{n+1}-1} z^{k-1}\right) \asymp \left\|\sum_{k=2^n}^{2^{n+1}-1} z^{k-1}\right\|_{H^\infty} = 2^n.$$

Furthermore, using a well-know inequality, we deduce that

$$\begin{aligned} M_\infty\left(1-\frac{1}{2^n}, \sum_{k=2^n}^{2^{n+1}-1} z^{k-1}\right) &\lesssim \left(\frac{1}{2^n}-\frac{1}{2^{n+1}}\right)^{-1/p} M_p\left(1-\frac{1}{2^{n+1}}, \sum_{k=2^n}^{2^{n+1}-1} z^{k-1}\right) \\ &\lesssim \left(\frac{1}{2^n}\right)^{-1/p} \left\|\sum_{k=2^n}^{2^{n+1}-1} z^{k-1}\right\|_{H^p}, \end{aligned}$$

that is, $\left\|\sum_{k=2^n}^{2^{n+1}-1} z^{k-1}\right\|_{H^p}^p \gtrsim 2^{n(p-1)}$, which together with (3.5.2) implies (3.5.1). This finishes the proof. □

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