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Alliance polynomial and hyperbolicity in regular graphs

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to my little Walter Jr.

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Escribo aquí en castellano, debido a que es la mejor forma de expresar mi agradecimiento a todos aquellos que de una forma u otra han contribuido a la preparación y el desarrollo de esta tesis.

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Thank you.

Resumen

Uno de los problemas abiertos en la teoría de grafos es la caracterización de cualquier grafo por un polinomio. La investigación en este área ha sido impulsada en gran parte por las ventajas que ofrece el uso de las computadoras que hacen que trabajar con grafos sea más simple. En esta Tesis introducimos el polinomio de alianza de un grafo. El polinomio de alianza de un grafo G con orden n y grado máximo δ_1 es el polinomio $A(G;x)=\sum_{k=-\delta_1}^{\delta_1} A_k(G)\,x^{n+k}$, donde $A_k(G)$ es el número de k alianzas defensivas exactas en G. También desarrollamos e implementamos un algoritmo que calcula de manera eficiente el polinomio de alianza.

En este trabajo obtenemos algunas propiedades de $A(G;x)$ y sus coeficientes para:

- Grafos caminos, ciclos, completos y estrellas. En particular, hemos demostrado que se caracterizan mediante sus polinomios de alianza.
- Grafos cúbicos (grafos con todos sus vértices de grado 3), ya que son una clase muy interesante de grafos con muchas aplicaciones. Hemos demostrado que sus polinomios de alianza verifican unimodalidad. Además, calculamos el polinomio de alianza para grafos cúbicos de orden pequeño, los cuales satisfacen unicidad.
- Grafos regulares (grafos con todos sus vértices de igual grado). En particular, se caracteriza el grado de los grafos regulares por el número de coeficientes distintos de cero de su polinomio de alianza. Además, se demuestra que la familia de polinomios de alianza de grafos conexos Δ-regulares con grado pequeño es muy especial, ya que no contiene polinomios de alianza de grafos conexos que no sean Δ-regulares.

Si X es un espacio métrico geodésico y $x_1,x_2,x_3\in X$, un triángulo geodésico $T=\{x_1,x_2,x_3\}$ es la unión de tres geodésicas $[x_1x_2]$, $[x_2x_3]$ y $[x_3x_1]$ de X. El espacio X es δ-hiperbólico (en el sentido de Gromov) si todo lado de todo triángulo geodésico T de X está contenido en la δ-vecindad de la unión de los otros dos lados. Se denota por $\delta(X)$ la constante de hiperbolicidad óptima de X, es decir, $\delta(X):=\inf\{\delta\geqslant 0:\ X\ \text{es } \delta\text{-hiperbólico}\}$. El estudio de los grafos hiperbólicos es un tema interesante dado que la hiperbolicidad de un espacio métrico geodésico es equivalente a la hiperbolicidad de un grafo más sencillo asociado al espacio.

Hemos obtenido información acerca de la constante de hiperbolicidad de los grafos cúbicos; dichos grafos son muy importantes en el estudio de la hiperbolicidad, ya que para cualquier grafo G con grado máximo acotado existe un grafo cúbico G^* tal que G es hiperbólico si y sólo si G^* es hiperbólico. En esta memoria conseguimos caracterizar los grafos cúbicos con constante de hiperbolicidad pequeña. Además, se obtienen cotas para la constante de hiperbolicidad del grafo complemento de un grafo cúbico; nuestro principal resultado dice que para cualquier grafo cúbico finito G no isomorfo a K_4 o $K_{3,3}$, se cumple la relación $5k/4\leqslant\delta(\overline{G})\leq 3k/2$, donde k es la longitud de todas las aristas en G.

Review

One of the open problems in graph theory is the characterization of any graph by a polynomial. Research in this area has been largely driven by the advantages offered by the use of computers which make working with graphs: it is simpler to represent a graph by a polynomial (a vector) that by the adjacency matrix (a matrix). We introduce the alliance polynomial of a graph. The alliance polynomial of a graph G with order n and maximum degree δ_1 is the polynomial $A(G;x) = \sum_{k=-\delta_1}^{\delta_1} A_k(G)\, x^{n+k}$, where $A_k(G)$ is the number of exact defensive k-alliances in G. Also, we develop and implement an algorithm that computes in an efficient way the alliance polynomial.

We obtain some properties of $A(G;x)$ and its coefficients for:

- Path, cycle, complete and star graphs. In particular, we prove that they are characterized by their alliance polynomials.

- Cubic graphs (graphs with all of their vertices of degree 3), since they are a very interesting class of graphs with many applications. We prove that they verify unimodality. Also, we compute the alliance polynomial for cubic graphs of small order, which satisfy uniqueness.

- Regular graphs (graphs with the same degree for all vertices). In particular, we characterize the degree of regular graphs by the number of non-zero coefficients of their alliance polynomial. Besides, we prove that the family of alliance polynomials of connected Δ-regular graphs with small degree is a very special one, since it does not contain alliance polynomials of graphs which are not connected Δ-regular.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-hyperbolic (in the Gromov sense) if any side of T is contained in the δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, *i.e.*, $\delta(X) := \inf\{\delta \geqslant 0 : \ X \text{ is } \delta\text{-hyperbolic}\}$. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it.

We obtain information about the hyperbolicity constant of cubic graphs. These graphs are also very important in the study of Gromov hyperbolicity, since for any graph G with bounded maximum degree there exists a cubic graph G^* such that G is hyperbolic if and only if G^* is hyperbolic. We find some characterizations for the cubic graphs which have small hyperbolicity constants. Besides, we obtain bounds for the hyperbolicity constant of the complement graph of a cubic graph; our main result of this kind says that for any finite cubic graph G which is not isomorphic either to K_4 or to $K_{3,3}$, the inequalities $5k/4 \leqslant \delta(\overline{G}) \leq 3k/2$ hold, if k is the length of every edge in G.

Contents

List of Figures

Introduction

Graph theory is a very old subject that has many modern applications. In this PhD Thesis we study two topics on graph theory: polynomials of graphs and hyperbolic graphs.

One of the open problems in graph theory is the characterization of any graph by a polynomial. In recent years there have been many works on graph polynomials. Research in this area has been largely driven by the advantages offered by the use of computers which make working with graphs: it is simpler to represent a graph by a polynomial (a vector) that by the adjacency matrix (a matrix).

Some parameters of a graph allow to define polynomials on the graph, for instance, the parameters associated to matching sets [47, 55], independent sets [28, 65], domination sets [3, 5, 4], chromatic numbers [104, 113] and many others. These polynomials are interesting since they compress information about the structure of the graph. Unfortunately, these polynomials do not solve the problem, since there are non-isomorphic graphs with the same polynomial.

In this work we choose the exact index of alliance in order to define the alliance polynomial of graph. We prove that this polynomial characterizes many classes of a graphs. In the light of these results, we conjecture that any graph can be characterized by our alliance polynomial, i.e., that non-isomorphic graphs have different polynomials.

In Chapter 3 we develop and implement an algorithm that computes in an efficient way the alliance polynomial. We also obtain several properties of alliance polynomials. In particular, we compute the alliance polynomial for some graphs and we study its coefficients; we show also that some of them are unimodal. We investigate the alliance polynomials of path, cycle, complete and complete bipartite graphs. Also we prove that the path, cycle, complete and star graphs are characterized by their alliance polynomials.

The main aim of Chapter 4 is to obtain further results about the alliance polynomial of cubic graphs (graphs with all of their vertices of degree 3), since they are a very interesting class of graphs with many applications (see, e.g., [25, 29, 42, 88]). In particular, we prove that the family of alliance polynomials of cubic graphs is a very special one, since it does not contain alliance polynomials of graphs which are not cubic. Furthermore, we obtain (computationally) the alliance polynomials of cubic graphs with small order and we prove that they satisfy uniqueness.

In Chapter 5 we obtain additional results on the alliance polynomial of regular graphs (graphs with all vertices with the same degree), since they are also a very interesting class of graphs. We prove that the family of alliance polynomials of connected Δ-regular graphs with small degree is a very special one, since it does not contain alliance polynomials of graphs which are not connected Δ-regular.

Finally, Chapter 7 deals with hyperbolic cubic graphs.

Gromov hyperbolicity was introduced by Gromov in the setting of geometric group theory [40], [54], [57], [58], but has played an increasing role in analysis on general metric spaces [12], [22], [23], with applications to the Martin boundary, invariant metrics in several complex variables [12] and extendability of Lipschitz mappings [79]. The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. Another important application of these spaces is secure transmission of information on the internet (see [68], [69],[70]). The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory.

Cubic graphs are very important in the study of Gromov hyperbolicity, since for any graph G with bounded maximum degree there exists a cubic graph G^* such that G is hyperbolic if and only if G^* is hyperbolic (see [18, Section 4] and [88, Theorem 2.2]). We find some characterizations for the cubic graphs which have small hyperbolicity constants, *i.e.*, the graphs which are like trees (in the Gromov sense). Besides, we obtain bounds for the hyperbolicity constant of the complement graph of a cubic graph; our main result of this kind says that for any finite cubic graph G which is not isomorphic either to K_4 or to $K_{3,3}$, the inequalities $5k/4 \le \delta(\overline{G}) \le 3k/2$ hold, if k is the length of every edge in G. This is a very precise result, since it implies that $\delta(\overline{G})$ is either $5k/4$ or $3k/2$, by [17, Theorem 2.6].

The results in this work appear in [34, 35, 99, 100]; these papers have been published or submitted to journals which appear in the Journal Citation Reports.

These results were presented in the following international and national conferences:

- Workshop of Young Researchers in Mathematics 2013, September 2013, Universidad Complutense de Madrid, Spain.

- VIII Encuentro Andaluz de Matemática Discreta, Octubre 2013, Universidad de Sevilla, Spain.

Our results will be presented also in the conference:

- IX Jornadas de Matemática Discreta y Algorítmica en Tarragona, Julio 2014, Universitat Rovira I Virgili.

Chapter 1

A brief introduction to graph theory

Graph theory is a very old topic, but it is used in many modern applications. Its basic ideas were introduced in the eighteenth century by the Swiss mathematician Leonhard Euler (1707-1783).

In 1736 Leonhard Euler published the article *Solutio problematis ad geometriam situs pertinentis* which gives a solution to a problem concerning the geometry of position, known by the name *The Seven Bridges of Königsberg Problem.* This work is considered the first article on what is now known as graph theory.

The city of Königsberg in Prussia (now Kaliningrad, Russia) was divided by the river Pregel in four zones. The problem was to find a walk through the city that would cross each bridge once and only once. Euler found that such a path was impossible, for its existence was necessary that at most two of the four land zones were joined by an odd number of bridges. Euler established, too, that this was not sufficient condition for a solution of the problem, however did not demonstrate these claims. It was not until 1873 that a demonstration was published. Its author, Hierholzer, unaware apparently the work of Euler.

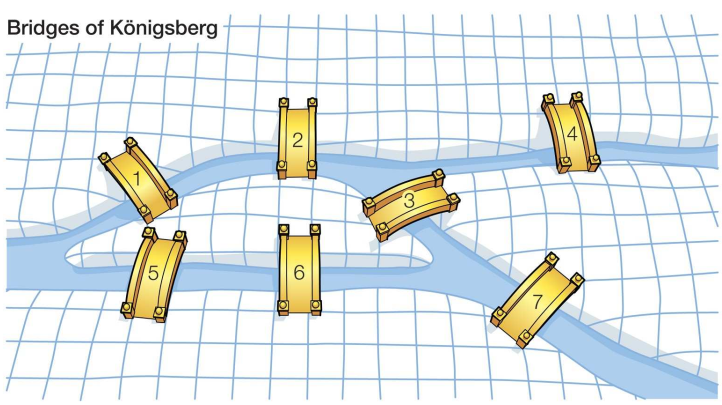


Figure 1.1: The Seven Bridges of Königsberg.

In recent years, discrete mathematics has undergone considerable development in the area of graph theory, framed in combinatory, but meanwhile has evolved enough to be considered an art in itself. This theory allows simple modeling any system in which there is a binary relation between objects, and this is why its scope is very broad and covers areas within the same mathematical, engineering, sociology, linguistics and so on. For example, a computer network can be represented and explored by a graph, where the vertices represent terminals and the edges represents connections (which, in turn, can be wired or wireless connections).

This chapter aims to present, in an organized manner, the concepts, terms and notations of graph theory that appear in different parts of this work.

1.1 Graphs

Many real-world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. For example, computers, roads, railways or electric networrks. Note that in this type of diagrams we are interested mainly if two given points are connected by a line, the way they come together is immaterial. The mathematical abstraction of situations of this type gives rise to the concept of graphs. The following definitions can be found in [21, 43].

Definition 1.1.1. (***Graph***)

A graph *G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G) \neq \emptyset$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges, together with an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to join u and v, and the vertices u and v are called the ends of e.*

Sometimes declare V and E without using the (G) unless if it is necessary to differentiate, then use $V(G)$ and $E(G)$.

We denote the numbers of vertices and edges in G by $n = |V(G)|$ and $m = |E(G)|$, respectively; these two basic parameters are called the *order* and *size* of G, respectively. We say that a graph G is *finite* if and only if $n < \infty$ and $m < \infty$. Otherwise we say that the graph is *infinite*. Since the edges are not ordered pairs of vertices, we are always dealing with non-oriented graphs.

An *edge* joining the vertices $u \in V(G)$ and $v \in V(G)$ on many occasions is denoted by $[uv]$, but we will use the notation $[u, v]$ to denote this edge, since the notation $[uv]$ will be used in this work for geodesics, which will be discussed in Chapter 7.

Any graph with just one vertex is referred to as *trivial graph*. All other graphs are *non-trivial*.

1.2 Adjacency of vertices, edges incidence and vertex degree

Graphs are so named because they can be represented graphically, and it is this graphical representation which helps us to understand many of their properties. Each vertex is indicated by a point, and each edge by a line joining the points representing its ends. Most of the definitions and concepts in graph theory are suggested by its graphical representation as illustrated in Figure 1.2.

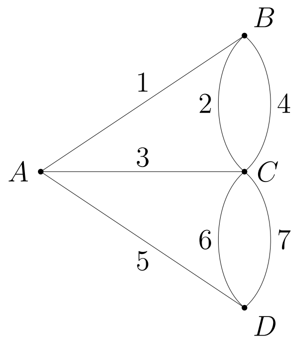


Figure 1.2: The graph of the bridges of Konigsberg.

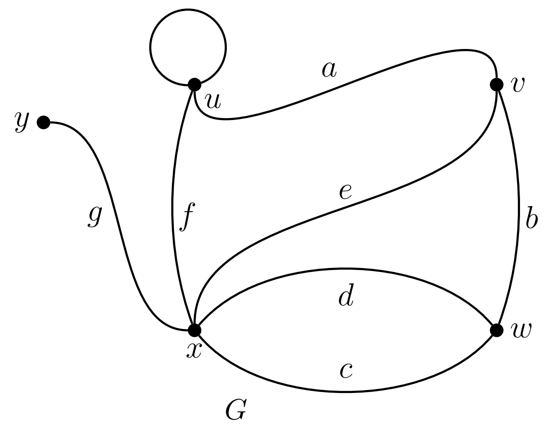
We support us in this representation. We say that two vertices $u \in V(G), v \in V(G)$ are *adjacent* or *neighbours* if $[u, v] \in E(G)$ and we also denote it by $u \sim v$; likewise, two edges are adjacent if they have one vertex in common; similarly, if $e = [u, v]$ we say that the edge $e \in E(G)$ is *incident* to the vertices u and v. The set of neighbours of a vertex v in a graph G is denoted by $N_G(v)$, i.e., $N_G(v) := \{u \in V(G) : [u, v] \in E(G)\}$.

Pairs of non-adjacent vertices or edges are said to be *independent*. More formally, a set of vertices and edges is independent (or stable) if none of its pairs of elements are adjacent. In Figure 1.2, the vertices B and D are an independent; besides, each vertex separately is an independent set.

1.3 Representations of graphs

Although drawings are a convenient means of specifying graphs, they are clearly not suitable for storing graphs in computers, or for applying mathematical methods to study their properties. For these purposes, we consider two matrices associated with a graph, its incidence matrix and its adjacency matrix.

Let $G = (E, V)$ with $n = |V|$ and $m = |E|$. The *incidence matrix* of G is the $n \times m$ matrix $M_G := (m_ve)$, where m_ve is the number of times (0, 1, or 2) that vertex v and edge e are incident. Clearly, the incidence matrix is just another way of specifying the graph. The *adjacency matrix* of G is the $n \times n$ matrix $A_G := (a_u v)$, where $a_u v$ is the number of edges joining vertices u and v, each loop counting as two edges. Incidence and adjacency matrices of the graph G are shown in Figure 1.3.



	a	b	c	d	e	f	g	h
u	1	0	0	0	0	1	0	2
v	1	1	0	0	1	0	0	0
w	0	1	1	1	0	0	0	0
x	0	0	1	1	1	1	1	0
y	0	0	0	0	0	0	1	0

M_G

	u	v	w	x	y
u	2	1	0	1	0
v	1	0	1	1	1
w	0	1	0	2	0
x	1	1	2	0	1
y	0	0	0	1	0

A_G

Figure 1.3: Representation of incidence and adjacency matrices of G.

Because most graphs have many more edges than vertices, the adjacency matrix of a graph is generally much smaller than its incidence matrix and thus requires less storage space. When dealing with simple graphs, an even more compact representation is possible. For each vertex v, the neighbours of v are listed in some order. A list $(N_G(v) : v \in V)$ of these lists is called an *adjacency list* of the graph. Simple graphs are usually stored in computers as adjacency lists. The adjacency matrix is the easiest way to save information from a graph, in the memory of a computer. It is vital to handle this type of representation for the development of this work.

1.4 Degree of a vertex

The *degree* of a vertex is the number of neighbors it has in the graph. The degree of $v \in V(G)$ is denoted by $\deg(v) := |N_G(v)|$. For a nonempty set $X \subseteq V(G)$, and a vertex $v \in V(G)$, $N_X(v)$ denotes the set of neighbors that v has in X, i.e., $N_X(v) := \{u \in X : u \sim v\}$, and the degree of v in X will be denoted by $\deg_X(v) = |N_X(v)|$. We denote the degree of a vertex

$v_i \in V(G)$ by $\deg(v_i) = \deg_G(v_i)$ (or by $\deg_i$ for short) and the *degree sequence* of G by $\{\deg_1, \deg_2, \ldots, \deg_n\}$ (ordered as follows: $\deg_1 \geq \deg_2 \geq \cdots \geq \deg_n$).

The number $\rho(G) := \min\{\deg(v) : v \in V(G)\}$ is the *minimum degree* of G and the number $\Delta(G) := \max\{\deg(v) : v \in V(G)\}$ is its *maximum degree.* In Figure 1.4, $\rho(G_1) = 0$ and $\Delta(G_2) = 4$.

If the degree of a vertex is 0, we say that is an *isolated vertex.* In Figure 1.4, the vertex D in the graph G_1 is an isolated vertex.

Figure 1.4: Simple graph G_1 and non-simple graph G_2.

Definition 1.4.1. (***Loop, Link***)

An edge with identical ends is called a loop*, and an edge with distinct ends a* link*. Two or more links with the same pair of ends are said to be* multiple edges*.*

In the graph G_2 of Figure 1.4, the edge c is a loop, and all other edges are links; the edges e and d are multiple edges.

A *simple graph* is one that has a single edge joining any two adjacent vertices, i.e., a graph without loops and multiple edges (see the graph G_1 in Figure 1.4).

Although some authors consider non-simple graphs (allowing loops and multiple edges), unless otherwise stated, we will work with simple graphs and then by graph we mean simple graph. In Chapter 2 we will address some polynomials such as: matching [47, 55], independent [28, 65], domination [3, 4], chromatic [20, 104] and clique polynomials [59, 65] which are defined on simple graphs. In Chapters 3, 4 and 5 we will introduce the alliance polynomial of simple graphs. In Chapter 6 and 7 we will study hyperbolic graphs. It has proven in [18] that the study of the hyperbolicity on graphs can be reduced to the study of the hyperbolicity on simpler graphs. In particular, the authors prove that the study of the hyperbolicity on a graph with loops and multiple edges can be reduced to the study of the hyperbolicity in the same graph without its loops and with simple edges replacing the multiple edges.

1.5 Subgraphs

Apart from the study of the characteristics or properties of a graph in its entirety, one can also consider only a region or a part thereof. For example, we can study arbitrary sets of vertices and edges of any graph. Moreover, in many cases, it is appropriate to consider graphs that are included "within" other. We will call them *subgraphs*.

Definition 1.5.1. (***Subgraph***)
If $G = (V, E)$ is a graph then $G_1 = (V_1, E_1)$ is a subgraph *of G if $\emptyset \neq V_1 \subseteq V$ and $E_1 \subseteq E$ where each edge E_1 is incident to vertices of V_1.*

See in Figure 1.5 the subgraphs G_1 and G_2 of the graph G. A especially relevant class of subgraphs in this work are the *induced subgraphs*.

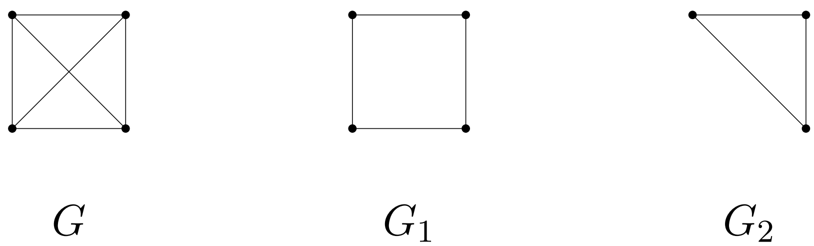


Figure 1.5: As subgraph G_1 and an induced subgraph G_2 of the graph G.

Definition 1.5.2. (***Induced subgraph***)
A subgraph obtained by vertex deletions only is called an induced subgraph. *If X is the set of vertices deleted, the resulting subgraph is denoted by $G - X$. Frequently, it is the set $Y := V \setminus X$ of vertices which remain that is the focus of interest.*

Particular types of subgraphs are obtained by removing in some graph a vertex or an edge. We have formalized this idea in the following definitions. Let v be a vertex of a graph $G = (V(G), E(G))$. The subgraph $G - v$ of G is that graph whose vertex set is $V(G) - \{v\}$ and edge set is $E(G-v)$ (all edges of the graph G except the incident edges to v). Therefore, $G - v$ is the subgraph of G induced by $V(G)\setminus\{v\}$.

In Figure 1.5, G_2 is an induced subgraph of G. We can see graphically that it is the result of removing a vertex in the graph G.

The subgraph of G induced by $Y \subseteq V(G)$ is denoted by $\langle Y \rangle$. Thus $\langle Y \rangle$ is the subgraph of G whose vertex set is Y and whose edge set consists of all edges of G which have both ends in Y.

1.6 Connectivity of graphs

One of the most significant property can have a graph, is to be connected. To understand this concept, it is necessary to give some definitions that describe us which means going from one vertex to another.

Definition 1.6.1. (***Path***)
A path *of a graph* $G = (V, E)$ *is a sequence of vertices* $P = \{v_0, v_1, v_2, \ldots, v_n\}$ *such that* v_{i-1} *is adjacent to* v_i*, for* $i = 1, 2, \ldots, n$*; a* simple path *is a path in which all vertices are different.*

Definition 1.6.2. (***Cycle***)
By cycle *we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.*

The *length* of a path or a cycle is the number of its edges. A path or cycle of length k is called a k-path or k-cycle, respectively; the path or cycle is odd or even according to the parity of k.

Definition 1.6.3. (***Connectivity***)
A graph is connected *if, for every partition of its vertex set into two nonempty sets* X *and* Y*, there is an edge with one end in* X *and one end in* Y*; otherwise, the graph is* disconnected *or* non-connected.

Given a connected graph $G = (V, A)$ and any two distinct vertices $u, v \in V$, we can find a path that connects them. Examples of connected and disconnected graphs are displayed in Figure 1.6.

Figure 1.6: Representation of a connected graph G_1 and a disconnect graph G_2.

A non-connected graph is formed by different "blocks" of vertices, each of which is a connected graph, what we call a *connected component*.

Definition 1.6.4. (***Connected component***)
A connected component *of a graph* G *is a connected subgraph of* G *which is not properly contained on any other connected subgraph of* G*, that is, a connected component of* G *is a subgraph that is maximal with respect to the property of being connected.*

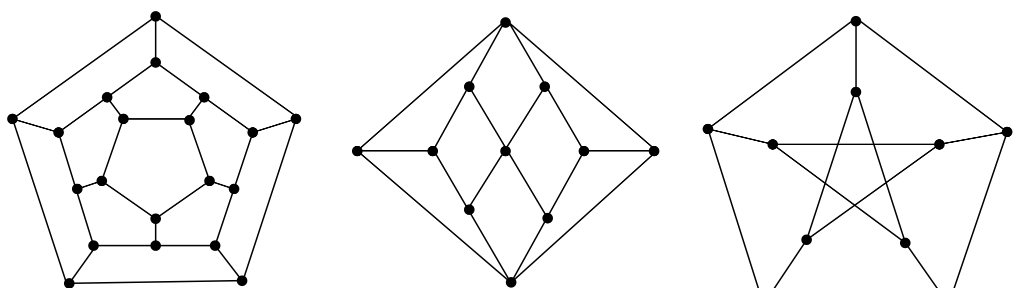
In a graph G we define the *distance* of two vertices u, v denoted by $d_G(u, v)$ or $d(u, v)$ as the length of a shortest $u - v$ path in G (a path joining u and v); if no such path exists, we set $d(u, v) := \infty$. In a connected graph G, for every $u, v \in V(G)$ we have $d_G(u, v) < \infty$. The greatest distance between any two vertices in G is the *diameter* of $V(G)$, denoted by $\operatorname{diam} V(G)$.

1.6.1 Hamiltonian cycle

In this section we give a brief introduction to the Hamiltonian paths and cycles. W. R. Hamilton (1805-1865) invented (and patented) a game in which it was to tour 20 cities (vertices) in the world without going through any more than once. The cities were connected by 30 edges, forming the graph of an icosahedron.

A *Hamiltonian cycle* is a cycle in a graph G that visits each vertex in $V(G)$ exactly once. *Hamiltonian path* is a non-closed path containing all vertices of the graph. A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph* (see, for example, Dodecahedron G_1 in Figure 1.7). The Herschel graph G_2 in Figure 1.7 is not Hamiltonian because it is bipartite and has an odd number of vertices.

The problem of finding a Hamiltonian cycle (or path) in an arbitrary graph is known to be NP-complete (there is no general method of resolution). The Petersen graph G_3 in Figure 1.7 is not a Hamiltonian graph but can not be easily deduced. We will see a sufficient condition for a graph to be Hamiltonian.



G_1 G_2 G_3

Figure 1.7: Dodecahedron G_1, Herschel graph G_2 and Petresen graph G_3.

The following theorem is a well known result in graph theory which will be useful.

Theorem 1.6.5 (Dirac 1952). *A graph with order $n \geqslant 3$ is Hamiltonian if every vertex has degree $n/2$ or greater.*

Bondy and Chvátal noted that the proof of Theorem 1.6.5 can be modified to obtain a stronger result.

Corollary 1.6.6 (Bondy and Chvátal 1974). *Let G be a graph and u and v non-adjacent vertices in G such that $\deg(u) + \deg(v) = n$. Then G is Hamiltonian if and only if $G + [u, v]$ is Hamiltonian.*

The following Ore's Theorem has as corollary Theorem 1.6.5.

Theorem 1.6.7 (Ore 1960). *Suppose that G is a graph with $n \geq 3$ and $\deg(u) + \deg(v) \geq n$ for each pair of non-adjacent vertices $u \neq v$. Then G is Hamiltonian.*

1.7 Some special graphs

Some graphs appear frequently in many applications and, hence, they have standard names.

Definition 1.7.1. (***Path graph***)
A path graph *is a non-empty graph* $P = (V, E)$ *with* $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 2$ *and* $E = \{[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n]\}$. *The path graph with* n *vertices is denoted by* P_n. *The vertices* v_1 *and* v_n *are linked by* P_n *and are called its ends; the vertices* $v_1, v_2, \ldots, v_{n-1}$ *are the inner vertices of* P_n.

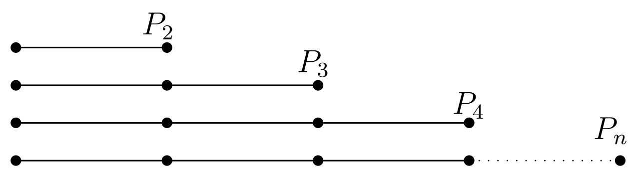


Figure 1.8: The paths graphs.

Definition 1.7.2. (***Cycle graph***)
A cycle graph *of* n *vertices is a graph* $G = (V, E)$ *with* $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$ *and* $E = \{[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1]\}$. *It is denoted by* C_n.

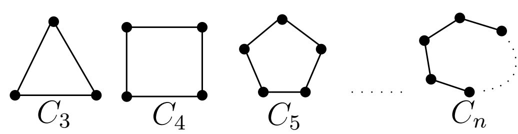


Figure 1.9: The Cicles graphs.

Definition 1.7.3. (***Complete graph***)
A complete graph *is a graph in which every pair of vertices are joined by exactly one edge, i.e., all pairs of vertices of* G *are adjacent. The complete graph with* n *vertices is denoted by* K_n. *At each vertex* $v \in V(G)$ *we have* $\deg_G(v) = n - 1$.

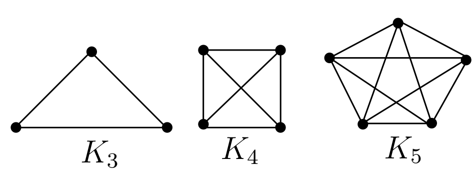


Figure 1.10: The complete graphs.

Definition 1.7.4. (***Empty graph***)
An empty graph *is a graph whose edge set is empty. We denote by E_n the empty graph with n vertices. In an empty graph all vertices have degree* 0.

Definition 1.7.5. (***Bipartite graph***)
A graph is bipartite *if its vertex set can be partitioned into two nonempty subsets V_1 and V_2 so that no edge has both ends in V_1 or both ends V_2.*

Definition 1.7.6. (***Complete bipartite graph***)
A bipartite graph is said to be a complete bipartite graph *if each vertex of V_1 is adjacent with each vertex of V_2. If $|V_1| = m$ and $|V_2| = n$, then this graph is denoted by $K_{m,n}$.*

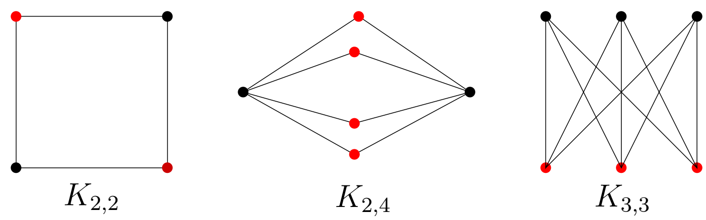


Figure 1.11: The complete bipartite graphs.

Definition 1.7.7. (***Star graph***)
The complete bipartite graph $K_{n-1,1}$ is called an n star graph *and it is denote by S_n.*

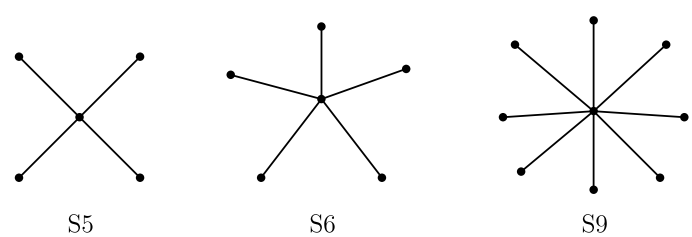


Figure 1.12: The stars graphs.

Definition 1.7.8. (***Wheel graph***)
The wheel graph *W_n is a graph with n vertices formed by connecting a single vertex to each vertex of a cycle C_{n-1}.*

Definition 1.7.9. (***Regular graph***)
A graph $G = (V, E)$ is regular *if all vertices have the same degree k, and we say that it is k-regular. Every regular graph G satisfies the equality $\rho(G) = \Delta(G)$.*

Definition 1.7.10. (***Tree***)
A tree *is an acyclic and connected graph, i.e., a connected graph without cycles.*

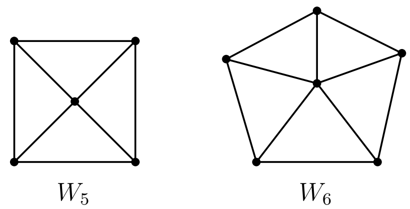


Figure 1.13: The wheels graphs.

1.8 Operations with graphs

In this section we define some of the most usual operations in graph theory and we will use them throughout the work. These operations produce new graphs from one or several graphs. We have unitary operations also called graph editing operations. They create a new graph from the original graph. Some examples of unitary operations are: adding or deleting a vertex or an edge, the contraction of an edge, line graph or graph complement. We also work with binary operations that create a new graph from two initial graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, such as: union of graphs or several kinds of products of graphs based on the Cartesian product of the set of vertices.

1.8.1 Unitary operations

Most of the subgraphs worthwhile studying are those that differ minimally from the initial graphs, because they retain much of their properties and have small differences that show important details.

The operations of *deletion* and *contraction* of an edge are essential in the study of the polynomials in graphs.

The graph obtained by *deleting an edge* $e \in E$ of a graph $G = (E, V)$, is the subgraph of G denoted $G - e$ or $G \setminus e$ defined as $G \setminus e = (V; E \setminus e)$. We say that a subgraph is *expansive* when it contains all the vertices of the initial graph.

The graph obtained by *contracting an edge* e in G, and denoted by G/e, results by identifying the endpoints of e followed by removing e. When e is a loop, G/e is the same as $G \setminus e$. It is not difficult to check that both deletion and contraction are commutative, and thus, for a subset of edges X, both $G \setminus X$ and G/X are well defined. Also, if $e \neq f$, then $(G \setminus e)/f$ and $(G/f) \setminus e$ are isomorphic; thus for disjoint subsets X, $X' \subseteq E(G)$, the graph $(G \setminus X)/X'$ is well-defined. A graph H isomorphic to $(G \setminus X)/X'$ for some choice of disjoint edge sets X and X' is called a *minor* of G.

Let us introduce another operation: *adding an edge* e of a graph G is the result of adding an edge to the set $E(G)$ connecting two vertices in $V(G)$; it is denoted by $G + e$.

Given a graph G with a finite number of connected components, an edge $e \in E(G)$ is a *bridge* or *cut edge* of G if the subgraph $G \setminus e$ has more connected components than G.

Proposition 1.8.1. *Let G be graph with a finite number of connected components. The edge $e \in E(G)$ is a bridge if and only if e does not belong to any cycle of G.*

Remove a vertex in a graph is not as simple as delete an edge, because when we remove a vertex all incident edges on it lose one end. Consequently, a good definition of this action is necessary: *Deleting a vertex* v of a graph G is to remove v from the set of vertices $V(G)$ and all the incident edges on v from the set of edges $E(G)$, obtaining a subgraph of G denoted by $G - v$ or $G \setminus v$.

Similarly, if G is a graph with a finite number of connected components, a vertex $v \in V(G)$ is a *cut vertex* of G if $G - v$ has more connected connected components than G.

We can obtain also the graph $G \cup \{v\}$ by adding to the graph G a single disjoint vertex v (i.e., $v \notin V(G)$). This operation is called *vertex addition*.

Definition 1.8.2. (***Complement***)

The complement $\overline{G}$ of the graph $G = (V, E)$ is the graph whose vertex set is V and whose edges are the pairs of non-adjacent vertices of G.

If $\mathtt{E} = \{[u, v] | u, v \in V, u \neq v\}$ is the set of all possible edges and $\overline{E} = \mathtt{E} \setminus E$ denotes the complement with respect to E, then $\overline{G} = (V, \overline{E})$.

1.8.2 Binary operations

Now we will see other operations that are applied to two or more graphs giving rise to new graphs. One of the most basic ways of combining graphs is by *union*.

The *disjoint union* of graphs, sometimes referred to as simply *graph union* is defined as follows. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets V_1 and V_2 (and hence disjoint edge sets), their union is the graph $G_1 \cup G_2 := (V_1 \cup V_2, E_1 \cup E_2)$. It is a commutative and associative operation.

The *graph join* $G_1 \uplus G_2$ of two graphs is their graph union with all the edges that connect the vertices of the first graph G_1 with the vertices of the second graph G_2. It is a commutative operation. The figure shows graph join of the cycle graph C_3 and the path graph P_3.

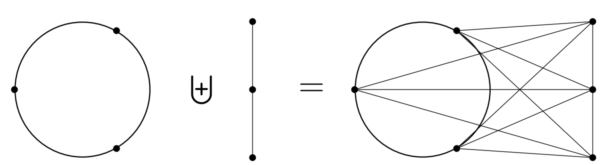


Figure 1.14: Join of a cycle C_3 and a path P_3.

The *Cartesian product* of the graphs G and H is the graph $G \Box H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $[(u_1, v_1), (u_2, v_2)]$ such that either $[u_1, u_2] \in E(G)$ and $v_1 = v_2$, or $[v_1, v_2] \in E(H)$ and $u_1 = u_2$ (see Figure 1.15).

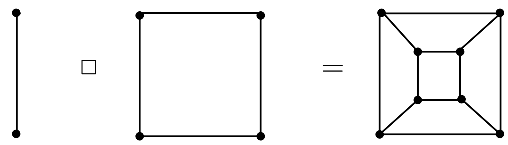


Figure 1.15: Cartesian product of a path P_2 and a cycle C_4.

We introduce now the corona of two graphs, defined by Frucht and Harary in 1970, see [52].

Definition 1.8.3. *Let G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. The* corona *of G_1 and G_2, denoted by $G_1 \diamond G_2$, is defined as the graph obtained by taking one copy of G_1 and a copy of G_2 for each vertex $v \in V(G_1)$, and then joining each vertex $v \in V(G_1)$ to every vertex in the v-th copy of G_2.*

From the definition, it clearly follows that the corona product of two graphs is a non-commutative and non-associative operation.

Figure 1.16 shows the corona of the graphs C_4 and C_3.

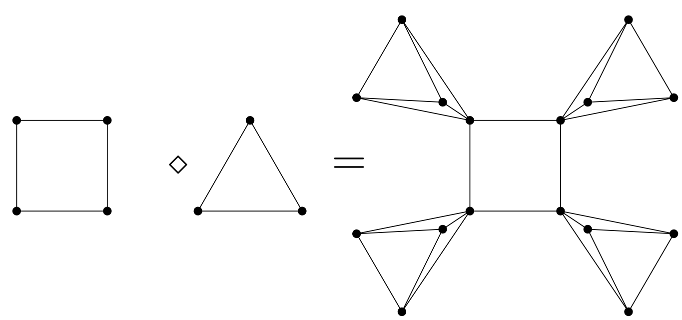


Figure 1.16: Corona $C_4 \diamond C_3$.

We will use also the strong product of graphs defined by Sabidussi in [105].

Definition 1.8.4. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The strong product $G_1 \boxtimes G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices $(u_1; v_1)$ and $(u_2; v_2)$ of $G_1 \boxtimes G_2$ are adjacent if either $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$, or $[u_1, u_2] \in E(G_1)$ and $v_1 = v_2$, or $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.*

Note that the strong product of two graphs is commutative.

1.9 Isomorphisms in graphs

It is evident that the importance of a graph is not in the names of the vertices nor in they way that we draw it. The characteristic property of a graph is the way in which the vertices

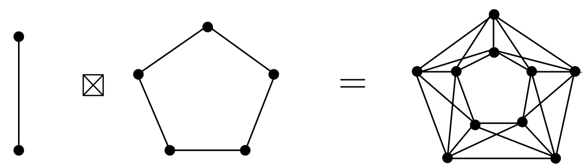


Figure 1.17: The strong product of a path P_2 and a cycle C_6.

are connected by edges. This motivates the following definition.

Definition 1.9.1. (***Isomorphisms***)
Let $G = (V, E)$ and $G^{'} = (V^{'}, E^{'})$ be two graphs. We say that G and $G^{'}$ are isomorphic, *and write $G \simeq G^{'}$, if there exists a bijection $\varphi : V \to V^{'}$ with $[u, v] \in E \Leftrightarrow [\varphi(u), \varphi(v)] \in E^{'}$ for all $u, v \in V$.*

In other words, a isomorphism is a bijective mapping between the vertices of V and $V^{'}$ preserving the connection of vertices. In this case, G and $G^{'}$ are mathematically identical; perhaps the appearance varies, but remain adjacency, structure, paths, cycles, number of vertices, number of edges, ...

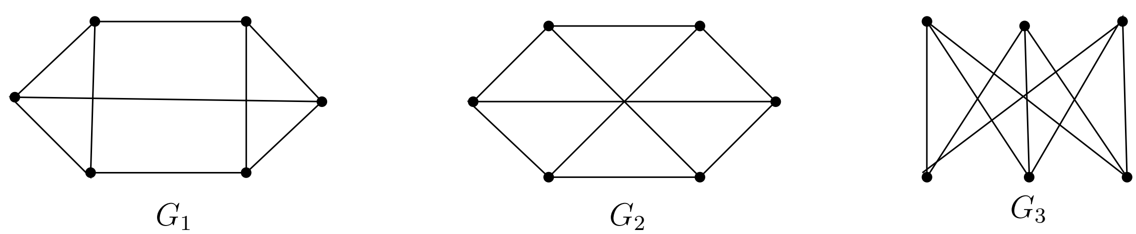


Figure 1.18: Isomorphic and non-isomorphic graphs.

In Figure 1.18 the graphs G_1 and G_2 are non-isomorphic, but the graphs G_2 and G_3 are isomorphic. Note that when two graphs are isomorphic we see them as the same graph. In fact, these two graphs are two possible graphical representations of the same space.

Chapter 2

Polynomials and alliances in graphs

One of the open problems in graph theory is the characterization of any graph by a polynomial. In recent years there have been many works on graph polynomials such as [4, 45, 65, 104, 106]. Research in this area has been largely driven by the advantages offered by the use of computers which make working with graphs: it is simpler to represent a graph by a polynomial (a vector) that by the adjacency matrix (a matrix).

Several polynomials have emerged in order to solve this problem, such as the characteristic polynomial of a graph [106], the independence and clique [28, 59], the chromatic [20, 104], the matching [45, 55], the domination [3, 4] and the Tutte [53, 56]. These polynomials are interesting since they compress information about the structure of the graph. Unfortunately, these polynomials do not solve the problem, since there are different graphs with the same polynomials.

In this chapter, we will make a brief introduction to these polynomials. At the end of this chapter we introduce the alliance polynomial. We prove in Chapter 3 that this polynomial characterizes many classes of graphs. In the light of these results, we conjecture that any graph G can be characterized by our alliance polynomial.

2.1 Some polynomials of a graph

2.1.1 The characteristic polynomial of a graph

Definition 2.1.1. (***Characteristic polynomial of a matrix***)

We call characteristic polynomial of a matrix A *to the determinant of the matrix* $\lambda I_n - A$*, and we denote it by* $PA(\lambda)$*:*

$$PA(\lambda) := \det(\lambda I_n - A),$$

where I_n *is the identity matrix of order* n*.*

We say that λ is an *eigenvalue* or *characteristic value* of A if $PA(\lambda) = 0$.

Let G be a graph of order n and A_G its adjacency matrix. As A_G is a square matrix of order n, we can compute the characteristic polynomial of A_G.

Definition 2.1.2. (***Characteristic polynomial of a graph***)
Let G be a graph of order n and A_G its adjacency matrix. The characteristic polynomial *of G is the characteristic polynomial of A_G and we denote it by $P(G;\lambda)$. Thus the characteristic polynomial of G is given by*

$$P(G;\lambda) := \det(\lambda I_n - A_G).$$

Example: Let G be the graph given by Figure 2.1.

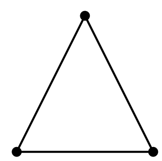


Figure 2.1: Graph K_3

The adjacency matrix of G is:

	v_1	v_2	v_3
v_1	0	1	1
v_2	1	0	1
v_3	1	1	0

The characteristic polynomial of A_G is given by:

$$P(G;\lambda) = \det(\lambda I_n - A_G) = \det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} = \lambda^3 - 3\lambda - 2.$$

Proposition 2.1.3. *Let G be a graph of order n, with connected components of $G_1, G_2, \ldots, G_r$. Then*

$$P(G;\lambda) = P(G_1 \cup G_2 \cup \cdots \cup G_r;\lambda) = P(G_1;\lambda)P(G_2;\lambda)\cdots P(G_r;\lambda).$$

2.1.2 The independence polynomial of a graph

A *stable* or *independent set* in a graph is a set of pairwise non-adjacent vertices. The *stability number* $\alpha(G)$ is the size of a maximum stable set in the graph G. There are three different kinds of structures that one can see by observing the behavior of stable sets of a graph: the enumerative structure, the intersection structure, and the exchange structure.

Definition 2.1.4. (***Independence polynomial of a graph***) *The* independence polynomial *of G is defined as*

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k = s_0 + s_1 x + s_2 x^2 + \cdots + s_{\alpha(G)} x^{\alpha(G)},$$

where s_k is the number of stable sets of cardinality k in G. By convention we assume that $s_0 = 1$.

The independence polynomial was defined by Gutman and Harary (1983), and it is a good representative of the enumerative structure. In [67] a number of general properties of the independence polynomial of a graph are presented. As examples, we mention that:

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x),$$

$$I(G_1 \uplus G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$

The following equalities are very useful in order to compute of the independence polynomial of many graphs (see [66, 67]).

Theorem 2.1.5. *For all graph G and H, we have:*

(i) $I(G;x) = I(G-v;x) + x \cdot I(G - N(v);x)$ *for every $v \in V(G)$.*

(ii) $I(G_1 \diamond H; x) = (I(H;x))^n \cdot I(G; \frac{x}{I(H;x)})$, *where $n = |V(G)|$ and $G \diamond H$ is the corona of G and H.*

Independence polynomial was defined as a generalization of *matching polynomial* of a graph (see Section 2.1.4), because the simple matching polynomial of a graph G and the independence polynomial of its *line graph* are identical. Recall that given a graph G, its line graph $L(G)$ is the graph whose vertex set is the edge set of G, and two vertices in $L(G)$ are adjacent if they share an end in G.

Figure 2.2: The graph G_1 and its line graph G_2.

For instance, the graphs G_1 and G_2 in Figure 2.2 satisfy $G_2 = L(G_1)$ and, hence, $I(G_2;x) = 1 + 6x + 7x^2 + x^3 = M(G_1;x)$, where $M(G_1;x)$ is the matching polynomial of the graph G_1.

2.1.3 The dependence and clique polynomial of a graph

The dependence polynomial was first introduced by Fisher [47], who studied the following problem: How many n letter words can be made from an m letter alphabet if certain pairs of letters commute? Fisher and Solow [48] defined the *dependence polynomial* as follows:

Definition 2.1.6. (***Dependence polynomial of a graph***)
Let c_j be the number of complete subgraphs of size j in a graph G. Then the dependence polynomial *of G is*

$$f_G(x) := 1 - c_1 x + c_2 x^2 - c_3 x^3 + \cdots + (-1)^k c_k x^k,$$

where k, is the clique number *of the graph G, i.e., the size of a greatest complete subgraph in G. Note that c_j is the number of independent sets of vertices of size j in $\overline{G}$, the complement of G.*

For a set S of words with an operation on them we assign a graph G_S such that $V(G_S) = S$ and two vertices are joined if they commute. Fisher [47] proved that the generating function for the above problem is precisely $\frac{1}{f_{G_S}(x)}$.

If we change the signs of all negative coefficients in $f_G(x)$ to positive signs, we obtain a polynomial which is called the *clique polynomial* of G. We denote it by $C(G; x)$.

Definition 2.1.7. (***Clique polynomial of a graph***)
Let c_j be the number of complete subgraphs of size j in a graph G. Then the clique polynomial *of G is*

$$C(G; x) = 1 + c_1 x + c_2 x^2 - c_3 x^3 + \cdots + c_k x^k.$$

We can deduce from the definition of the independence polynomial that $c_k(G) = s_k(\overline{G})$ we have the identity and, hence,

$$C(G; x) = I(\overline{G}; x).$$

Obviously, we also have

$$C(G; x) + C(\overline{G}; x) = I(G; x) + I(\overline{G}; x).$$

The following results are easily obtained. If G_1 and G_2 are two vertex-disjoint graphs, then:

$$C(G_1 \cup G_2; x) = C(G_1; x) + C(G_2; x) - 1 \text{ and } I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x),$$

$$C(G_1 \uplus G_2; x) = C(G_1; x) \cdot C(G_2; x) \text{ and } I(G_1 \uplus G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$

.

2.1.4 The matching polynomial of a graph

A spanning subgraph of a graph G is a subgraph of G which contains every vertex of G. By a *matching* M of a graph G we will mean a spanning subgraph of G consisting of vertices and edges only, i.e., the connected components of M are either vertices or edges. If M contains k edges, then M will be called a k-matching. If G has n vertices and M contains $k = [n/2]$ (the integer part of $n/2$) edges, then M will be called a *maximal matching*. If n is even then the maximal matching will be called a *perfect or complete matching*. It is clear that if G contains n vertices, then a k-matching in G contains $n-2k$ vertices.

Let M be a k-matching in G, and let us assign "weights" w_1 and w_2 to each node and edge respectively in M. Let us associate with M the weight $w_1^{n-2k}w_2^k$. If a_k is the number of k-matchings in G, then the total weight of the k-matchings in G will be $a_k w_1^{n-2k}w_2^k$. By adding the weights of all k-matchings in G, for all possible values of k, we will obtain a polynomial in w_1 and w_2. This polynomial will be called the *matching polynomial* of G [45, 55, 67].

Definition 2.1.8. (***Matching polynomial of a graph***)
The matching polynomial *of G, denoted by $M(G;w)$, is defined by*

$$M(G;w) := \textstyle\sum_{k=0}^{n/2} a_k w_1^{n-2k} w_2^k.$$

Here, $w = (w_1, w_2)$ is called the weight vector *associated with the matching polynomial.*

If we put $w_1 = w_2 = w$, then the resulting polynomial in the single variable w is called the *simple matching polynomial* of G.

The coeficient of $w_1^{n-2k}w_2^k$ is the number of sets of k independent edges in G. Since a matching in one component of a graph cannot affect matchings in other components, we get the following result [45].

Proposition 2.1.9. *Let G be a graph consisting of r components $G_l, G_2, \ldots, G_r$. Then*

$$M(G;w) = \textstyle\prod_{i=1}^{r} M(G_i;w).$$

2.1.5 The chromatic polynomial of a graph

The chromatic polynomial of a graph was introduced by Birkhoff and Lewis in their attack to the problem of the four colors. This classical problem emerged 147 years ago and was resolved in 1978 by Appel and Haken [10].

A *coloring* (or vertex coloring) is an assignment of colors to the vertices of a graph such that two vertices that share an edge have different colors.

The terminology of using colors to tag vertices comes from the problem of coloring maps. Tags as red or blue are only used when the number of colors is small, and usually the colors are represented by the integers $(1, 2, 3, \ldots, k)$.

A coloring using at most k colors is called a (proper) k-coloring. The smallest number of colors needed to color a graph G is called the *chromatic number* of G and it is denoted by $\chi(G)$. A graph with a (proper) k-coloring is *k-colorable* and it is *k-chromatic* if its chromatic number is exactly k. A subset of vertices with the same color is called a *color class*. Each class is an independent set. That is, a k-coloring is the same as a partition of the vertex set into k independent sets, and terms k-partite and k-colorable have the same meaning.

The *chromatic polynomial* counts the number of ways in which a graph can be colored given a number of colors.

Definition 2.1.10. (***Chromatic polynomial of a graph***)
Let G be a graph and let k positive integer with $1 \leq k \leq n$. Let $P_G(k)$ be the number of ways of coloring the graph G using the colors of the collection $(1, \ldots, k)$.

For example, the graph in Figure 2.3 can not be colored with only 2 colors. Using 3 colors, the graph can be colored in 12 different ways. With 4 colors, can be colored $24+4 \cdot 12$ different ways: using all four colors together, there are $4! = 24$ valid colorings (every assignment with 4 colors to a graph with four vertices is a proper coloring); and for each choice of 3-coloring of the four colors, there are 12 valid colorings.

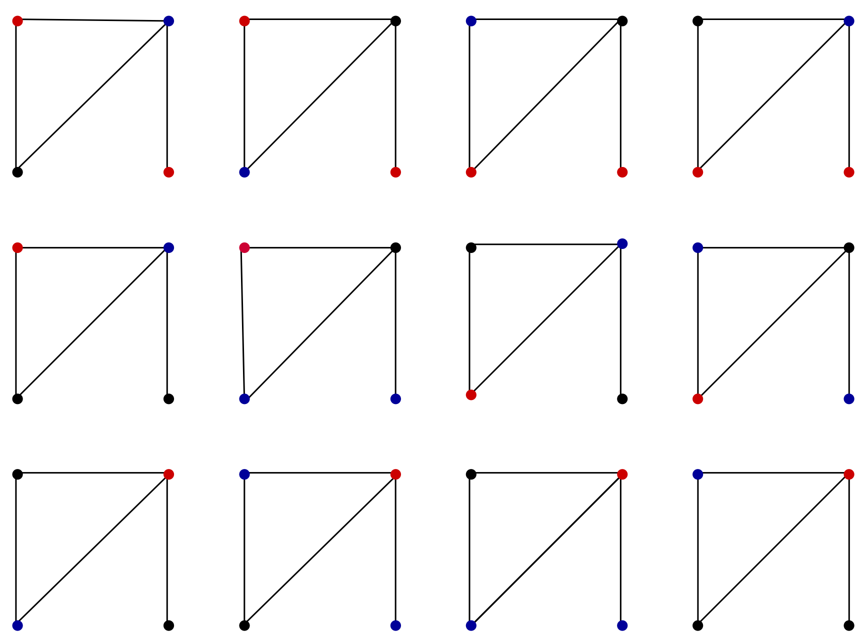


Figure 2.3: This graph can be 3-colored in 12 different ways.

The *chromatic polynomial* is a polynomial function $P(G;t)$ which counts the number of t-colorations in G, i.e., $P(G,k) = P_G(k)$ for $1 \leq k \leq n$. For the graph in Figure 2.3 we have $P(G,t) = t(t-1)2(t-2)$ and $P(G,4) = 72$.

The following are chromatic polynomials of some graphs:

$$P(K_n,t)=t(t-1)(t-2)\dots(t-(n-1)).$$

$$P(G,t)=t(t-1)^{n-1}, \text{ if } G \text{ is a tree with } n \text{ vertices.}$$

$$P(C_n,t)=(t-1)^{n-1}+(-1)^n(t-1).$$

$$P(G,t)=t(t-1)(t-2)(t^7-12t^6+67t^5-230t^4+529t^3-814t^2+775t-352), \text{ if } G \text{ is the Petersen graph.}$$

Two graphs are said to be chromatically equivalent if they have the same chromatic polynomial. Figure 2.4 shows chromatically equivalent non-isomorphic graphs.

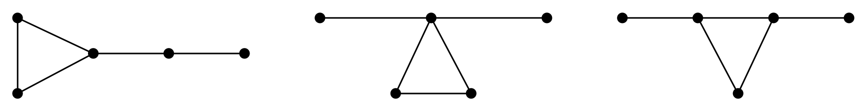


Figure 2.4: Three graphs with chromatic polynomial $(t-2)(t-1)^3t$.

2.1.6 The Tutte polynomial of a graph

The *Tutte polynomial*, also called the dichromate or the Tutte Whitney polynomial, is a polynomial in two variables which plays an important role in graph theory and computer science. It contains information about how the graph is connected.

Definition 2.1.11. (***Tutte polynomial of a graph***)
For a graph $G=(V,E)$ we define the Tutte polynomial *as*

$$T_G(x,y):=\textstyle\sum_{A\subseteq E}(x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|},$$

where $k(A)$ denotes the number of connected components of the graph (V,A).

Tutte's original definition of T_G is equivalent but less easily stated. For connected G we set

$$T_G(x,y)=\textstyle\sum_{i,j}t_{ij}x^iy^j,$$

where t_{ij} denotes the number of spanning trees of "internal activity i and external activity j".

At $y=0$, the Tutte polynomial specialises to the chromatic polynomial, $P(G,t)=(-1)^{|V|-k(G)}t^{k(G)}T_G(1-t,0)$, where $k(G)$ denotes the number of connected components of G. In particular, $T_G(2,0)=(-1)^{|V|}P(G,-1)$ gives the number of acyclic orientations.

2.1.7 The domination polynomial of a graph

In this section we state the definition of domination polynomial and some of its properties. In graph theory, a *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The *domination number* $\gamma(G)$ is the number of vertices in a smallest dominating set for G.

Definition 2.1.12. (***Domination polynomial of a graph***)
Let $D(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |D(G, i)|$. The domination polynomial *$D(G, x)$ of G is defined as*

$$D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i,$$

where $\gamma(G)$ is the domination number of G.

For example, the path graph with 4 vertices P_4 has one dominating set of cardinality 4, and four dominating sets of cardinalities 3 and 2; its domination polynomial is then $D(P_4, x) = x^4 + 4x^3 + 4x^2$. As another example, it is easy to see that $D(K_n, x) = (1+x)^n - 1$, for every $n \in \mathbb{N}$.

Theorem 2.1.13. *If a graph G has r components $G_1, \ldots, G_r$, then*

$$D(G, x) = D(G_1, x) \cdots D(G_r, x).$$

2.2 Defensive alliances in graphs

The study of the mathematical properties of alliances in graphs started in [78]. The defensive alliances in graphs is a topic of recent and increasing interest in graph theory; see, for instance [31, 50, 63, 95, 96, 107, 108, 109, 110]. The study of defensive alliances as a graph-theoretic concept has recently attracted a great deal of attention due to some interesting applications in a variety of areas, including quantitative analysis of secondary RNA structures [64] and national defense [94]. Besides, defensive alliances are the mathematical model of web communities. Adopting the definition of Web community proposed recently in [49], "a Web community is a set of web pages having more hyperlinks (in either direction) to members of the set than to non-members".

Consider a (not necessarily connected) graph $G = (V, E)$ of order $|V| = n$. Recall that we denote two adjacent vertices u and v by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors that v has in X: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of v in X will be denoted by

$$\delta_X(v) = |N_X(v)|.$$

We denote the degree of a vertex $v_i \in V$ by $\delta(v_i) = \delta_G(v_i)$ (or by δ_i for short) and the degree sequence of G by $\{\delta_1, \delta_2, \dots, \delta_n\}$ (ordered as follows $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$; then $\delta_1 = \Delta(G)$ and $\delta_n = \rho(G)$ are the maximum and minimum degree of G, respectively).

The subgraph induced by $S \subset V$ will be denoted by $\langle S \rangle$ and the complement of the set $S \subset V$ will be denoted by $\bar{S} = V \setminus S$.

A nonempty set $S \subseteq V$ is a *defensive k-alliance* in $G = (V, E)$, $k \in [-\delta_1, \delta_1] \cap \mathbb{Z}$, if $\langle S \rangle$ is connected and, for every $v \in S$,

$$\delta_S(v) \geqslant \delta_{\bar{S}}(v) + k. \tag{2.2.1}$$

A vertex $v \in S$ is said to be *k-satisfied* by the set S, if (2.2.1) holds. Notice that (2.2.1) is equivalent to

$$\delta(v) \geqslant 2\delta_{\bar{S}}(v) + k \tag{2.2.2}$$

and

$$2\delta_S(v) \geqslant \delta(v) + k. \tag{2.2.3}$$

Note that we just consider the value of k in the set of integers $\mathcal{K} := [-\delta_1, \delta_1] \cap \mathbb{Z}$. In some graphs G, there are some values of $k \in \mathcal{K}$, such that do not exist defensive k-alliances in G. For instance, in the star graph S_n do not exist defensive k-alliances for $k \geqslant 2$. Besides, $V(G)$ is a defensive δ_n-alliance in G. Notice that for any S there exists some $k \in \mathcal{K}$ such that it is a defensive k-alliance in G.

Given $S \subseteq V$ with $\langle S \rangle$ connected, we define

$$k_S := \max\{k \in \mathcal{K} \,:\, S \text{ is a defensive } k\text{-alliance}\}. \tag{2.2.4}$$

We say that k_S is the *exact index of alliance* of S, or also, S is an *exact defensive k_S-alliance* in G, see e.g. [31].

Proposition 2.2.1. *Let G be a graph and let $S \subset V$. The following statements are equivalents:*

1. *k is the exact index of alliance of S.*
2. *S is a defensive k-alliance in G with one vertex $v \in S$ such that $\delta_S(v) = \delta_{\overline{S}}(v) + k$.*
3. *S is a defensive k-alliance but it is not a defensive $(k+1)$-alliance in G.*

Proof. **(1) $\Longrightarrow$ (2)** Seeking for a contradiction assume that for all $v \in S$ we have $\delta_S(v) > \delta_{\overline{S}}(v) + k$, then we obtain $\delta_S(v) \geq \delta_{\overline{S}}(v) + (k+1)$ for every $v \in S$; then S is a defensive $(k+1)$-alliance. This is the contradiction we were looking for, since k is a maximum; so, there is $v \in S$ such that $\delta_S(v) = \delta_{\overline{S}}(v) + k$.

(2) $\Longrightarrow$ (3) Since there exists $v \in S$ with $\delta_S(v) = \delta_{\overline{S}}(v) + k$, we have that S is not a defensive $(k+1)$-alliance in G.

(3) $\Longrightarrow$ (1) It is easy to check that $k = k_S$.

$\square$

Remark 2.2.2. *The exact index of alliance of S in G is*

$$k_S = \min_{v \in S}\{\delta_S(v) - \delta_{\overline{S}}(v)\}. \tag{2.2.5}$$

2.2.1 The alliance polynomial of a graph

Definition 2.2.3. (***Alliance polynomial of a graph***)

Let G be a graph with order n. We define the alliance polynomial *of G with variable x as follows:*

$$A(G;x) = \sum_{S \subseteq V} \sigma_G(S) \cdot x^{n+k_S}, \tag{2.2.6}$$

where $\sigma_G(S) = 1$ if $\langle S \rangle$ is nonempty and connected in G, and $\sigma_G(S) = 0$ otherwise.

Other expression for this alliance polynomial is the following:

$$A(G;x) = x^n \sum_{k \in \mathcal{K}} A_k(G) x^k, \tag{2.2.7}$$

with $A_k(G)$ the number of exact defensive k-alliances in G.

In Chapters 3, 4 and 5 we will obtain interesting properties of these alliance polynomials.

Chapter 3

Properties of the alliance polynomial

In this chapter we will get some properties of alliance polynomial (see Section 3.1).

A finite sequence of real numbers $(a_0, a_1, a_2, ..., a_n)$ is said to be *unimodal* if there is some $k \in \{0, 1, ..., n\}$, called the *mode* of the sequence, such that

$$a_0 \leq ... \leq a_{k-1} \leq a_k \quad \text{and} \quad a_k \geq a_{k+1} \geq ... \geq a_n;$$

the mode is unique if $a_{k-1} < a_k$ and $a_k > a_{k+1}$. A polynomial is called unimodal if the sequence of its coefficients is unimodal.

We present in Section 3.1 an algorithm that computes in an efficient way the alliance polynomial. In Section 3.2, we compute the alliance polynomial for some graphs and study its coefficients; in particular, we show that some of them are unimodal. We investigate the alliance polynomials of path, cycle, complete and complete bipartite graphs. Also we prove that the path, cycle, complete and star graphs are characterized by their alliance polynomials. Finally, in Section 3.3 we show that the alliance polynomial characterizes many graphs that are not distinguished by other usual polynomials of graphs.

3.1 Alliance polynomials

Let G be a graph with order n. Recall that we define the *alliance polynomial* of a graph G with variable x as follows:

$$A(G;x) = \sum_{S \subseteq V} \sigma_G(S) \cdot x^{n+k_S}, \tag{3.1.1}$$

where $\sigma_G(S) = 1$ if $\langle S \rangle$ is nonempty and connected in G, and $\sigma_G(S) = 0$ otherwise.

Other expression for this alliance polynomial is the following:

$$A(G;x) = x^n \sum_{k \in \mathcal{K}} A_k(G) x^k, \tag{3.1.2}$$

with $A_k(G)$ the number of exact defensive k-alliances in G.

As an example, we compute now the alliance polynomial of the complete bipartite graph $K_{3,3}$.

Note that since $K_{3,3}$ is a cubic graph, we have $A_k(K_{3,3}) = 0$ for $k \in \{-2, 0, 2\}$. In order to obtain $A(K_{3,3}; x)$, we compute its non-zero coefficients.

$A_{-3}(K_{3,3}) = 6$: Since $K_{3,3}$ is a cubic graph, the number of exact defensive (-3)-alliances is $|V(K_{3,3})| = 6$.

$A_{-1}(K_{3,3}) = 33$: We have that $S \subset V(K_{3,3})$ is an exact defensive (-1)-alliance, if both parts of $K_{3,3}$ have some vertex in S and one of them has just one vertex. Then a combinatorial argument gives the result.

$A_1(K_{3,3}) = 15$: We have that $S \subset V(K_{3,3})$ is an exact defensive 1-alliance, if $S \neq V(K_{3,3})$ and S contains at lest two vertices of both parts of $K_{3,3}$. Thus, we obtain the result from combinatorial arguments.

$A_3(K_{3,3}) = 1$: The unique exact defensive 3-alliance is the set of vertices of $K_{3,3}$.

Then, we obtain

$$A(K_{3,3}; x) = 6x^3 + 33x^5 + 15x^7 + x^9.$$

We propose now an algorithm that facilitates the efficient computation of the alliance polynomial of a graph G with order n. Let $W = \{S_1, \dots, S_{2^n-1}\}$ be the collection of nonempty subsets of V.

Algorithm 3.1.1.
Input: adjacency matrix of G.
Output: alliance polynomial of G.

The algorithm starts with $A(G; x) = 0$ *and continues with the following steps, for* $1 \leqslant j \leqslant 2^n - 1$.

1. *If* $\langle S_j \rangle$ *is a connected subgraph, then go to step (2), else replace* j *by* $j + 1$ *and apply this step again.*

2. *Compute* k_{S_j}.

3. *Add one term* $x^{n+k_{S_j}}$ *to* $A(G; x)$.

4. *Replace* j *by* $j + 1$ *and apply step (1) again.*

This algorithm for computing the alliance polynomial of a graph shows a complexity $O(m2^n)$ where m is the number of edges of G; furthermore, when it is running on Δ-regular graphs its complexity is $O(n2^n)$. The algorithm looks for the $2^n - 1$ nonempty induced subgraphs of G. In step (1), for each induced subgraph, it analyzes if it is connected or not, using Depth-First Search (DFS) algorithm. It is a well known result that DFS algorithm complexity is $O(m)$. Furthermore, it is easy to check that step (2) has cost $O(n)$ and step (3) has cost $O(1)$.

Remark 3.1.2. *Let G_1 and G_2 be isomorphic graphs. Then $A(G_1; x) = A(G_2; x)$.*

The following proposition shows general properties of the alliance polynomials.

Proposition 3.1.3. *Let G be a graph. Then $A(G; x)$ satisfies the following properties:*

i) All real zeros of $A(G; x)$ are non-positive numbers.

ii) The value 0 is a zero of $A(G; x)$ with multiplicity $n - \delta_1 \geq 1$.

iii) $\sum_{i=k}^{\delta_1} A_i(G)$ is the number of defensive k-alliances in G for every $k \in \mathcal{K}$.

iv) If G has at least an edge and its degree sequence has exactly r different values $\{c_1, c_2, \ldots, c_r\}$, then $A(G; x)$ has at least $r + 1$ terms: $x^{n-c_1}, \ldots, x^{n-c_r}, x^{n+\delta_n}$.

v) $A(G; x)$ is a symmetric polynomial (either an even or an odd function) if and only if the degree sequence of G has either all values even or all odd.

Proof. We prove separately each item.

i) Since the coefficients of $A(G; x)$ are non-negatives, we have the result.

ii) Since $n + k \geqslant n - \delta_1$ for any $k \in \mathcal{K}$, we have a common factor $x^{n-\delta_1}$ in $A(G; x)$ and $A_{-\delta_1}(G) \neq 0$.

iii) If S is an exact defensive r-alliance in G with $r \geqslant k$, then we have $\delta_S(v) \geq \delta_{\overline{S}}(v) + r \geqslant \delta_{\overline{S}}(v) + k$ for all $v \in S$; in fact, S is a defensive k-alliance in G. This finishes the proof, since an exact defensive r-alliance in G with $r < k$ is not a defensive $(r + 1)$-alliance and $r + 1 \leqslant k$.

iv) Consider $v_1, v_2, \dots, v_r \in V$ with $\delta_G(v_i) = c_i$ for all $i = 1, \dots, r$. Note that $\{v_i\}$ for $i = 1, \dots, r$ is an exact defensive $(-c_i)$-alliance, since $0 = \delta_{S_i}(v_i) = \delta_{\overline{S_i}}(v_i) - c_i = c_i - c_i$. Therefore, that makes appear the term x^{n-c_i} in $A(G;x)$ for all $i = 1, \dots, r$. Consider now a connected component S of G and u a vertex in S with $\delta_G(u) = \delta_n$. Hence, S is an exact defensive δ_n-alliance in G, since we have

$$\delta_S(v) = \delta_G(v) \geq \delta_{\overline{S}}(v) + \delta_n = \delta_n, \quad \forall v \in S \tag{3.1.3}$$

and $\delta_S(u) = \delta_n$. So, that makes appear the term $x^{n+\delta_n}$ in $A(G;x)$.

v) In order to prove the directed implication assume that $A(G;x)$ is an even polynomials (the case odd is analogous). Let c be any element of the degree sequence of G and $v \in V$ with $\delta(v) = c$. By item v) we have $A_{-c}(G) \neq 0$, then $n - c$ is even and $c \cong n (\text{mod } 2)$. So, we conclude that the elements in the degree sequence of G are either all even or all odd numbers.

Finally, we prove the converse implication. Consider $S \subseteq V$ an exact defensive k-alliance. By Proposition 2.2.1, there exists $v \in S$ with

$$2\delta_S(v) = \delta_G(v) + k.$$

This finishes the proof since $\delta_G(v) + k$ is even.

□

In Chapter 1 we have given the definition of a cut vertex. Now we define a *cut vertex set* of a graph $G = (V, E)$ as a subset $X \subsetneq V$ such that $\langle V \setminus X \rangle$ is a non-connected graph.

Theorem 3.1.4. *Let G be any graph with order n. Then we have the following statements hold.*

1. *$A(G;1) < 2^n$, and it is the number of connected induced subgraphs $\langle S \rangle$ in G.*
2. *The number of cut vertex sets of G is $2^n - 1 - A(G;1)$.*

Proof. By (3.1.1), we have

$$A(G;1) = \sum_{S \subset V} \sigma_G(S).$$

Thus, $A(G;1)$ is the number of connected induced subgraph $\langle S \rangle$ in G; this amount is less that 2^n, since we have $2^n - 1$ nonempty subsets of V.

Let $c_k(G)$ be the number of cut vertex sets of cardinality k for $0 \leqslant k < n$ and $s_k(G)$ be the number of connected induced subgraphs of G with order k for $0 < k \leqslant n$. Note that X is a cut vertex set if and only if $V(G) \setminus X$ induces a non-connected subgraph.Conversely, if

$X \subset V(G)$ is not a cut vertex set of G then $\langle V(G) \setminus X\rangle$ is connected. Then, we have the following equality for every $0 < k \leqslant n$

$$c_{n-k}(G) + s_k(G) = \binom{n}{k}.$$

Finally, we obtain the result since $A(G;1) = \sum_{k=1}^{n} s_k(G)$. □

The following theorem shows some properties of coefficients and degree of any alliance polynomial.

Theorem 3.1.5. *Let $A(G;x)$ be the alliance polynomial of a graph G with $\mathrm{Deg}_{\min}(A(G;x))$ and $\mathrm{Deg}(A(G;x))$ the minimum degree and maximum degree of its terms, respectively. Then $A(G;x)$ satisfies the following statements:*

i) $\mathrm{Deg}_{\min}(A(G;x)) = n - \delta_1$ *and its coefficient* $A_{-\delta_1}(G)$ *is the number of vertices in G with degree δ_1.*

ii) $A_{-\delta_1+1}(G)$ *is the number of vertices in G with degree $\delta_1 - 1$.*

iii) $A_{\delta_n}(G) > 0$.

iv) $n + \delta_n \leq \mathrm{Deg}(A(G;x)) \leq n + \delta_1$.

v) $A_{\delta_1}(G)$ *is equal to the number of connected components in G which are δ_1-regular.*

vi) There not exist defensive k-alliances in G for $k > \mathrm{Deg}(A(G;x)) - n$.

Proof. We prove separately each item.

i) The minimum value of $\mathcal{K}$ is $-\delta_1$, so $\mathrm{Deg}_{\min}(A(G;x)) \geqslant n - \delta_1$. Consider now the sets $S_v = \{v\}$ with $\delta_G(v) = \delta_1$, then $\langle S_v\rangle$ is connected and S_v is an exact defensive $(-\delta_1)$-alliance. Finally, it is clear that any $S \in V$ with more than one vertex is not an exact defensive $(-\delta_1)$-alliance, since for any $v \in S$ we have

$$\delta_S(v) - \delta_{\overline{S}}(v) \geq 1 - (\delta_1 - 1) > -\delta_1 + 1. \tag{3.1.4}$$

Then, $A_{-\delta_1}(G)$ is the number of vertices in G with degree δ_1. Note that, consequently, $A_{-\delta_1}(G) \leqslant n$ and $A_{-\delta_1}(G) = n$ if and only if G is a regular graph.

ii) Similarly to the previous item, we consider the sets $S_v = \{v\}$ with $\delta_G(v) = \delta_1 - 1$ and we obtain $A_{-\delta_1+1}(G) \geq NV_{\delta_1-1}$ where $NV_i := \{\text{number of vertices in } G \text{ with degree } i\}$; therefore, we obtain the equality since any $S \subset V$ with more than one vertex is an exact defensive k-alliance for $k \geq -\delta_1 + 2$ by (3.1.4).

iii) This is a consequence of Proposition 3.1.3 iv).

iv) Item iii) gives the first inequality. The second one holds since δ_1 is the maximum value of $\mathcal{K}$.

v) By (3.1.2), $A_{\delta_1}(G)$ is the number of defensive δ_1-alliance in G. We characterize this by the number of connected components in G which is δ_1-regular. First, note that if S is a defensive δ_1-alliance, then S is an exact defensive δ_1-alliance since δ_1 is the maximum value in $\mathcal{K}$. Clearly, any connected component in G which is δ_1-regular is an exact defensive δ_1-alliance.

Now, consider an exact defensive δ_1-alliance S in G. Hence, for any $v \in S$ we have

$$\delta_S(v) \geq \delta_{\overline{S}}(v) + \delta_1 \quad \Longrightarrow \quad \delta_1 \geq \delta_S(v) \geq \delta_{\overline{S}}(v) + \delta_1 \geq \delta_1.$$

Then, we have $\delta_S(v) = \delta_G(v) = \delta_1$ for every $v \in S$ and conclude that S is a connected component in G which is δ_1-regular.

vi) Suppose that there is a defensive k-alliance S in G, in fact, $k_S \geq k$. Then, that makes appear the term x^{n+k_S} in $A(G;x)$ and so,

$$n + k \leq n + k_S \leq \mathrm{Deg}(A(G;x)).$$

□

Proposition 3.1.6. *Let G be any connected graph. Then G is regular if and only if*

$$A_{\delta_1}(G) = 1. \tag{3.1.5}$$

Proof. If G is regular, then by Theorem 3.1.5 v) we obtain $A_{\delta_1}(G) = 1$. Besides, if $A_{\delta_1}(G) = 1$, then there is an exact defensive δ_1-alliance S in G with $\delta_S(v) \geqslant \delta_{\bar{S}}(v) + \delta_1 \geqslant \delta_1$ for every $v \in S$ (i.e., $\delta_S(v) = \delta_1$ for every $v \in S$). So, the connectivity of G gives that G is a δ_1-regular graph. □

Proposition 3.1.7. *Let G be any graph and G_1 any proper subgraph of G. Then*

$$A(G;x) \neq A(G_1;x).$$

Proof. Since G_1 is a proper subgraph of G, all connected induced subgraph of G_1 is a connected induced subgraph of G and at less one edge e (with endpoints $u, v \in V$) of G is not contained in G_1. Hence, since $\langle\{u, v\}\rangle$ is connected in G but is no connected in G_1, we have $A(G;1) > A(G_1;1)$ by Theorem 3.1.4. □

Theorem 3.1.8. *Let $G = G_1 \cup \ldots \cup G_r$ be the disjoint union of the graphs $G_1, \ldots, G_r$ ($r \geqslant 2$) with orders $n_1, \ldots, n_r$, respectively. Then we have*

$$A(G;x) = x^{n-n_1} A(G_1;x) + \ldots + x^{n-n_r} A(G_r;x), \tag{3.1.6}$$

where $n := n_1 + \ldots + n_r$.

Proof. Since every connected induced subgraph of G is a connected induced subgraph of G_i for some $1 \leqslant i \leqslant r$, and every exact defensive k-alliance in G is an exact defensive k-alliance in G_i for some $1 \leqslant i \leqslant r$, we have that $\mathcal{K}(G) = \bigcup_{i=1}^{r} \mathcal{K}(G_i)$ and

$$A_k(G) = A_k(G_1) + \ldots + A_k(G_r), \quad \text{for } k \in \mathcal{K}(G).$$

So, we have

$$A_k(G)x^{n+k} = x^{n-n_1} A_k(G_1)x^{n_1+k} + \ldots + x^{n-n_r} A_k(G_r)x^{n_r+k}, \quad \text{for } k \in \mathcal{K}(G).$$

Finally, if we sum in $k \in \mathcal{K}(G)$, then we obtain the result. □

This result allows to obtain the alliance polynomial of the graph $G \cup \{v\}$ obtained by adding to the graph G a single disjoint vertex v (i.e., $v \notin V(G)$). This operation is called *vertex addition*.

Corollary 3.1.9. *Let G be any graph with order n and let v be a vertex such that $v \notin V(G)$. Then*

$$A(G \cup \{v\};x) = x\, A(G;x) + x^{n+1}.$$

The n-vertex edgeless graph or *empty graph* is the complement graph for the complete graph K_n; it is commonly denoted as E_n for $n \geqslant 1$.

Corollary 3.1.10. *Let n be a natural number with $n \geqslant 1$. If $A(G;x) = nx^n$, then G is an isomorphic graph to E_n.*

Proof. Note that the empty graph E_1 satisfies $A(E_1;x) = x$. So, by Theorem 3.1.8 or Corollary 3.1.9 we have that

$$A(E_{n+1}) = xA(E_n;x) + x^{n+1}, \quad \forall n \geq 1.$$

This implies that $A(E_n;x) = nx^n$. The uniqueness follows from items iii) and iv) in Theorem 3.1.5. □

Corollary 3.1.11. *Let G be any graph with order n. Then*

$$A(G \cup E_m;x) = x^m\, A(G;x) + mx^{n+m}.$$

Theorem 3.1.12. *Let G_1, G_2 be two graphs with order n_1 and n_2, respectively. Then*

$$A(G_1 \uplus G_2; x) = A(G_1; x) + A(G_2; x) + \widetilde{A}(G_1, G_2; x),$$

where $\widetilde{A}(G_1, G_2; x)$ is a polynomials with $\widetilde{A}(G_1, G_2; 1) = (2^{n_1} - 1)(2^{n_2} - 1)$ and

$Deg\big(\widetilde{A}(G_1, G_2; x)\big) = Deg\big(A(G_1 \cup G_2; x)\big)$.

Proof. Let us define $\widetilde{A}(G_1, G_2; x) = A(G_1 \uplus G_2; x) - A(G_1; x) - A(G_2; x)$. First, if S_1 is a defensive alliance in G_1 which provides a term $x^{n_1+k_{S_1}}$ in $A(G_1; x)$, then S_1 provides a term $x^{n_1+n_2+k_{S_1}-n_2} = x^{n_1+k_{S_1}}$ in $A(G_1 \uplus G_2; x)$. It follows immediately that we obtain $A(G_1; x)$ as an addend in $A(G_1 \uplus G_2; x)$ when S_1 runs on the defensive alliances in G_1. Similarly, we obtain $A(G_2; x)$ as an addend in $A(G_1 \uplus G_2; x)$ when we consider the defensive alliances in G_2.

In order to complete the summation in $A(G_1 \uplus G_2; x)$ we consider $R_1 \subseteq V(G_1)$ (being either a defensive alliance in G_1 or not) with $1 \leqslant r_1 \leqslant n_1$ elements and $R_2 \subseteq V(G_2)$ (being either a defensive alliance in G_2 or not) with $1 \leqslant r_2 \leqslant n_2$ elements. Note that any $R_1 \cup R_2$ is a defensive alliance in $G_1 \uplus G_2$ since $\langle R_1 \cup R_2 \rangle$ is connected. By Theorem 3.1.4, we have

$$\widetilde{A}(G_1, G_2; 1) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \binom{n_1}{i} \binom{n_2}{j} = \left(\sum_{i=1}^{n_1} \binom{n_1}{i} \right) \left(\sum_{j=1}^{n_2} \binom{n_2}{j} \right) = (2^{n_1} - 1)(2^{n_2} - 1).$$

However, the exact index of alliance of $R_1 \cup R_2$ in $G_1 \uplus G_2$ depends strongly on the particular geometry (topology) of G_1 and G_2. In general, we can not determine the exact index of alliance of $R_1 \cup R_2$ given its cardinality and degree sequence.

It is obvious that terms in $A(G_1 \uplus G_2; x)$ provided from every $R_1 \cup R_2$ with maximum degree are obtained from R_1^* and R_2^* defensive alliances with $\langle R_1^* \rangle$, $\langle R_2^* \rangle$ connected subgraphs and highest exact index of alliance in G_1 and G_2, respectively. Hence,

$$Deg\big(\widetilde{A}(G_1, G_2; x)\big) = n_1 + n_2 + \max\{k_{R_1^*}, k_{R_2^*}\},$$

where the maximum is taken over all R_1^*, R_2^* defensive alliances in G_1, G_2, respectively. So, (3.1.6) finishes the proof since

$$\begin{aligned} Deg\big(A(G_1 \cup G_2; x)\big) &= \max\big\{n_2 + Deg\big(A(G_1; x)\big), n_1 + Deg\big(A(G_2; x)\big)\big\} \\ &= n_1 + n_2 + \max\{k_{R_1^*}, k_{R_2^*}\}, \end{aligned}$$

where the maximum is taken over all R_1^*, R_2^* defensive alliances in G_1, G_2, respectively. ☐

Theorem 3.1.12 allows to obtain the following result which will be useful (see Section 3.2.2). We denote by $\overline{G}$ the *complement graph* of G (recall that $\overline{K_n}$ is isomorphic to the empty graph E_n).

Theorem 3.1.13. *Let n, m be two positive integers. Then we have*

$$A(K_n \uplus \overline{K}_m; x) = A(K_n; x)\widetilde{A}_m(x) + mx^m \tag{3.1.7}$$

where $\widetilde{A}_m(x)$ is a polynomial which just depend of m, in fact,

$$\widetilde{A}_m(x) = \sum_{r=0}^{m} \binom{m}{r} x^{\min\{2r, m+1\}}.$$

Proof. First, we fix $S \subset V(K_n)$ with $1 \leqslant s \leqslant n$ elements. Note that S provides a term x^{2s-1} in $A(K_n; x)$. Consider $R \subset V(\overline{K}_m)$ with $0 \leqslant r \leqslant m$ elements. Now we compute the exact index of alliance of $H_R = S \cup R$ in $K_n \uplus \overline{K}_m$. We have

$$\delta_{H_R}(v) - \delta_{\overline{H_R}}(v) = (r + s - 1) - (n - s + m - r) = 2s - 1 - (n + m) + 2r, \quad \text{for every } v \in S$$

and

$$\delta_{H_R}(v) - \delta_{\overline{H_R}}(v) = s - (n - s) = 2s - 1 - (n + m) + m + 1, \quad \text{for every } v \in R.$$

Then, H_R provides a term $x^{2s-1+\min\{2r,m+1\}}$ for each R. Therefore, for each S we obtain the polynomial $x^{2s-1} \cdot \widetilde{A}_m(x)$ when R runs in the subsets of $V(\overline{K}_m)$. In order to complete the sum, note that the defensive alliances without elements of $V(K_n)$ are just the single vertices of $V(\overline{K}_m)$. Then (3.1.1) gives the result. ☐

Also, we can compute the alliance polynomials of $K_n \uplus K_m$ (see Proposition 3.2.7) and $\overline{K}_n \uplus \overline{K}_m$ (see Proposition 3.2.13).

3.2 Characterization of some classes of graphs by their alliance polynomials

In this section we obtain the explicit formulae for alliance polynomials of some classical classes of graphs using combinatorial arguments. We also study fundamental properties such as unimodality and the uniqueness of these polynomials.

Figure 3.1 shows two graphs G_1 and G_2 with the same order, size, degree sequence and number of induced subgraphs; however, these graphs have different alliance polynomials. A simple computation gives $A(G_1; x) = 2x^7 + 4x^8 + 27x^9 + 50x^{10} + 11x^{11}$ and $A(G_2; x) = 2x^7 + 4x^8 + 30x^9 + 47x^{10} + 11x^{11}$.

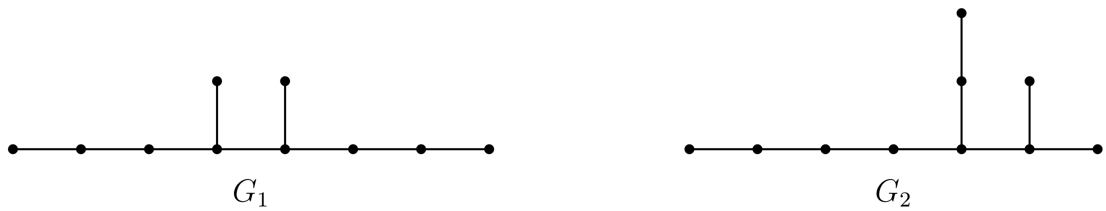


Figure 3.1: Graphs with same order, size, degree sequence and number of connected induced subgraphs such that $A(G_1; x) \neq A(G_2; x)$.

3.2.1 Polynomials for path and cycle graphs

Proposition 3.2.1. *Let P_n be a path graph with order $n \geq 2$. Then*

$$A(P_n; x) = (n-2)\, x^{n-2} + 2\, x^{n-1} + \frac{(n-2)(n+1)}{2}\, x^n + x^{n+1}. \tag{3.2.8}$$

Proof. We analyze the subsets with different cardinality separately.

Let us consider any subset S of $V(P_n)$ with connected induced subgraph $\langle S \rangle$, and $|S| = r$ with $r = 1, \ldots, n$.

If $r = 1$, then there are n alliances.

- Since there are two vertices with degree 1, we have 2 exact defensive (-1)-alliances. So, that makes appear the term
$$2x^{n-1}.$$
- Since there are $n-2$ vertices with degree 2, we have $n-2$ exact defensive (-2)-alliances. So, that makes appear the term
$$(n-2)x^{n-2}.$$

Consider now the case $2 \leq r \leq n-1$. The connectivity of $\langle S \rangle$ allows to compute k_S since it is a sub-path with r vertices. Then we have $n - r + 1$ exact defensive 0-alliances, since at least one endpoint of any induced P_r attains the exact index of alliance $k_{P_r} = 0$. So, we have the terms

$$(n - r + 1)x^n, \qquad \text{for every } 2 \leq r \leq n - 1.$$

Finally, if $r = n$, then $S = V(P_n)$. We have just one exact defensive 1-alliance, with the term

$$x^{n+1}.$$

Then, we obtain

$$\begin{aligned} A(P_n; x) &= (n-2)\, x^{n-2} + 2\, x^{n-1} + \sum_{r=2}^{n-1} (n - r + 1)\, x^n + x^{n+1}, \\ &= (n-2)\, x^{n-2} + 2\, x^{n-1} + \frac{(n-2)(n+1)}{2}\, x^n + x^{n+1}. \end{aligned}$$

□

We have the following consequences of Proposition 3.2.1.

Corollary 3.2.2. *Let P_n be the path graph with n vertices. Then $A(P_n;x)$ is unimodal if and only if $2 \leqslant n \leqslant 4$.*

Proof. By simple computation we can check that $A(P_n;x)$ is unimodal for $2 \leqslant n \leqslant 4$, since $A(P_2;x) = 2x + x^3$, $A(P_3;x) = x + 2x^2 + 2x^3 + x^4$ and $A(P_4;x) = 2x^2 + 2x^3 + 5x^4 + x^5$. But, for $n > 4$ we have that $A_{-2}(P_n) = n - 2 > 2 = A_{-1}(P_n) < (n-2)(n+1)/2 = A_0(P_n)$. □

Now we characterize graphs G with $A(G;x) = A(P_t;x)$.

Theorem 3.2.3. *Let t be a natural number with $t \geqslant 2$. If $A(G;x) = A(P_t;x)$, then G is an isomorphic graph to P_t.*

Proof. Let us consider a graph G with $A(G;x) = A(P_t;x)$; denote by n the order of G and by Δ_G the maximum degree of G.

Assume first that $t \geqslant 3$. By items i) and ii) in Theorem 3.1.5, $n - \Delta_G = t - 2$, G has $t-2$ vertices of degree Δ_G, and 2 vertices of degree $\Delta_G - 1$. So, we have $n \geqslant t$.

Assume now that $t = 2$. Then $A(G;x) = A(P_2;x) = 2x + x^3$. By Theorem 3.1.5 i), $n - \Delta_G = 1$ and G has 2 vertices of maximum degree Δ_G. So, we have $n \geqslant t$.

Hence, $n \geqslant t$ for every $t \geqslant 2$.

By Theorem 3.1.5 iv), we have $t + 1 \geq n + \delta_G$ where δ_G is the minimum degree of G. So, δ_G is either 0 or 1. Hence, if $n > t$, then $n = t + 1$ and $\delta_G = 0$. Besides, the maximum degree of $A(G;x)$ is greater than $t + 1$ since G has a connected component with vertex of positive degree. This is a contradiction, thus $n = t$ and then $t - \Delta_G = t - 2$ if $t \geqslant 3$, and $2 - \Delta_G = 1$ if $t = 2$; therefore, $\Delta_G = 2$ if $t \geqslant 3$, and $\Delta_G = 1$ if $t = 2$.

Hence, if $t = 2$, G is an isomorphic graph to P_2. If $t \geqslant 3$, then G has $t - 2$ vertices of degree 2 and 2 vertices of degree 1. If G is disconnected, then $A(G;x)$ has at least two terms x^k with $k > t$, one for each connected component. But this is a contradiction since $A(G;x) = A(P_t;x)$. So, G is connected and this implies that G is an isomorphic graph to P_t. □

Proposition 3.2.4. *Let C_n be a cycle graph with order $n \geq 3$. Then*

$$A(C_n;x) = n\,x^{n-2} + n(n-2)\,x^n + x^{n+2}. \tag{3.2.9}$$

Proof. We analyze the subsets with different cardinality separately.

Let us consider any subset S of $V(C_n)$ with connected induced subgraph $\langle S \rangle$, and $|S| = r$ with $r = 1, \ldots, n$.

If $r = 1$, then we have n exact defensive (-2)-alliances. So, that makes appear the term

$$nx^{n-2}.$$

Consider now the case $2 \leq r \leq n-1$. The connectivity of $\langle S\rangle$ allows to compute k_S since it is a path with r vertices. Then we have n exact defensive 0-alliances, since the end vertices of the induced P_r attain the exact index of alliance $k_{P_r} = 0$. So, we have the term

$$nx^n, \qquad \text{for every } 2 \leqslant r \leqslant n-1.$$

Finally, if $r = n$, then $S = V(C_n)$. We have an exact defensive 2-alliance with the term

$$x^{n+2}.$$

Then, we obtain $A(C_n; x) = n\, x^{n-2} + n(n-2)\, x^n + x^{n+2}$. □

Corollary 3.2.5. *Let C_n be a cycle graph with order $n \geqslant 3$. Then $A(C_n; x)$ is unimodal.*

Here we want to characterize graphs G with $A(G; x) = A(C_t; x)$.

Theorem 3.2.6. *Let t be a natural number with $t \geqslant 3$. If $A(G; x) = A(C_t; x)$, then G is an isomorphic graph to C_t.*

Proof. Let us consider a graph G with order n such that $A(G; x) = A(C_t; x)$; denote by Δ_G the maximum degree of G and by δ_G its minimum degree. By Theorem 3.1.5 i), G has t vertices of degree Δ_G, so $n \geq t$. Besides, $n + \delta_G \leq t + 2 \leq n + \Delta_G$. Hence, $\delta_G \leq 2$.

Assume that $n > t$. Then δ_G is either 0 or 1.

If $\delta_G = 0$, then Proposition 3.1.3 iv) makes appear the term x^n. Since x^{t+1} does not appear in $A(C_t; x)$, we obtain $n \geq t + 2$. Furthermore, it appears one term, associated to one connected component with vertices of positive degree, with exponent $n + \Delta_G > n$, but this is impossible since $A(C_t; x)$ has degree $t + 2$.

Hence $\delta_G = 1$ and $n = t + 1$. So, by Theorem 3.1.5 i), G has t vertices of degree $\Delta_G = 3$ and one vertex of degree 1. Denote by v the vertex of G with degree 1 and by S the connected component of G containing v. Clearly, S is an exact defensive 1-alliance in G, and then the term $x^{(t+1)+1}$ appears in $A(G; x)$; but $S \setminus \{v\}$ is an exact defensive 1-alliance in G. This is a contradiction since there is just one term x^{t+2} in $A(G; x)$.

Hence, we have $n = t$. Besides, by Theorem 3.1.5 i), G is a regular graph and $\Delta_G = 2$. Since $A(C_t; x)$ is a monic polynomial with degree $t+2$, the number of connected components of G is 1 by Theorem 3.1.5 v), and so, G is connected. □

3.2.2 Polynomials for complete graphs

Since K_{n+1} is an isomorphic graph to $K_n \uplus \overline{K_1}$ for every $n \geqslant 1$, Theorem 3.1.13 has the following consequences.

Proposition 3.2.7. *Let K_n be a complete graph with order $n \geq 1$. Then*

$$A(K_n; x) = \frac{(x^2+1)^n - 1}{x}. \tag{3.2.10}$$

Proposition 3.2.7 gives the following results.

Corollary 3.2.8. *Let K_n be the complete graph with order n. Then $A(K_n; x)$ is unimodal.*

Now we characterize graphs G with $A(G; x) = A(K_t; x)$.

Theorem 3.2.9. *If $A(G; x) = A(K_t; x)$, then G is an isomorphic graph to K_t.*

Proof. Consider a graph G with order n such that $A(G; x) = A(K_t; x)$. By Theorem 3.1.5 i), G has t vertices of maximum degree $\Delta_G = n-1$, so $n \geq t$. Denote by $v_1, v_2, \dots, v_t$ the vertices of G with maximum degree $n-1$. Hence, we have that G contains a clique $\langle\{v_1, v_2, \dots, v_t\}\rangle$ isomorphic to K_t. If $n > t$, then Proposition 3.1.7 gives $A(G; x) \neq A(K_t; x)$. So, we obtain that $n = t$. Finally, since $n = t$, G is an $(t-1)$-regular graph. Therefore, G is an isomorphic graph to K_t. □

Since a complete graph without one of its edges K_n/e is isomorphic to $K_{n-2} \uplus \overline{K}_2$ for every $n \geqslant 3$, Theorem 3.1.13 has the following consequence.

Proposition 3.2.10. *Let K_n/e be a complete graph without one of its edges, with $n \geq 2$ vertices. Then,*

$$A(K_n/e; x) = \frac{(x^2+1)^n - (x^4 - x^3)(x^2+1)^{n-2} + x^3 - 2x^2 - 1}{x}. \tag{3.2.11}$$

Proposition 3.2.10 gives the following results.

Corollary 3.2.11. *Let K_n/e be the complete graph with $n \geq 2$ vertices, without one of its edges. Then $A(K_n/e; x)$ is unimodal if and only if $2 \leqslant n \leqslant 4$.*

Proof. We can check that $A(K_n/e; x)$ is unimodal for $2 \leqslant n \leqslant 4$, since $A(K_2/e; x) = A(E_2; x) = 2x^2$, $A(K_3/e; x) = A(P_3; x) = x + 2x^2 + 2x^3 + x^4$ and $A(K_4/e; x) = 2x + 2x^2 + 5x^3 + 2x^4 + 2x^5 + x^6$. But, for $n > 4$ we have that $A_{-(n-1)}(K_n/e) = n - 2 > 2 = A_{-(n-2)}(K_n/e) < \binom{n}{2} - 1 = A_{-n+3}(K_n/e)$. □

Now we characterize graphs G with $A(G; x) = A(K_t/e; x)$.

Theorem 3.2.12. *Let t be a natural number with $t \geqslant 2$. If $A(G; x) = A(K_t/e; x)$, then G is an isomorphic graph to K_t/e.*

Proof. If $t = 2$, then the result follows from Corollary 3.1.10. Assume now that $t \geqslant 3$.

Let us consider a graph G with order n such that $A(G; x) = A(K_t/e; x)$. By items i) and ii) in Theorem 3.1.5, G has $t-2$ vertices of maximum degree $\Delta_G = n-1$ and 2 vertices of degree $n-2$, so $n \geq t$. Denote by $v_1, \dots, v_{t-2}$ the vertices of G with maximum degree $n-1$ and by w_1, w_2 the vertices with degree $n-2$. Hence, we have that G contains a subgraph $\langle\{v_1, \dots, v_{t-2}, w_1, w_2\}\rangle$ which is either a clique or an isomorphic graph to K_t/e, depending on whether or not w_1 is adjacent to w_2 in G. If $n > t$, then Proposition 3.1.7 gives $A(G;x) \neq A(K_t;x)$. So, we obtain that $n = t$.

Note that any nonempty subset S of $V(G)$ induces a connected subgraph $\langle S\rangle$ of G, if $S \neq \{w_1, w_2\}$. Obviously, $A(G;1) = 2^t - 2$ and this is a characterization of the graph K_t/e, since a graph with one more induced connected subgraph is isomorphic to K_t. Furthermore, any graph G with order t obtained from K_t by removing at least two edges, does not satisfy the condition $A(G;1) = 2^t - 2$. Since $A(G;x) = A(K_t/e;x)$ and G has order t, then G is isomorphic to K_t/e. □

3.2.3 Polynomials for completed bipartite graphs

Since $\overline{K}_n \uplus \overline{K}_m = K_{n,m}$, an argument similar to the ones in the proofs of Theorems 3.1.12 and 3.1.13 allows to obtain $A(\overline{K}_n \uplus \overline{K}_m;x)$.

Proposition 3.2.13. *Let $K_{n,m}$ be a complete bipartite graph with $n, m \geqslant 1$. Then*

$$A(K_{n,m};x) = n\,x^n + m\,x^m + \sum_{k=2}^{n+m} \sum_{i,j>0\,,\,i+j=k} \binom{n}{i}\binom{m}{j} x^{n+m+\min\{2i-n,2j-m\}}. \qquad (3.2.12)$$

Proof. Fix $n \geq 1$ and $m \geq 1$. Let us consider any subset S of $V(K_{n,m})$ with connected induced subgraph $\langle S\rangle$ and $|S| = k$ with $k = 1, \dots, n+m$.

If $k = 1$, then there are $n+m$ alliances.

- If S is a vertex associated to n, we have n exact defensive $(-m)$-alliances. So, that makes appear the term
$$nx^{n+m-m}.$$
- If S is a vertex associated to m, we have m exact defensive $(-n)$-alliances. So, that makes appear the term
$$mx^{n+m-n}.$$

Consider now the case $2 \leq k \leq n+m$. Obviously, any subset S of $V(G)$ with $k \geq 2$ elements induces a connected subgraph of $K_{n,m}$, if and only if it contains elements in both parts. Then, we have $\binom{n}{i}\binom{m}{j}$ exact defensive $\min\{j-(m-j), i-(n-i)\}$-alliances for each couple $i, j > 0$ such that $i + j = k$ (by choosing i vertices associated to n and j vertices associated to m).

So, we have the terms

$$\sum_{i,j>0,\,i+j=k} \binom{n}{i}\binom{m}{j} x^{n+m+\min\{2j-m,2i-n\}}.$$

Then, we obtain

$$A(K_{n,m};x) = n\,x^n + m\,x^m + \sum_{k=2}^{n+m} \sum_{i,j>0\,,\,i+j=k} \binom{n}{i}\binom{m}{j} x^{n+m+\min\{2i-n,2j-m\}}.$$

□

The complete bipartite graph $K_{n-1,1}$ is called an n *star graph* S_n. We have the following consequence of Theorem 3.1.13 (since S_n is an isomorphic graph to $K_1 \uplus \overline{K}_{n-1}$ for every $n \geqslant 2$) or Proposition 3.2.13.

Corollary 3.2.14. *Let S_n be star graph with order $n \geq 2$. Then*

$$A(S_n;x) = A(K_{n-1,1};x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1}{k} x^{2k+1} + (n-1)x^{n-1} + x^{n+1} \sum_{k=\lceil n/2 \rceil}^{n-1} \binom{n-1}{k}. \tag{3.2.13}$$

Here we want to characterize graphs G with $A(G;x) = A(S_t;x)$.

Theorem 3.2.15. *Let t be a natural number with $t \geqslant 2$. If $A(G;x) = A(S_t;x)$, then G is an isomorphic graph to S_t.*

Proof. If $t = 2$ then Theorem 3.2.3 gives the result. Fix $t \geqslant 3$.

Let us consider a graph G with order n such that $A(G;x) = A(S_t;x)$. Since $\text{Deg}_{min}(A(G;x)) = 1$, there is $v \in V(G)$ such that $v \sim w$ for all $w \in V(G) \setminus \{v\}$. Therefore, G is a connected graph, δ_G (the minimum degree of G) is greater that 0 and G contains an isomorphic subgraph G_S of S_n. Hence, any $S \subseteq V(G)$ which induces a connected subgraph $\langle S \rangle$ in G_S, induces a connected subgraph in G, too. So,

$$A(G;1) \geq A(G_S;1) = A(S_n;1). \tag{3.2.14}$$

Since $\text{Deg}(A(G;x)) = t+1$, we have $n + \delta_G \leq t+1$, and so, $n \leq t$. But, by (3.2.13), we have

$$2^n > A(G;1) = t - 1 + \sum_{k=0}^{t-1} \binom{t-1}{k} = 2^{t-1} + t - 1 > 2^{t-1},$$

and this condition implies that $n \geq t$. Thus, $n = t$.

Seeking for a contradiction assume that there are $w_1, w_2 \in V(G) \setminus \{v\}$ such that $w_1 \sim w_2$. Then, $\{w_1, w_2\}$ induces a connected subgraph in G, but not in G_S; and so,

$$A(G;1) > A(S_t;1) \quad \Longrightarrow \quad A(G;x) \neq A(S_t;x).$$

This is the contradiction we were looking for, and so, G is isomorphic to S_t. □

3.3 Distinctive power of alliance polynomial

In this section we explain the distinctive power of the alliance polynomial of a graph. This is an interesting difference with others well-known polynomials of graphs.

We denote by $D(G;x)$ the domination polynomial of G (see [3]), by $I(G;x)$ the independence polynomial of G (see [65]), by $M(G;x)$ the matching polynomial (see [47]), by $P(G;x)$ the characteristic polynomial, by $T(G;x,y)$ the Tutte polynomial (see [114]), by $P_{chr}(G;x,y)$ the bivariate chromatic polynomial introduced in [104], and by $Q(G;x,y)$ the subgraph component polynomial introduced in [112].

We say that a graph G is characterized by a graph polynomial f if for every graph G' such that $f(G') = f(G)$ we have that G' is isomorphic to G. The class of graphs K is characterized by a graph polynomial f if every graph $G \in K$ is characterized by f.

This notion has been studied in [82, 86], for the chromatic polynomial, the Tutte polynomial and the matching polynomial. It is shown, e.g., that several well-known families of graphs are determined by their Tutte polynomial, among them the class of wheels, squares of cycles, complete multipartite graphs, ladders, Möbius ladders, and hypercubes. In Section 3.2.1, we have proved that path, cycle, complete and star graphs are characterized by their alliance polynomials. In [100] the authors prove that the family of alliance polynomials of cubic graphs is a special one, since it does not contain alliance polynomials of graphs which are not cubic; and they also prove that the cubic graphs with at most 10 vertices are characterized by their alliance polynomials. Furthermore, in [35] the authors prove a similar result for the family of alliance polynomials of Δ-regular connected graphs with $\Delta \leqslant 5$, i.e., it does not contain alliance polynomials of graphs which are not connected Δ-regular.

G_3 G_4

Figure 3.2: Graphs with same characteristic polynomial.

We denote by $G_1 \square G_2$ and $G_1 \boxtimes G_2$ the Cartesian and the strong products of G_1 and G_2, respectively.

Proposition 3.3.1. *For the graphs G_i, $i = 1, ..., 6$, from Figures 3.1, 3.2, 3.3 and for P_4, $K_{1,3}$, P_5, $P_2 \cup C_3$, $K_{3,3}$, $P_2 \square C_3$, $P_2 \boxtimes P_3$ and $E_2 \uplus P_4$ we have*

(1) $P(G_3;x) = P(G_4;x)$ *but* $A(G_3;x) \neq A(G_4;x)$.

(2) $M(P_2 \cup C_3;x) = M(P_5;x)$ *but* $A(P_2 \cup C_3;x) \neq A(P_5;x)$.

(3) $I(P_2 \boxtimes P_3;x) = I(E_2 \uplus P_4;x)$ *but* $A(P_2 \boxtimes P_3;x) \neq A(E_2 \uplus P_4;x)$.

G_5 G_6

Figure 3.3: Graphs with same bivariate chromatic polynomial.

(4) $D(K_{3,3}; x) = D(P_2 \square C_3; x)$ *but* $A(K_{3,3}; x) \neq A(P_2 \square C_3; x)$.

(5) $P_{chr}(G_5; x, y) = P_{chr}(G_6; x, y)$ *but* $A(G_5; x) \neq A(G_6; x)$.

(6) $T(P_4; x, y) = T(K_{1,3}; x, y)$ *but* $A(P_4; x) \neq A(K_{1,3}; x)$.

(7) $Q(G_1; x, y) = Q(G_2; x, y)$ *but* $A(G_1; x) \neq A(G_2; x)$.

Proof. Proposition 3.1.3 v) gives that $A(G_3; x)$, $A(P_2 \boxtimes P_3; x)$ and $A(G_5; x)$ are symmetric polynomials, but $A(G_4; x)$, $A(E_2 \uplus P_4; x)$ and $A(G_6; x)$ are not symmetric; then $A(G_3; x) \neq A(G_4; x)$, $A(P_2 \boxtimes P_3; x) \neq A(E_2 \uplus P_4; x)$ and $A(G_5; x) \neq A(G_6; x)$. Besides, by Theorem 3.2.3 we have that P_4 and P_5 are characterized by their alliance polynomials, and so, $A(P_2 \cup C_3; x) \neq A(P_5; x)$ and $A(P_4; x) \neq A(K_{1,3}; x)$. Furthermore, by [100, Proposition 3.1] we have $A(K_{3,3}; x) \neq A(P_2 \square C_3; x)$. Besides, $A(G_1; x) \neq A(G_2; x)$ (see the beginning of Section 3.2). A simple computation gives $P(G_3; x) = P(G_4; x)$, $M(P_2 \cup C_3; x) = M(P_5; x)$, $I(P_2 \boxtimes P_3; x) = I(E_2 \uplus P_4; x)$ and $D(K_{3,3}; x) = D(P_2 \square C_3; x)$. So, items (1), (2), (3) and (4) hold. Item (5) follows from [44]. Since Tutte polynomial does not distinguish trees of the same size, we deduce item (6). Finally, $Q(G_1; x, y) = Q(G_2; x, y)$ follows from [112], and we have item (7). $\square$

The results in Section 3.2 and [35, 100] suggest the conjecture that every graph can be characterized by its alliance polynomial, although it seems hard to be proved.

However, if our conjecture turned out to be false, we think that the study of the following problem could be of interest.

Problem 3.3.2. *Are there graphs distinguished by* $P(G; x)$, $M(G; x)$, $I(G; x)$, $D(G; x)$, $P_{chr}(G; x, y)$, $T(G; x, y)$ *or* $Q(G; x, y)$ *which are not distinguished by* $A(G; x)$*?*

Chapter 4

Alliances polynomial of cubic graphs

The main aim of this chapter is to obtain further results about the alliance polynomial of cubic graphs (graphs with all of their vertices of degree 3), since they are a very interesting class of graphs with many applications (see, e.g., [25, 29, 42, 88]).

In Section 4.1 we obtain some properties of alliance polynomials of cubic graphs; besides, we prove that the family of alliance polynomials of cubic graphs is a very special one, since it does not contain alliance polynomials of graphs which are not cubic (see Theorem 4.1.6). Finally, in Section 4.2 we obtain (computationally) the alliance polynomials of cubic graphs with small order and we prove that they satisfy uniqueness. Recall that the subgraph induced by $S \subset V$ will be denoted by $\langle S \rangle$ and the complement of the set $S \subset V$ will be denoted by $\overline{S} = V \backslash S$.

4.1 Computing the alliance polynomials of cubic graphs

In this section we study the alliance polynomials of cubic graphs. We recall some previous results for alliance polynomials of general graphs (not necessarily cubic) which appear in Chapter 3 and that will be useful (see Proposition 3.1.3, Theorem 3.1.5 and Theorem 3.1.4).

Theorem 4.1.1. *Let G be any graph. Then, $A(G;x)$ satisfies the following properties:*

i) $A(G;x)$ does not have zeros in the interval $(0,\infty)$.

ii) $A(G;1) < 2^n$, and it is the number of connected induced subgraph $\langle S \rangle$ in G.

iii) $A(G;x)$ is a symmetric polynomial (i.e., an even or odd function of x) if and only if the degree sequence of G has either all values odd or all even.

iv) The monomial with minimum degree of $A(G;x)$ has exponent $n-\delta_1$ and the coefficients $A_{-\delta_1}(G)$ and $A_{-\delta_1+1}(G)$ are the number of vertices in G with degree δ_1 and $\delta_1 - 1$, respectively.

v) $n + \delta_n \le \mathrm{Deg}(A(G;x)) \le n + \delta_1$.

vi) $A_{\delta_1}(G)$ *is equal to the number of connected components of* G *which are* δ_1*-regular.*

Recall that by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex. The following lemma is a well known result of graph theory.

Lemma 4.1.2. *If* $r \geqslant 2$ *is a natural number and* G *is any graph with* $\delta(v) \geqslant r$ *for every* $v \in V(G)$*, then there exists a cycle* η *in* G *with* $L(\eta) \geqslant r+1$.

Theorem 4.1.3. *Let* G *be any cubic graph. Then,*

$$A(G;x) = A_{-3}(G)\, x^{n-3} + A_{-1}(G)\, x^{n-1} + A_1(G)\, x^{n+1} + A_3(G)\, x^{n+3},$$

with $A_{-3}(G) = n < m \leqslant A_{-1}(G)$ *and* $A_1(G) \geqslant A_3(G)$.

Proof. Since G is 3-regular, by Theorem 4.1.1 iv) we obtain that $A_{-3}(G) = n$ and by item iii) we have $A_{-2}(G) = A_0(G) = A_2(G) = 0$ and then

$$A(G;x) = n\, x^{n-3} + A_{-1}(G)\, x^{n-1} + A_1(G)\, x^{n+1} + A_3(G)\, x^{n+3}.$$

Note that if $u, v \in V(G)$ with $u \sim v$, then $\{u, v\}$ is an exact defensive (-1)-alliance in G. Thus, we obtain $A_{-1}(G) \geqslant m$. Besides, by Theorem 4.1.1 vi) we have that $A_3(G)$ is equal to the number of connected components in G. Assume that G has $r \geqslant 1$ connected components. Let us consider $\{G_i\}_{i=1}^{r}$ the connected components of G, and denote by g_i the girth of G_i for $1 \leqslant i \leqslant r$. Note that $g_i > 0$ for $1 \leqslant i \leqslant r$, by Lemma 4.1.2. Besides, we have that $V(g_i)$ is an exact defensive 1-alliance in G since $\delta_{V(g_i)}(v) = 2$ and $\delta_{\overline{V(g_i)}}(v) = 1$ for every $v \in V(g_i)$. So, for each connected component in G there is at least one exact defensive 1-alliance, and then $A_1(G) \geqslant A_3(G)$. $\square$

Corollary 4.1.4. *The alliance polynomial* $A(G;x)$ *is unimodal for every cubic graph* G.

The n-vertex edgeless graph or *empty graph* is a graph without edges and with n vertices, and it is commonly denoted as E_n for $n \geqslant 1$.

The following result which appear in Chapter 3 will be useful.

Theorem 4.1.5. *The empty graph* E_n *with* n *vertices is the unique graph such that*

$$A(E_n;x) = nx^n. \tag{4.1.1}$$

Theorem 4.1.6. *Let* G *be any cubic graph and* G^* *any graph. If* $A(G^*;x) = A(G;x)$ *then* G^* *is cubic with the same order, size and number of connected components of* G.

Proof. Since G is a cubic graph with order n, by Theorems 4.1.3 and 4.1.1 vi) we have

$$A(G^*; x) = A(G; x) = nx^{n-3} + A_{-1}(G)x^{n-1} + A_1(G)x^{n+1} + A_3(G)x^{n+3},$$

with $A_3(G)$ is the number of connected components of G. Let us denote by n_1 the order of G^* and by δ and Δ the minimum and the maximum degree of G^*, respectively.

By Theorem 4.1.1 iv), we have

$$n_1 - \Delta = n - 3$$

and

$$n \leq n_1.$$

Hence, $n_1 \geq n$ and $\Delta \geq 3$. Also we have

$$n_1 + \delta \leqslant n + 3$$

by Theorem 4.1.1 v).

Furthermore, if $\Delta = 3$, then $n_1 = n$ and so, G^* is 3-regular since $A_{-3}(G^*) = n$. Since G^* and G are cubic graphs with the same order, they also have the same size; Theorem 4.1.1 vi) gives that they have the same number of connected components.

We will finish the proof by checking that $\Delta = 3$.

Seeking a contradiction, assume that $\Delta > 3$ (then $n_1 > n$) and denote by $k = n_1 - n = \Delta - 3$.

Assume that $\Delta \geqslant 6$ (i.e., $k \geqslant 3$). By Theorem 4.1.5 there exists a connected component G_0 of G^* with $\delta_{G_0}(v) = \delta(v) \geqslant 1$ for every $v \in V(G_0)$; if $S = V(G_0)$, then $\delta_S(v) = \delta(v) \geqslant 1$, and so, $k_S^{(G^*)} \geqslant 1$. Hence, $A(G^*; x)$ has at least one term with exponent greater than n_1,

$$\mathrm{Deg}(A(G^*; x)) \geq n_1 + k_S^{(G^*)} > n_1 \geqslant n + 3 = \mathrm{Deg}(A(G; x)),$$

and $A(G^*; x) \neq A(G; x)$, which is a contradiction. Thus, $\Delta = 4$ or $\Delta = 5$.

Assume that $\Delta = 5$, then $n_1 = n+2$. By Theorem 4.1.1 iv), we have that G^* has exactly n vertices with degree 5; and so, by Theorem 4.1.1 iii), we have that the other two vertices of G have degree 1 or 3. Since $n_1 + \delta \leqslant n + 3$, we obtain $\delta = 1$.

Assume that G^* has two vertices v_1 and v_2 with degree 1. In this case, if $v_1 \sim v_2$, then G^* is a disconnected graph with at least one connected component which is 5-regular since $V(G^*) \setminus \{v_1, v_2\}$ induces a 5-regular subgraph G_1 of G^*. Since $V(G_1)$ is an exact defensive 5-alliance, $\mathrm{Deg}(A(G^*; x)) \geqslant n_1+5$ and we have $\mathrm{Deg}(A(G^*; x)) \geqslant n_1+5 > n+3 = \mathrm{Deg}(A(G; x))$. If $v_1 \nsim v_2$ but there exists $w \in V(G^*)$ such that $w \sim v_1$ and $w \sim v_2$, then let us consider the connected component G_2 of G^* containing $\{v_1, v_2, w\}$. The set $S = V(G_2) \setminus \{v_1, v_2, w\}$ is a defensive 3-alliance in G^*, since for any $v \in S$ we have $\delta_S(v) \geq 4$ and $\delta_{\overline{S}}(v) \leq 1$. Then, $\mathrm{Deg}(A(G^*; x)) \geq n_1 + 3 > n + 3 = \mathrm{Deg}(A(G; x))$. If $v_1 \nsim v_2$ but there not exists $w \in V(G^*)$

with $w \sim v_1$ and $w \sim v_2$, then let us consider the connected component G_3 of G^* containing v_1 and $S = V(G_3) \setminus \{v_1, v_2\}$. The set S is a defensive 3-alliance in G^*, since for all $v \in S$ we have $\delta_S(v) \geq 4$ and $\delta_{\overline{S}}(v) \leq 1$. Then, $\mathrm{Deg}(A(G^*; x)) \geq n_1 + 3 > n + 3 = \mathrm{Deg}(A(G; x))$.

Consider now the case of G^* containing two vertices v_1 and v_2 with degree 1 and 3, respectively. If $v_1 \sim v_2$, then let us consider the connected component G_4 of G^* containing $\{v_1, v_2\}$ and $S = V(G_4) \setminus \{v_1, v_2\}$. Then, S is a defensive 3-alliance in G^*, since for all $v \in S$ we have $\delta_S(v) \geq 4$ and $\delta_{\overline{S}}(v) \leq 1$. Then, $\mathrm{Deg}(A(G^*; x)) \geq n_1 + 3 > n + 3 = \mathrm{Deg}(A(G; x))$. If $v_1 \nsim v_2$, let G_5 be the connected component of G^* containing v_1 and $S = V(G_5) \setminus \{v_1\}$. Hence, S is an exact defensive 3-alliance in G^*, since $\delta_S(v_2) - \delta_{\overline{S}}(v_2) = 3 - 0$ if $v_2 \in S$ and $\delta_S(v) - \delta_{\overline{S}}(v) \geqslant 4 - 1$ for any $v \in S \setminus \{v_2\}$. Then, $\mathrm{Deg}(A(G^*; x)) \geq n_1 + 3 > n + 3 = \mathrm{Deg}(A(G; x))$.

So, it is not possible to have $\Delta = 5$.

Assume that $\Delta = 4$, then $n_1 = n + 1$. If G^* is a disconnected graph, then there exists a connected component $\langle S^* \rangle$ of G^* such that $\langle S^* \rangle$ is 4-regular and so, S^* is an exact defensive 4-alliance in G^*. Therefore, $\mathrm{Deg}(A(G^*; x)) = n_1 + 4 > n + 3 = \mathrm{Deg}(A(G; x))$. Thus, G^* is connected, and $\delta = 2$ by Theorem 4.1.1 iii). So, we have that G^* has exactly n vertices with degree 4 and other vertex w with degree 2. Let $v_1, v_2 \in V(G^*) \setminus \{w\}$ with $v_1 \neq v_2$, $v_1 \sim w$ and $v_2 \sim w$. Consider $\{u_1, \ldots, u_{n-2}\} := V(G^*) \setminus \{w, v_1, v_2\}$. Let G_i be the connected component of $\langle V(G^*) \setminus \{u_i\} \rangle \subset G^*$ containing w, and $S_i = V(G_i)$, for each $1 \leqslant i \leqslant n - 2$. Note that S_i is an exact defensive 2-alliance since $\delta_{S_i}(w) - \delta_{\overline{S_i}}(w) = 2$, for each $1 \leqslant i \leqslant n - 2$. Note that if $i \neq j$ and $u_j \notin S_i$ then $u_i \in S_j$, and so, $S_i \neq S_j$ since $u_i \notin S_i$; furthermore, if $u_j \in S_i$ then $S_i \neq S_j$ since $u_j \notin S_j$. Then, we obtain that $A_2(G^*) \geqslant n - 1$, and thus $A_3(G) \geqslant n - 1$. This contradicts Theorem 4.1.1 vi) since G is a cubic graph with order n.

So, it is not possible to have $\Delta = 4$. □

This result allows to obtain the uniqueness of the alliance polynomial of a cubic graph by computing the alliance polynomials of every cubic graph with the same order.

4.2 Computing the alliance polynomials for cubic graphs with small order

In this section, we obtain (computationally) the alliance polynomials of cubic graphs with small order, showing that they satisfy uniqueness. Theorem 4.1.6 suggest to compute the alliance polynomials of cubic graphs using Algorithm 3.1.1.

We compute the alliance polynomial of cubic graphs with orders at most 10. Since their alliance polynomials are different, these cubic graphs with small orders are characterized by their alliance polynomials.

Let G be a cubic graph with order n.

If $n = 4$ then G is isomorphic to K_4 and Theorem 4.1.6 gives the uniqueness.

If $n = 6$ then G is isomorphic either to $K_{3,3}$ or to the Cartesian product $P_2 \square C_3$; hence, Theorem 4.1.6 gives the uniqueness of their alliance polynomials since $A(K_{3,3}; x) = 6x^3 + 33x^5 + 15x^7 + x^9$ and $A(P_2 \square C_3; x) = 6x^3 + 33x^5 + 11x^7 + x^9$. Notice that these alliance polynomials are equal except for the coefficient of x^7; it is an interesting fact since many parameters of these graphs are different.

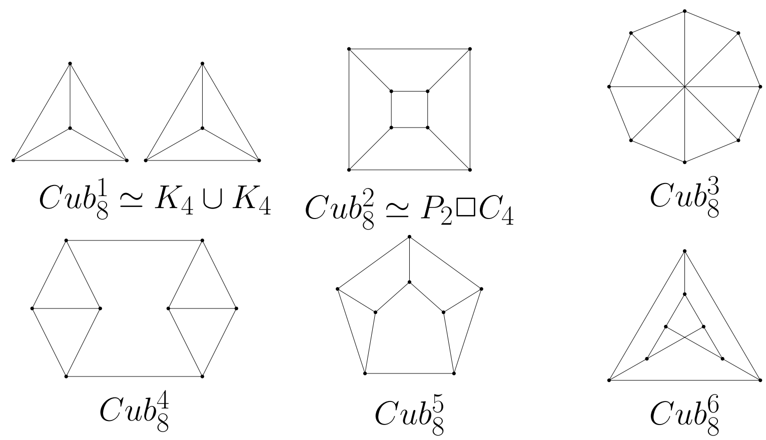


Figure 4.1: Cubic graphs with order 8.

Graph	Alliance polynomial	Graph	Alliance polynomial
Cub_8^1	$8x^5 + 12x^7 + 8x^9 + 2x^{11}$	Cub_8^4	$8x^5 + 94x^7 + 20x^9 + x^{11}$
Cub_8^2	$8x^5 + 128x^7 + 30x^9 + x^{11}$	Cub_8^5	$8x^5 + 118x^7 + 24x^9 + x^{11}$
Cub_8^3	$8x^5 + 132x^7 + 32x^9 + x^{11}$	Cub_8^6	$8x^5 + 126x^7 + 28x^9 + x^{11}$

Table 4.1: Alliance polynomials of cubic graph with order 8.

Figure 4.1 shows the cubic graphs with order 8 and Table 4.1 their alliance polynomials; since they are different, Theorem 4.1.6 gives their uniqueness.

Notice that except for Cub_8^1 (a non-connected graph), the coefficients of the others are quite alike; Cub_8^3 and Cub_8^4 have the largest and smallest coefficients, respectively. Also, these alliance polynomials are equal except for the coefficients of x^7 and x^9; besides, their coefficients except for the leading one are even numbers.

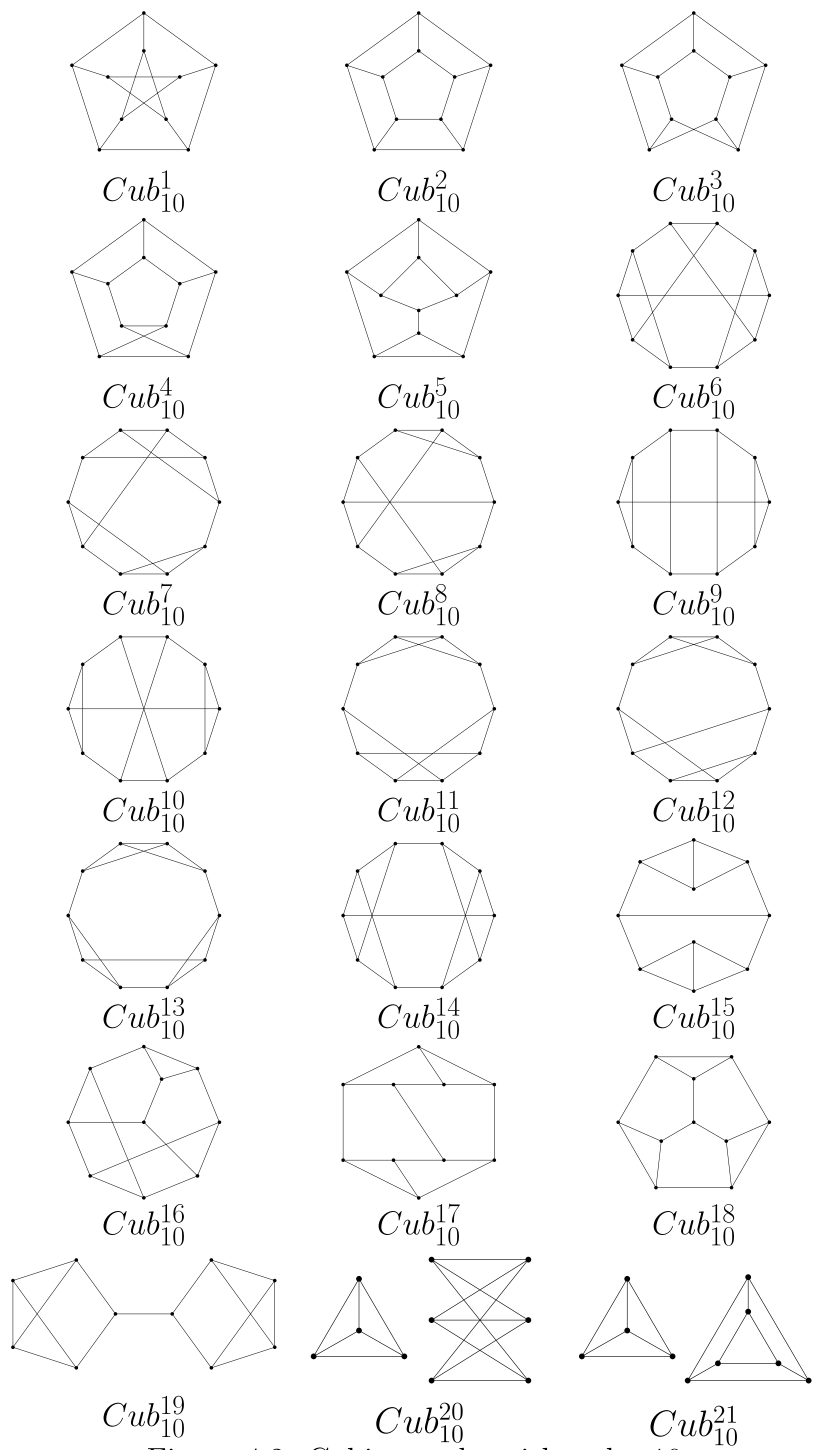


Figure 4.2: Cubic graphs with order 10.

Graph	Alliance polynomial	Graph	Alliance polynomial	Graph	Alliance polynomial
Cub_{10}^{1}	$10x^7+480x^9+77x^{11}+x^{13}$	Cub_{10}^{8}	$10x^7+407x^9+56x^{11}+x^{13}$	Cub_{10}^{15}	$10x^7+272x^9+42x^{11}+x^{13}$
Cub_{10}^{2}	$10x^7+425x^9+67x^{11}+x^{13}$	Cub_{10}^{9}	$10x^7+357x^9+53x^{11}+x^{13}$	Cub_{10}^{16}	$10x^7+419x^9+62x^{11}+x^{13}$
Cub_{10}^{3}	$10x^7+435x^9+65x^{11}+x^{13}$	Cub_{10}^{10}	$10x^7+387x^9+55x^{11}+x^{13}$	Cub_{10}^{17}	$10x^7+372x^9+54x^{11}+x^{13}$
Cub_{10}^{4}	$10x^7+451x^9+69x^{11}+x^{13}$	Cub_{10}^{11}	$10x^7+307x^9+55x^{11}+x^{13}$	Cub_{10}^{18}	$10x^7+351x^9+50x^{11}+x^{13}$
Cub_{10}^{5}	$10x^7+404x^9+61x^{11}+x^{13}$	Cub_{10}^{12}	$10x^7+304x^9+48x^{11}+x^{13}$	Cub_{10}^{19}	$10x^7+176x^9+36x^{11}+x^{13}$
Cub_{10}^{6}	$10x^7+462x^9+67x^{11}+x^{13}$	Cub_{10}^{13}	$10x^7+267x^9+43x^{11}+x^{13}$	Cub_{10}^{20}	$10x^7+39x^9+19x^{11}+2x^{13}$
Cub_{10}^{7}	$10x^7+393x^9+61x^{11}+x^{13}$	Cub_{10}^{14}	$10x^7+424x^9+67x^{11}+x^{13}$	Cub_{10}^{21}	$10x^7+39x^9+15x^{11}+2x^{13}$

Table 4.2: Alliance polynomials of cubic graph with order 10.

Figure 4.2 shows the cubic graphs with order 10 and Table 4.2 their alliance polynomials. Since they are different, Theorem 4.1.6 gives their uniqueness.

There are two non-connected cubic graphs Cub_{10}^{20} and Cub_{10}^{21}; notice that their alliance polynomials are equal except for the coefficient of x^{11}, this is an expected result since $Cub_{10}^{20} \simeq K_4 \cup K_{3,3}$, $Cub_{10}^{21} \simeq K_4 \cup P_2 \square C_3$ and $A(K_{3,3};x)$, $A(P_2 \square C_3;x)$ are equal except for one coefficient.

Notice that, except for the two non-connected graphs, the coefficients of the polynomials of cubic graphs with order 10 are similar; Cub_{10}^{1} (Petersen's graph) and Cub_{10}^{19} have the largest and smallest coefficients, respectively. Furthermore, the coefficients of x^9 of the alliance polynomials of connected graphs are different; however, some of them are equal except for this coefficient. For example, $A(Cub_{10}^{2};x)$, $A(Cub_{10}^{6};x)$ and $A(Cub_{10}^{14};x)$ have a term of the form $67x^{11}$; note that $A(Cub_{10}^{2};x) = A(Cub_{10}^{14};x) + x^9$. There are two other couples of these alliance polynomials with just one different coefficient: $A(Cub_{10}^{5};x)$, $A(Cub_{10}^{7};x)$ and $A(Cub_{10}^{10};x)$, $A(Cub_{10}^{11};x)$.

Proposition 4.2.1. *Every cubic graph of order at most* 10 *is characterized by its alliance polynomial.*

The alliance polynomial $A(G;x)$ is of special interest, since it counts the connected induced subgraphs of G. In Chapter 3 Theorem 3.1.4 ii) will be useful to obtain the following results.

The previous computations of alliance polynomials of cubic graphs with at most 10 vertices allow to obtain the following consequence, see Table 4.3.

Corollary 4.2.2. *Any two cubic graphs with order at most* 10 *have different number of connected induced subgraphs, and different number of cut vertex sets.*

G	$A(G;1)$	$2^n-1-A(G;1)$	G	$A(G;1)$	$2^n-1-A(G;1)$	G	$A(G;1)$	$2^n-1-A(G;1)$
Cub_4^1	15	0	Cub_6^1	55	8	Cub_6^2	51	12
Cub_8^1	30	225	Cub_8^2	167	88	Cub_8^3	173	82
Cub_8^4	123	132	Cub_8^5	151	104	Cub_8^6	163	92
Cub_{10}^1	568	455	Cub_{10}^2	503	520	Cub_{10}^3	511	512
Cub_{10}^4	531	492	Cub_{10}^5	476	547	Cub_{10}^6	540	483
Cub_{10}^7	465	558	Cub_{10}^8	474	549	Cub_{10}^9	421	602
Cub_{10}^{10}	453	570	Cub_{10}^{11}	373	650	Cub_{10}^{12}	363	660
Cub_{10}^{13}	321	702	Cub_{10}^{14}	502	521	Cub_{10}^{15}	325	698
Cub_{10}^{16}	492	531	Cub_{10}^{17}	437	586	Cub_{10}^{18}	412	611
Cub_{10}^{19}	223	800	Cub_{10}^{20}	70	953	Cub_{10}^{21}	66	957

Table 4.3: Numbers of connected induced subgraphs and cut vertex sets of cubic graphs.

We recall the distinctive power of the alliance polynomial of a graph. This is an interesting difference with others well-known polynomials of graphs.

We say that a graph G is characterized by a graph polynomial f if for every graph G' such that $f(G')=f(G)$ we have that G' is isomorphic to G. A set of graphs K is characterized by a graph polynomial f if every graph $G\in K$ is characterized by f.

In Chapter 3 we prove that path, cycle, complete, complete without one edge and star graphs are characterized by its alliance polynomials. In this work, we prove that the family of alliance polynomials of cubic graphs is a special one, since it does not contain alliance polynomials of graphs which are not cubic; and Proposition 4.2.1 gives that the cubic graphs with at most 10 vertices are characterized by their alliance polynomials. Furthermore, in the next Chapter we prove a similar result for the family of alliance polynomials of Δ-regular connected graphs with $\Delta\leqslant 5$, i.e., it does not contain alliance polynomials of graphs which are not connected Δ-regular.

The computations in this section and the results in Chapter 3 suggest the conjecture that every graph can be characterized by its alliance polynomial, although it seems hard to be proved.

Chapter 5

Alliances polynomial of regular graphs

The main aim of this chapter is to obtain further results about the alliance polynomial of regular graphs (graphs with all vertices with the same degree), since they are a very interesting class of graphs.

We study the alliance polynomials of regular graphs and their coefficients in Section 5.1. In Section 5.2 we focus on the alliance polynomials of connected regular graphs; besides, we prove that the family of alliance polynomials of connected Δ-regular graphs with small degree is a very special one, since it does not contain alliance polynomials of graphs which are not connected Δ-regular.

5.1 Computing the alliance polynomials of regular graphs

Below, a quick reminder of some previous results for alliance polynomials of general graphs (not necessarily regular) which appear in Chapter 3 and that will be useful (see Proposition 3.1.3, Theorem 3.1.5 and Theorem 3.1.4). We denote by $\mathrm{Deg}(p)$ the degree of the polynomial p.

Theorem 5.1.1. *Let G be any graph. Then $A(G;x)$ satisfies the following properties:*

i) $A(G;x)$ does not have zeros in the interval $(0,\infty)$.

ii) $A(G;1) < 2^n$, and it is the number of connected induced subgraphs in G.

iii) If G has at least an edge and its degree sequence has exactly r different values, then $A(G;x)$ has at least $r+1$ terms.

iv) $A(G;x)$ is a symmetric polynomial (i.e., an even or an odd function of x) if and only if the degree sequence of G has either all values odd or all even.

v) $A_{-\Delta}(G)$ *and* $A_{-\Delta+1}(G)$ *are the number of vertices in* G *with degree* Δ *and* $\Delta - 1$*, respectively.*

vi) $A_{\Delta}(G)$ *is equal to the number of connected components in* G *which are* Δ*-regular.*

vii) $n + \delta \leq \mathrm{Deg}(A(G;x)) \leq n + \Delta$.

We show now some results about the alliance polynomial of regular graphs and their coefficients. If p is a polynomial we denote by $\mathrm{Deg}_{min}(p)$ the minimum degree of their non-zero coefficients.

Theorem 5.1.2. *For any* Δ*-regular graph* G*, its alliance polynomial* $A(G;x)$ *satisfies the following properties:*

i) $A_{-\Delta+2i}(G)$ *is the number of connected induced subgraphs of* G *with minimum degree* i $(0 \leqslant i \leqslant \Delta)$.

ii) $\mathrm{Deg}_{\min}\big(A(G;x)\big) = n - \Delta$ *and* $A_{-\Delta}(G) = n$.

iii) $\mathrm{Deg}\big(A(G;x)\big) = n + \Delta$. *Furthermore,*

$$n = \frac{\mathrm{Deg}_{\min}\big(A(G;x)\big) + \mathrm{Deg}\big(A(G;x)\big)}{2} \tag{5.1.1}$$

and

$$m = A_{-\Delta}(G)\frac{\mathrm{Deg}\big(A(G;x)\big) - \mathrm{Deg}_{\min}\big(A(G;x)\big)}{4} = \frac{\mathrm{Deg}^2\big(A(G;x)\big) - \mathrm{Deg}^2_{\min}\big(A(G;x)\big)}{8}.$$

iv) $1 \leqslant A_{\Delta}(G) \leqslant n/(\Delta+1)$. *Furthermore,* G *is a connected graph if and only if* $A_{\Delta}(G) = 1$.

v) If $\Delta > 0$*, then* $A_{-\Delta+2}(G) \geqslant m$ *and* $A_{\Delta-2}(G) \geqslant n+n_0$ *with* n_0 *the number of cut vertices; in particular,* $A_{\Delta-2}(G) \geqslant n$.

vi) $A(G;x)$ *is either an even or an odd function of* x*. Furthermore,* $A(G;x)$ *is an even function of* x *if and only if* $n + \Delta$ *is even.*

vii) The unique real zero of $A(G;x)$ *is* $x = 0$*, and its multiplicity is* $n - \Delta$.

Proof. We prove each item separately.

i) Let us consider $S \subset V$ with S an exact defensive $(2i - \Delta)$-alliance in G. Then, we have for all $v \in S$

$$2\delta_S(v) \geqslant \delta(v) + 2i - \Delta = \Delta + 2i - \Delta \quad \Leftrightarrow \quad \delta_S(v) \geqslant i,$$

besides, the equality holds at some $w \in S$. We have the result since $A_{-\Delta+2i}(G)$ is the number of exact defensive $(2i - \Delta)$-alliance in G.

ii) One can check directly that if S is a single vertex, then S is an exact defensive $(-\Delta)$-alliance; furthermore, it is clear that any $S \subseteq V$ with $\langle S\rangle$ connected and more than one vertex is not an exact defensive $(-\Delta)$-alliance, since for any $v \in S$ we have

$$\delta_S(v) - \delta_{\overline{S}}(v) \geq 1 - (\Delta - 1) = -\Delta + 2. \tag{5.1.2}$$

Consequently $A_{-\Delta}(G) = n$, since G is a Δ-regular graph.

iii) The maximum value in $\mathcal{K}$ is Δ, so $Deg\big(A(G;x)\big)$ is at most $n+\Delta$. We have that each connected component of G is an exact defensive Δ-alliance since $\delta(v) = \Delta$ for any vertex v. Then, $A_\Delta(G) > 0$ and $Deg\big(A(G;x)\big) = n + \Delta$. Besides, the other results are consequences of the well known fact $2m = n\Delta$ and the previous results.

iv) By item i), $A_\Delta(G)$ is the number of connected induced subgraphs of G with minimum degree Δ; hence, $A_\Delta(G)$ is the number of connected components of G. Besides, since any connected component has cardinality greater than Δ, we obtain the upper bound of $A_\Delta(G)$.

v) If $u, v \in V$ with $u \sim v$, then $S := \{u, v\}$ is an exact defensive $(2-\Delta)$-alliance since $1 = \delta_S(u) = \delta_{\overline{S}}(u) + 2 - \Delta$ and $1 = \delta_S(v) = \delta_{\overline{S}}(v) + 2 - \Delta$. Thus, we obtain $A_{-\Delta+2}(G) \geqslant m$. Note that if $\Delta = 1$, we have the second inequality. Assume that $\Delta \geqslant 2$. Without loss of generality we can assume that G is connected; otherwise, it suffices to analyze each connected component of G. Let us define $S_v := V \setminus \{v\}$ for any $v \in V$. Since $\delta_{S_v}(u) \geqslant \Delta - 1$, $\delta_{\overline{S_v}}(u) \leqslant 1$ for every $u \in S_v$ and both equalities hold for every $w \in N(v)$, we have that S_v is an exact defensive $(\Delta-2)$-alliance if v is a non-cut vertex, or contains at least two exact defensive $(\Delta-2)$-alliances if v is a cut vertex.

vi) The first statement is a consequence of Theorem 5.1.1 iv). Consider an exact defensive k-alliance S in G. So, there exists $v \in S$ with

$$2\delta_S(v) = \delta(v) + k = \Delta + k.$$

Then, $\Delta \equiv k \pmod 2$, $n + k \equiv n + \Delta \pmod 2$ and we have the result.

vii) Since $Deg_{\min}\big(A(G;x)\big) = n - \Delta$, we have that $x = 0$ is a zero of $A(G;x)$ with multiplicity $n - \Delta$. The positivity of all coefficients of $A(G;x)$ gives $A(G;x) \neq 0$ for every $x > 0$. Finally, by item vi), $A(G;x) = (-1)^{n+\Delta} A(G;-x) \neq 0$ for every $x < 0$.

$\square$

Theorem 5.1.3. *Let G be any connected graph. Then G is regular if and only if $A_\Delta(G) = 1$.*

Proof. If G is regular, then by Theorem 5.1.1 vi) we obtain $A_\Delta(G) = 1$. Besides, if $A_\Delta(G) = 1$, then there is an exact defensive Δ-alliance S in G with $\delta_S(v) \geqslant \delta_{\bar{S}}(v) + \Delta \geqslant \Delta$ (i.e., $\delta_S(v) = \Delta$ and $\delta_{\overline{S}}(v) = 0$) for every $v \in S$. So, the connectivity of G gives that G is a Δ-regular graph. □

Theorem 5.1.4. *Let G_1, G_2 be two regular graphs. If $A(G_1; x) = A(G_2; x)$, then G_1 and G_2 have the same order, size, degree and number of connected components.*

Proof. Let n_1, n_2 be the orders of G_1 and G_2, respectively, and Δ_1, Δ_2 the degrees of G_1 and G_2, respectively. Then, by Theorem 5.1.2 ii) and iii) we have

$$n_1 - \Delta_1 = n_2 - \Delta_2 \quad \text{and} \quad n_1 + \Delta_1 = n_2 + \Delta_2$$

and we conclude

$$n_1 = n_2 \quad \text{and} \quad \Delta_1 = \Delta_2.$$

Hence, both graphs have the same size. Finally, since $A_{\Delta_1}(G_1) = A_{\Delta_2}(G_2)$, they have the same number of connected components by Theorem 5.1.1 vi). □

Corollary 5.1.5. *Let G_1, G_2 be two regular graphs with orders n_1 and n_2, and degrees Δ_1 and Δ_2, respectively. If $n_1 \neq n_2$ or $\Delta_1 \neq \Delta_2$, then $A(G_1; x) \neq A(G_2; x)$.*

The next theorem characterizes the degree of any regular graph by the number of non-zero coefficients of its alliance polynomial.

Theorem 5.1.6. *Let G be any Δ-regular graph with order n. Then $A(G; x)$ has $\Delta + 1$ non-zero coefficients. Furthermore,*

$$A(G; x) = \sum_{i=0}^{\Delta} A_{\Delta-2i}(G)\ x^{n+\Delta-2i},$$

with $A_{-\Delta}(G) = n$, $A_\Delta(G) \geqslant 1$, and

$$A_{\Delta-2i}(G) \geqslant \frac{n\binom{\Delta}{i}}{\min\{\Delta, n-i\}} \quad \textit{for} \ \ 1 \leqslant i \leqslant \Delta - 1 \ \textit{if} \ \Delta > 0.$$

Proof. Since G is Δ-regular, by Theorem 5.1.2 we have $A_{-\Delta}(G) = n$, $A_\Delta(G) \geqslant 1$ and $A(G; x)$ is an even or an odd function of x. Assume now that $\Delta > 0$ and fix $1 \leqslant i \leqslant \Delta - 1$. Let us consider $u \in V$ and $v_1, \ldots, v_i$ different vertices in $N(u)$. Denote by $S_u := V \setminus \{v_1, \ldots, v_i\}$. Then, we have that $\delta_{S_u}(v) \geqslant \Delta - i$ and $\delta_{\overline{S_u}}(v) \leqslant i$ for every $v \in S_u$; furthermore, the equalities hold at u. Let $S_u^* \subset S_u$ such that $\langle S_u^* \rangle$ is the connected component of $\langle S_u \rangle$ which contains u. So, S_u^* is an exact defensive $(\Delta - 2i)$-alliance and $A_{\Delta-2i}(G) > 0$. Since each set S_u^* can appear at most $n - i$ times (once for each S_w^* with $w \in V \setminus \{v_1, \ldots, v_i\}$), and at most Δ times (once for each S_w^* with $w \sim v_1$), we obtain $A_{\Delta-2i}(G) \geqslant n\binom{\Delta}{i}/\min\{\Delta, n-i\}$. □

Recall that *Hamiltonian cycle* is a cycle in a graph that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*. The following theorem is a well known result in graph theory which will be useful.

In what follows we will use the following notation: for any $A, B \subset V$, we denote by $N(A, B)$ the number of edges with one endpoint in A and the other endpoint in B.

Theorem 5.1.7. *Let G be any Δ-regular graph with order $n < 2\Delta$. Then $A_{\Delta-2}(G) = n$.*

Proof. Notice that $\Delta \geqslant 2$, since otherwise, such a graph G does not exist; furthermore, $n \geqslant \Delta + 1 \geqslant 3$. We have that G is a Hamiltonian graph by Theorem 1.6.5. Besides, by Theorem 5.1.2 i), we have that $A_{\Delta-2}(G)$ is the number of connected induced subgraphs of G with minimum degree $\Delta - 1$. Let us consider $u \in V$ and define $S_u := V \setminus \{u\}$. Since G is a Hamiltonian graph, $\langle S_u \rangle$ is connected. Besides, we have $\delta_{S_u}(v) \geqslant \Delta - 1 \geqslant \delta_{\overline{S_u}}(v) + \Delta - 2$ for all $v \in S_u$ and the equality holds at $w \in N(u)$. So, S_u is an exact defensive $(\Delta - 2)$-alliance in G and $A_{\Delta-2}(G) \geqslant n$.

Seeking for a contradiction assume that there is an exact defensive $(\Delta-2)$-alliance $S \subset V$ with $|S| \leqslant n - 2$. Notice that $|S| \geqslant \Delta > n/2$, by Theorem 5.1.2 i). Then, since any vertex in S has degree Δ in G with at most one edge among S and $\overline{S}$, we have

$$N(S, S) + N(S, \overline{S}) = \frac{|S|\Delta}{2} + \frac{N(S, \overline{S})}{2} \leqslant \frac{|S|\Delta}{2} + \frac{|S|}{2} = \frac{|S|(\Delta + 1)}{2}.$$

Besides, since $|\overline{S}| = n - |S|$, we have

$$N(\overline{S}, \overline{S}) \leqslant \frac{(n - |S|)(n - |S| - 1)}{2}.$$

If m denotes the size of G, then

$$\begin{aligned} 0 &= 2\left(N(S, S) + N(S, \overline{S}) + N(\overline{S}, \overline{S})\right) - 2m \\ &\leqslant |S|(\Delta + 1) + (n - |S|)(n - |S| - 1) - n\Delta \\ &= |S|^2 + |S|(\Delta + 2 - 2n) + n^2 - n - n\Delta. \end{aligned}$$

Define $P(x) := x^2 + x(\Delta + 2 - 2n) + n^2 - n - n\Delta$; then $P(|S|) \geqslant 0$. Since

$$\begin{aligned} P\left(\frac{n}{2}\right) &= \frac{n^2}{4} + \frac{n}{2}(\Delta + 2 - 2n) + n^2 - n - n\Delta \\ &= \frac{n^2}{4} + \frac{n\Delta}{2} + n - n^2 + n^2 - n - n\Delta \\ &= \frac{n}{4}(n - 2\Delta) < 0 \end{aligned}$$

and

$$\begin{aligned} P(n - 2) &= (n - 2)^2 + (n - 2)(\Delta + 2 - 2n) + n^2 - n - n\Delta \\ &= (n - 2)^2 + (n - 2)(\Delta - n) - (n - 2)^2 + n^2 - n - n\Delta \\ &= n - 2\Delta < 0, \end{aligned}$$

we obtain that $P(|S|) < 0$. This is the contradiction we were looking for, so, there not exists an exact defensive $(\Delta-2)$-alliance S with $|S| \leqslant n-2$. This finishes the proof since V is an exact defensive Δ-alliance. $\square$

Lemma 5.1.8. *Let G be any Δ-regular graph with order n, $\Delta \geqslant 3$ and $2\Delta \leqslant n \leqslant 2\Delta+1$. If G contains two cliques of cardinality Δ, then these cliques are disjoint. In particular, G contains at most two cliques of cardinality Δ.*

Proof. Seeking for a contradiction, assume that there exist $S_1, S_2 \subset V$ cliques of cardinality Δ with $S_1 \cap S_2 \neq \emptyset$. Denote by r the number $r := |S_1 \cap S_2|$; then $1 \leqslant r \leqslant \Delta - 1$. Note that for any $v \in S_1 \cap S_2$ we have $\delta_{S_1 \cup S_2}(v) = |S_1| - 1 + |S_2 \setminus S_1| = \Delta - 1 + \Delta - r$, so, we obtain $r = \Delta - 1$. Then, we have $|S_1 \cup S_2| = \Delta + 1$ and $\Delta - 1 \leqslant |\overline{S_1 \cup S_2}| \leqslant \Delta$. Besides, we have $N(S_1 \cup S_2, \overline{S_1 \cup S_2}) = 2 = |(S_1 \cup S_2) \setminus (S_1 \cap S_2)|$ and, since $|\overline{S_1 \cup S_2}| \leqslant \Delta$, $N(\overline{S_1 \cup S_2}, S_1 \cup S_2) \geqslant |\overline{S_1 \cup S_2}| \geqslant \Delta - 1$. Since $N(S_1 \cup S_2, \overline{S_1 \cup S_2}) = N(\overline{S_1 \cup S_2}, S_1 \cup S_2)$, we obtain $\Delta = 3$ and $n = 6$; therefore, G is a graph isomorphic to either $K_{3,3}$ or the Cartesian product $P_2 \square K_3$. Thus, we obtain that there are not two non-disjoint cliques in G with cardinality Δ. This finishes the proof since, by $n \leqslant 2\Delta + 1$, it is impossible to have three disjoint cliques of cardinality Δ contained in G. $\square$

Remark 5.1.9. *If G is a Δ-regular graph with $n \leqslant 2\Delta+1$, then G does not contain a clique of cardinality greater than Δ, since $2(\Delta+1) > 2\Delta + 1 \geqslant n$.*

Remark 5.1.10. *Let G be any Δ-regular graph with order n and $\Delta \geqslant 1$ such that G has two disjoint cliques of cardinality Δ. Then*

1. *If $n = 2\Delta$, then G is isomorphic to the Cartesian product graph $P_2 \square K_\Delta$.*
2. *If $n = 2\Delta + 1$, then Δ is even (since $n\Delta = 2m$) and G can be obtained from $P_2 \square K_\Delta$ by removing $\Delta/2$ copy edges of P_2 and connecting the Δ vertices with degree $\Delta - 1$ with a new vertex. In particular, if S is a clique of cardinality Δ in G, then $\overline{S}$ is not an exact defensive $(\Delta-2)$-alliance.*

Theorem 5.1.11. *Let G be any Δ-regular graph with order n, size m, $\Delta \geqslant 3$ and $2\Delta \leqslant n \leqslant 2\Delta + 1$. Then $n \leqslant A_{\Delta-2}(G) \leqslant n + m + 2$.*

Proof. Note that if $\Delta = 3$ then $n = 6$, and G is a graph isomorphic to either $K_{3,3}$ or $P_2 \square K_3$. Thus, a simple computation gives $6 \leqslant A_1(K_{3,3}) = 15 \leqslant 6 + 9 + 2$ and $6 \leqslant A_1(P_2 \square K_3) = 11 \leqslant 6 + 9 + 2$.

Assume now that $\Delta \geqslant 4$. Clearly, G is a connected graph and $\operatorname{diam} G = 2$, since $2\Delta > n - 2$.

First we prove that G does not have cut vertices. If $n = 2\Delta$, then G is a Hamiltonian graph by Theorem 1.6.5. If $n = 2\Delta+1$, seeking for a contradiction assume that there is a cut vertex w in G. Let $S_1, S_2 \subset V$ with $S_1 \cup S_2 \cup \{w\} = V$ such that $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are disjoint. Without loss of generality we can assume that $|S_1| \leqslant \Delta \leqslant |S_2|$. Since $\delta_{S_1}(w), \delta_{S_2}(w) \geqslant 1$,

$\delta_{S_1}(w)+\delta_{S_2}(w)=\Delta$ and $\delta_{S_1}(u)\leqslant |S_1|-1\leqslant \Delta-1$ for all $u\in S_1$, we have $\delta_{S_1}(w)=|S_1|$ and $\delta_{S_1}(u)=\Delta-1$ for all $u\in S_1$. Then, we obtain that $|S_1|=\Delta$, but this is a contradiction since $\delta_{S_1}(w)=\Delta-\delta_{S_2}(w)\leqslant \Delta-1<\Delta=|S_1|=\delta_{S_1}(w)$. Then, G does not have cut vertices.

By Theorem 5.1.2 i), we have that $A_{\Delta-2}(G)$ is the number of connected induced subgraphs of G with minimum degree $\Delta-1$; thus, any exact defensive $(\Delta-2)$-alliance S in G verifies $|S|\geqslant\Delta$. Let us consider $u\in V$ and denote by $S_u:=V\setminus\{u\}$. Since G does not have cut vertices, $\langle S_u\rangle$ is connected. Besides, we have $\delta_{S_u}(v)\geqslant\Delta-1\geqslant\delta_{\overline{S_u}}(v)+\Delta-2$ for all $v\in S_u$ and the equality holds for every $v\in N(u)$; so, S_u is an exact defensive $(\Delta-2)$-alliance in G. Thus, $A_{\Delta-2}(G)\geqslant n$.

Let us consider $u_1,u_2\in V$ with $u_1\neq u_2$ and define $S_{u_1,u_2}:=V\setminus\{u_1,u_2\}$. If $u_1\nsim u_2$, then there is $w\in V$ with $u_1,u_2\in N(w)$ since $\delta(u_1)+\delta(u_2)=2\Delta>|S_{u_1,u_2}|$; in fact, S_{u_1,u_2} is not a defensive $(\Delta-2)$-alliance in G. So, S_{u_1,u_2} may be an exact defensive $(\Delta-2)$-alliance in G, if $u_1\sim u_2$; then there are at most m exact defensive $(\Delta-2)$-alliances with $n-2$ vertices. Consider now $u_1,\dots,u_r\in V$ with $3\leqslant r\leqslant\Delta-1$ and $u_i\neq u_j$ if $i\neq j$. Note that $S_r:=V\setminus\{u_1,\dots,u_r\}$ is not a defensive $(\Delta-2)$-alliance in G if $r>3$, since $N(\overline{S_r},S_r)\geqslant r(\Delta-r+1)=2\Delta-r+(r-2)(\Delta-r)>2\Delta+1-r\geqslant|S_r|$. Besides, if $r=3$ and $\Delta\geqslant5$ (thus $\Delta-r\geqslant2$) we have the same inequality and then S_r is not a defensive $(\Delta-2)$-alliance in G. Note that, if $r=3$ and $n=2\Delta$, then $N(\overline{S_r},S_r)\geqslant 2\Delta-r+(r-2)(\Delta-r)>2\Delta-r=n-r\geqslant|S_r|$ and we also conclude that S_r is not a defensive $(\Delta-2)$-alliance in G. However, if $r=3$, $\Delta=4$ and $n=2\Delta+1$ (thus, $n=9$), then S_r may be an exact defensive $(\Delta-2)$-alliance in G. But a simple computation gives that these five graphs G verify $A_2(G)<9+18+2$.

We analyze separately the cases $n=2\Delta$ and $n=2\Delta+1$. Assume first that $n=2\Delta$. We only need to compute the possible exact defensive $(\Delta-2)$-alliances in G with cardinality Δ, since every defensive $(\Delta-2)$-alliance has at least Δ vertices and $n=2\Delta$. If S is an exact defensive $(\Delta-2)$-alliance in G, then S is a clique of cardinality Δ and by Lemma 5.1.8 there are at most 2 exact defensive $(\Delta-2)$-alliances with Δ vertices. Assume now that $n=2\Delta+1$. So, Δ is even. We only need to compute the possible exact defensive $(\Delta-2)$-alliances in G with cardinalities Δ and $\Delta+1$. If S is an exact defensive $(\Delta-2)$-alliance in G with $|S|=\Delta+1$, then $\delta_S(u)\geqslant\Delta-1$ for every $u\in S$ and $\delta_S(u_0)=\Delta$ for some $u_0\in S$, since otherwise $\delta_S(u)=\Delta-1$ for every $u\in S$ and we conclude $(\Delta+1)(\Delta-1)=|S|(\Delta-1)=2m_S$, with m_S the size of $\langle S\rangle$, which is not possible since Δ is even. Hence, $N(\overline{S},S)\leqslant\Delta$; furthermore, since $|\overline{S}|=\Delta$, $\delta_S(v)\geqslant1$ for all $v\in\overline{S}$, and so, $\overline{S}$ is a clique. If S is an exact defensive $(\Delta-2)$-alliance in G with $|S|=\Delta$, then $\delta_S(u)\geqslant\Delta-1$ for every $u\in S$ and S is a clique of cardinality Δ. Lemma 5.1.8 completes the proof since if G has two cliques of cardinality Δ, then they are disjoint and Remark 5.1.10 gives that $\overline{S}$ is not an exact defensive $(\Delta-2)$-alliance in G. □

Theorem 5.1.12. *Let G be a Δ-regular connected graph with order n and let G^* be a graph with order n_1 and, minimum and maximum degrees δ_1 and Δ_1, respectively. If $A(G^*;x)=A(G;x)$, then G^* is a connected graph with exactly n vertices of degree $\Delta_1=\Delta+n_1-n$,*

$n_1 \geqslant n$, $\Delta_1 \geqslant \Delta$ and $\delta_1 \equiv \Delta_1 (mod\ 2)$.

Furthermore, if $n_1 > n$, then the following inequalities hold:

$$\frac{\Delta_1 + \delta_1 + 2}{2} \leqslant \Delta. \tag{5.1.3}$$

$$\delta_1 + 2 < \Delta < \Delta_1, \tag{5.1.4}$$

$$\Delta + 1 \leqslant \Delta_1 \leqslant 2\Delta - 3, \tag{5.1.5}$$

$$\delta_1 + 4 \leqslant \Delta_1. \tag{5.1.6}$$

Proof. Since $A(G^*; x) = A(G; x)$ is a symmetric polynomial by Theorem 5.1.2 vi), we conclude that $\delta_1 \equiv \Delta_1(\text{mod } 2)$ by Theorem 5.1.1 iv). By Theorems 5.1.1 v) and 5.1.2 ii), G^* has n vertices of maximum degree Δ_1, so, $n_1 \geq n$; besides, $n_1 - \Delta_1 = n - \Delta$. Note that if $n_1 = n$ then G^* is a Δ-regular graph with $A_\Delta(G^*) = 1$, so, Theorem 5.1.3 gives that G^* is a connected graph.

Assume that $n_1 > n$. Denote by $t := n_1 - n = \Delta_1 - \Delta$. Let $v_1, \ldots, v_n \in V(G^*)$ be the vertices in G^* with degree Δ_1 and define $S := \{v_1, \ldots, v_n\}$. Note that for any $v \in S$ we have $\delta_S(v) \geqslant \Delta_1 - t = t + (\Delta_1 - 2t) \geqslant \delta_{\overline{S}}(v) + \Delta_1 - 2t$; hence, S contains a defensive $(\Delta_1 - 2t)$-alliance S_1 and $k_{S_1} \geqslant \Delta_1 - 2t$. Therefore, there is at least one term of degree greater or equal than $n_1 + \Delta_1 - 2t$ in $A(G^*; x)$. Since $x^{n_1+\Delta_1-2t} = x^{n+\Delta}$, S_1 is an exact defensive $(\Delta_1 - 2t)$-alliance in G^*. Finally, note that if $\langle S \rangle$ is not a connected subgraph (i.e., $S_1 \neq S$), then in $A(G^*; x)$ appear at least two terms $x^{n+\Delta}$, but this is a contradiction since $A(G; x)$ is a monic polynomial by Theorem 5.1.1 vi). Hence, $\langle S \rangle$ is connected. Since the degree of $A(G^*; x)$ is $n + \Delta = n_1 + \Delta_1 - 2t$, then S is an exact defensive $(\Delta_1 - 2t)$-alliance in G^*; therefore, there exists $1 \leqslant j \leqslant n$ such that $\Delta_1 = \delta(v_j) = 2\delta_{\overline{S}}(v_j) + \Delta_1 - 2t$, and we have $\delta_{\overline{S}}(v_j) = t$. Since $|S| = n = n_1 - t$ and $|\overline{S}| = t$, $\overline{S} \subseteq N(v_j)$ and G^* is a connected graph.

Also, since G^* is connected, $A(G^*; x) = A(G; x)$, $k_S = \Delta_1 - 2t$ and $k_{V(G^*)} = \delta_1$, we have $\delta_1 \leqslant \Delta_1 - 2t$. We are going to prove $\delta_1 < \Delta_1 - 2t$; seeking for a contradiction assume that $\delta_1 = \Delta_1 - 2t$. Since G^* is connected, $k_{V(G^*)} = \delta_1 = \Delta_1 - 2t = k_S$ and this contradicts that $A(G^*; x)$ is a monic polynomial. Therefore, $\delta_1 < \Delta_1 - 2t$. But, since $\delta_1 \equiv \Delta_1(\text{mod } 2)$ we obtain $\delta_1 + 2 \leqslant \Delta_1 - 2(\Delta_1 - \Delta) = 2\Delta - \Delta_1$, so (5.1.3) holds.

Besides, since $\Delta_1 > \Delta$, (5.1.3) gives $\delta_1 + 2 < \Delta$, and so, (5.1.4) holds. Furthermore, we have $\Delta + 1 \leqslant \Delta_1$ and (5.1.3) gives (5.1.5), since $\delta_1 \geqslant 1$. Finally, since $\Delta \leqslant \Delta_1 - 1$, (5.1.3) provides (5.1.6). □

5.2 Alliance polynomials of regular graphs with small degree

The theorems in this section can be seen as a natural continuation of the studies in [34, 100] in the sense of showing the distinctive power of the alliance polynomial of a graph. In particular, we show that the family of alliance polynomials of Δ-regular graphs with small degree Δ is a special family of alliance polynomials since there not exists a non Δ-regular graph with alliance polynomial equal to one of their members, see Theorems 5.2.1 and 5.2.4.

Theorem 5.2.1. *Let G be a Δ-regular graph with $0 \leqslant \Delta \leqslant 3$ and G^* another graph. If $A(G^*; x) = A(G; x)$, then G^* is a Δ-regular graph with the same order, size and number of connected components of G.*

Proof. In Chapter 4 Theorem 4.1.5 we obtain the uniqueness of the alliance polynomials of 0-regular graphs (the empty graphs).

By Theorems 5.1.1 iii) and 5.1.6 we have that 1-regular graphs are the unique graphs which has exactly two non-zero terms in its alliance polynomial; besides, Theorems 5.1.1 vi) and 5.1.2 ii) give the uniqueness of these alliance polynomials.

In order to obtain the result for $\Delta = 2$, denote by n, n_1, the orders of G, G^*, respectively, and let δ_1, Δ_1 be the minimum and maximum degree of G^*. By Theorem 5.1.6 we have $A(G; x) = nx^{n-2} + A_0(G)x^n + A_2(G)x^{n+2}$, thus, by Theorem 5.1.1 iii) the degree sequence of G^* has at most two different values. If G^* is regular then Theorem 5.1.4 gives the result. Therefore, seeking for a contradiction assume that the degree sequence of G^* has exactly two different values (i.e., G^* is bi-regular). By Theorems 5.1.1 iv) and 5.1.2 vi) we have $\delta_1 \equiv \Delta_1 (\text{mod } 2)$. By Theorems 5.1.1 v) and 5.1.2 ii) we have $A_{-\Delta_1}(G^*) = A_2(G) = n < n_1$ and $n - 2 = n_1 - \Delta_1$, so, we have $\Delta_1 > 2$. By Theorems 5.1.1 vii) and 5.1.2 iii) we have $n_1 + \delta_1 \leqslant n + 2$, so, we obtain $0 \leqslant \delta_1 \leqslant 1$. If $\delta_1 = 0$, then there is a connected component G' of G^* which is Δ_1-regular. So, $k_{V(G')} = \Delta_1$ and $\text{Deg}\left(A(G^*; x)\right) = n_1 + \Delta_1 > n + 2$, which is a contradiction. Thus, we can assume that $\delta_1 = 1$. Then, we have $n_1 = n + 1$; and so, $\Delta_1 = 3$. We prove now that $A_1(G^*) \geqslant n$. Let u_0, v_0 be the vertices of G^* with $\delta(u_0) = 1$ and $v_0 \sim u_0$. If G^* is not connected, then it has a 3-regular connected component G_0^*; since $V(G_0^*)$ is an exact defensive 3-alliance, then $\text{Deg}\left(A(G^*; x)\right) \geqslant n_1 + 3 > n + 2 = \text{Deg}\left(A(G; x)\right)$, which is a contradiction and we conclude that G^* is connected. Let us define $S_v := V(G^*) \setminus \{v\}$ for any $v \in V(G^*) \setminus \{v_0\}$. Since $\delta_{S_v}(u) \geqslant 2$, $\delta_{\overline{S_v}}(u) \leqslant 1$ for every $u \in S_v \setminus \{u_0\}$ and both equalities hold for every $w \in N(v)$, and $\delta_{S_v}(u_0) = 1$, $\delta_{\overline{S_v}}(u_0) = 0$, we have that S_v is an exact defensive 1-alliance or contains an exact defensive 1-alliance if v is a cut vertex. Thus, $A_1(G^*) \geqslant n$. Besides, Theorem 5.1.2 iv) gives $A_2(G) \leqslant n/3 < n \leqslant A_1(G^*)$, so, $A(G; x) \neq A(G^*; x)$. This is the contradiction we were looking for, and so, we conclude $n_1 = n$ and $\Delta_1 = 2$, and we obtain the result for $\Delta = 2$.

Finally, in Chapter 4 Theorem 4.1.6 gives the result for $\Delta = 3$. $\square$

Now we prove a similar result for Δ-regular graphs with $\Delta > 3$. First, we prove some technical results which will be useful.

Lemma 5.2.2. *Let G_1 be a graph with minimum and maximum degree δ_1 and Δ_1, respectively, and let $n \geqslant 3$ be a fixed natural number. Assume that G_1 has order $n_1 > n$ with exactly n vertices of degree Δ_1, and such that its alliance polynomial $A(G_1; x)$ is symmetric. The following statements hold:*

1. *If $\delta_1 = 1$, then $A(G_1; x)$ is not a monic polynomial of degree $2n - n_1 + \Delta_1$.*

2. *If $\delta_1 = 2$, then we have $2n_1 < 2\Delta_1 + n$ or $A(G_1; x)$ is not a monic polynomial of degree $2n - n_1 + \Delta_1$.*

Proof. Seeking for a contradiction assume that $A(G_1; x)$ is a monic polynomial with degree $2n - n_1 + \Delta_1$. By hypothesis, we have n different vertices $v_1, \dots, v_n$ in G_1 with degree Δ_1. Denote by S the set $S := \{v_1, \dots, v_n\}$. The argument in the proof of Theorem 5.1.12 gives that G_1 is a connected graph, S is an exact defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance in G_1 and there is $w \in S$ with $\overline{S} \subseteq N(w)$. Let $u \in \overline{S}$ with $\delta(u) = \delta_1$.

First assume that $\delta_1 = 1$. So, $S_w := S \setminus \{w\}$ contains a defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance since $\delta_{S_w}(v) \geqslant \Delta_1 - \big(|\overline{S} \cup \{w\}| - |\{u\}|\big) = \Delta_1 - (n_1 - n)$ and $\delta_{\overline{S_w}}(v) \leqslant |\overline{S} \cup \{w\}| - 1 = n_1 - n$ for all $v \in S_w$; thus, in $A(G_1; x)$ appears at least one term of degree greater or equal than $2n - n_1 + \Delta_1$ associated to S_w, but this is impossible since $A(G_1; x)$ is monic of degree $2n - n_1 + \Delta_1$. This is the contradiction we were looking for.

Assume now that $\delta_1 = 2$. Let $w' \in V(G_1) \setminus \{w\}$ with $w' \sim u$. If $w' \in \overline{S}$ then S_w is a defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance since $u \notin N(v)$ for every $v \in S_w$. This implies a contradiction as above. So, we can assume that $w' \in S_w$. Note that if $w' \nsim w$ then S_w is a defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance since $\delta_{S_w}(w') - \delta_{\overline{S_w}}(w') \geqslant (\Delta_1 - n_1 + n) - (n_1 - n)$ and $\delta_{S_w}(v) - \delta_{\overline{S_w}}(v) \geqslant (\Delta_1 - n_1 + n) - (n_1 - n)$ for all $v \in S_w \setminus \{w'\}$, but this is impossible since $A(G_1; x)$ is a monic polynomial of degree $n_1 + \Delta_1 - 2(n_1 - n)$. Then, we can assume that $w' \sim w$. Note that if $\delta_{\overline{S}}(w') < n_1 - n$ then S_w is a defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance, but this is impossible, too. So, we can assume that $\overline{S} \subseteq N(w')$. Notice that if there is $u' \in S$ with $d(u', \{w, w'\}) \geqslant 2$, then we can check that $S \setminus \{u'\}$ is a defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance, which is impossible. Thus, we can assume that $S \subseteq N(w) \cup N(w')$; in fact,

$$n - 2 = |S \setminus \{w, w'\}| \leqslant \delta_{S \setminus \{w'\}}(w) + \delta_{S \setminus \{w\}}(w') = 2[\Delta_1 - (n_1 - n) - 1].$$

Since $S \subseteq N(w) \cup N(w')$, if $n - 2 = 2[\Delta_1 - (n_1 - n) - 1]$ then $S \cap N(w) \cap N(w') = \emptyset$, and

$$\delta_{S \setminus \{w,w'\}}(v) \geqslant \Delta_1 - (n_1 - n) \text{ and } \delta_{\overline{S \setminus \{w,w'\}}}(v) \leqslant n_1 - n, \quad \text{for every } v \in S \setminus \{w, w'\}.$$

Hence, $S \setminus \{w, w'\}$ is a defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance, which is impossible. Then $n - 2 < 2[\Delta_1 - (n_1 - n) - 1]$ and this finishes the proof. □

Lemma 5.2.3. *Let G_1 be a graph with minimum and maximum degree 2 and Δ_1, respectively, and let $n \geqslant 3$ be a fixed natural number. Assume that G_1 has order $n_1 > n$ with exactly n vertices of degree Δ_1, and such that its alliance polinomial $A(G_1; x)$ is symmetric. If $n < 2[\Delta_1 - (n_1 - n)]$ and $A(G_1; x)$ is a monic polynomial of degree $2n - n_1 + \Delta_1$, then $A_{2(n-n_1)+\Delta_1-2}(G_1) > n$.*

Proof. By hypothesis, there exist different vertices $v_1, \dots, v_n$ in G_1 with degree Δ_1. The arguments in the proof of Lemma 5.2.2 give that G_1 is a connected graph where $S :=$

$\{v_1,\dots,v_n\}$ is the unique exact defensive $\big[\Delta_1-2(n_1-n)\big]$-alliance in G_1 and there are $w,w'\in S$ with $\overline{S}\subset N(w)\cap N(w')$. Note that $S_u:=S\setminus\{u\}$ is a defensive $\big[\Delta_1-2(n_1-n)-2\big]$-alliance for any $u\in S$, since for all $v\in S_u$ we have

$$\delta_{S_u}(v)\geqslant\Delta_1-\big|\overline{S_u}\big|\quad\text{ and }\quad\delta_{\overline{S_u}}(v)\leqslant\big|\overline{S_u}\big|=n_1-n+1.$$

Note that $\delta_S(v)\geqslant\Delta_1-(n_1-n)>n/2$ for every $v\in S$. Since $\langle S\rangle$ is Hamiltonian by Theorem 1.6.5, we have that S_u induces a connected subgraph for any $u\in S$. Since S is the unique exact defensive $\big[\Delta_1-2(n_1-n)\big]$-alliance in G_1, S_u is an exact defensive $\big[\Delta_1-2(n_1-n)-2\big]$-alliance for any $u\in S$. Therefore, we have $A_{\Delta_1-2(n_1-n)-2}(G_1)\geqslant n$.

Denote by u' a vertex of G_1 with $\delta(u')=2$. Since $v\nsim u'$ for any $v\in S\setminus\{w,w'\}$ we have $|S|-1\geqslant\delta_S(v)\geqslant\delta_S(w)+1$, and so, $\delta_S(w)\leqslant|S|-2$ and there are $u_1,u_2\in S\setminus\{w,w'\}$ with $u_1,u_2\notin N(w)$; then $u_1,u_2\notin N(w)\cap N(w')$. Note that $S\setminus\{u_1,u_2\}$ is a defensive $\big[\Delta_1-2(n_1-n)-2\big]$-alliance in G_1, since

$$\delta_{S\setminus\{u_1,u_2\}}(w)-\delta_{\overline{S\setminus\{u_1,u_2\}}}(w)=\Delta_1-2\delta_{\overline{S\setminus\{u_1,u_2\}}}(w)\geqslant\Delta_1-2(n_1-n+1),$$

$$\delta_{S\setminus\{u_1,u_2\}}(w')-\delta_{\overline{S\setminus\{u_1,u_2\}}}(w')=\Delta_1-2\delta_{\overline{S\setminus\{u_1,u_2\}}}(w')\geqslant\Delta_1-2(n_1-n+1),$$

and

$$\delta_{S\setminus\{u_1,u_2\}}(v)-\delta_{\overline{S\setminus\{u_1,u_2\}}}(v)\geqslant\Delta_1-2(n_1-n+1)\quad\text{ for all } v\in S\setminus\{u_1,u_2,w,w'\}.$$

Then $S\setminus\{u_1,u_2\}$ is an exact defensive $\big[\Delta_1-2(n_1-n)-2\big]$-alliance and this finishes the proof. □

Theorem 5.2.4. *Let G be a connected Δ-regular graph with $\Delta\leqslant 5$ and G^* another graph. If $A(G^*;x)=A(G;x)$, then G^* is a connected Δ-regular graph with the same order and size of G.*

Proof. If $0\leqslant\Delta\leqslant 3$, then the result follows from Theorem 5.2.1. Assume that $4\leqslant\Delta\leqslant 5$. Let n,n_1 be the orders of G,G^*, respectively, and let δ_1,Δ_1 be the minimum and maximum degree of G^*, respectively. By Theorem 5.1.12, G^* is a connected graph and $n_1\geqslant n$. Seeking for a contradiction assume that $n_1>n$.

Assume first $\Delta=4$. By Theorem 5.1.12 we have $n_1=n+\Delta_1-4$, $\Delta_1>4$ and $\Delta_1+\delta_1\leqslant 6$. Thus, we have $\Delta_1=5$ and $\delta_1=1$, and then $n_1=n+1$. Then, Theorem 5.1.12 and Lemma 5.2.2 give that $A(G;x)=A(G^*;x)$ is not a monic polynomial of degree $n_1+3=n+4$. This is the contradiction we were looking for, and we conclude $n_1=n$.

Assume now $\Delta=5$. By Theorem 5.1.12 we have $n_1=n+\Delta_1-5$, $\Delta_1>5$, $\Delta_1+\delta_1\leqslant 8$, $\delta_1+4\leqslant\Delta_1$ and $\delta_1\equiv\Delta_1(\text{mod }2)$. Thus, we have the following cases:

Case 1 $\delta_1=1$ and $\Delta_1=7$,

Case 2 $\delta_1=2$ and $\Delta_1=6$.

Lemma 5.2.2 gives that $A(G;x)$ is not a monic polynomial of degree $n+5$ in Case 1; this is the contradiction we were looking for, and we conclude $n_1 = n$. In Case 2 we have $n_1 = n+1$. Since $A(G;x)$ is a monic polynomial of degree $n+5$, Lemma 5.2.2 gives that $n < 10$. Hence, Lemma 5.2.3 gives that $A_2(G^*) > n$; however, Theorem 5.1.7 gives $A_3(G) = n$. This is the contradiction we were looking for, and we conclude $n_1 = n$. $\square$

Chapter 6

A brief introduction to Gromov hyperbolic graphs

6.1 A historical introduction to non-Euclidean geometries

Let us first briefly discuss the history of non-Euclidean geometries. Euclid's Elements consists of 13 books, written at about 300BC, that are mainly concerned with geometry (although they also contain some number theory and the method of exhaustion which is related to integration). It is the earliest known systematic discussion of geometry.

Book 1 begins with 23 definitions (of a point, line, etc.) and 10 axioms. Of these axioms, the following five are termed Postulates:

(1) Any two points can be joined by a straight line.

(2) Any straight line segment can be extended indefinitely in a straight line.

(3) Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

(4) All right angles are congruent.

(5) Parallel Postulate: If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

The Parallel Postulate is equivalent to the statement that for any given line R and point $p \notin R$, there is exactly one line through p that does not intersect R, i.e., that is parallel to the line R.

For two millenia, mathematicians were troubled by the Parallel Postulate of Euclid, principally because it is more complex and rather different from the other Postulates. For most of that time, mathematicians attempted to prove that it followed from the other postulates, as Proclus, Ibn al-Haytham (Alhacen), Omar Khayyám, Nasir al-Din al-Tusi, Witelo, Gersonides, Alfonso, and later Giovanni Gerolamo Saccheri, John Wallis, Johann Heinrich Lambert and Legendre. Some of them succeeded in finding a large variety of false "proofs" which all fail because they make some assumption that is equivalent to the Parallel Postulate. In fact, if we replace the Parallel Postulate by:

(a) for any line R and any point $p \notin R$, there exist at least two lines parallel to R passing through p,

or

(b) for any line R and any point $p \notin R$, there exists no line parallel to R passing through the point p,

we obtain different geometries: hyperbolic geometry or elliptic geometry, respectively.

In elliptic geometry, whose main model is any sphere in $\mathbb{R}^3$, there are no parallel lines at all. Elliptic geometry has a variety of properties that differ from those of classical Euclidean plane geometry. For example, the sum of the angles of any triangle is always greater than π.

In the nineteenth century, hyperbolic geometry was extensively explored by Janos Bolyai and Nikolai Ivanovich Lobachevsky, after whom it sometimes is named. Lobachevsky published in 1830, while Bolyai independently discovered it and published in 1832. Carl Friedrich Gauss also studied hyperbolic geometry, describing in a 1824 letter to Taurinus that he had constructed it, but did not publish his work. Initially, some mathematicians thought that this new geometry was not consistent; however, Eugenio Beltrami provided models of the hyperbolic geometry in 1868, and used this to prove that hyperbolic geometry is consistent provided that Euclidean geometry is. The term "hyperbolic geometry" was introduced by Felix Klein in 1871. For more history, see [29], [41], [83] and [117].

There are four models commonly used for hyperbolic geometry: the Klein model, the Poincaré disc, the Poincaré halfplane, and the Lorentz model. These models define a real hyperbolic space which satisfies the axioms of a hyperbolic geometry. Despite their names, the first three mentioned above were introduced as models of hyperbolic space by Beltrami, not by Poincaré or Klein. We are mainly interested in the two Poincaré models.

The Poincaré metric in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is given infinitesimally at a point $z = x + iy \in \mathbb{D}$ by

$$ds_{\mathbb{D}}^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2} = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}\,, \qquad ds_{\mathbb{D}} = \frac{2\,|dz|}{1 - |z|^2}\,,$$

and so the hyperbolic area element is

$$dA_{\mathbb{D}} = \frac{4\,dx\,dy}{(1 - (x^2 + y^2))^2} = \frac{4\,dx\,dy}{(1 - |z|^2)^2}\,.$$

Given $z_1, z_2 \in \mathbb{D}$, the associated distance function is

$$d_{\mathbb{D}}(z_1, z_2) = 2 \operatorname{arctanh}\left|\frac{z_1 - z_2}{1 - z_1\overline{z_2}}\right|.$$

The hyperbolic plane contains a unique geodesic between every pair of points. In the Poincaré disk $\mathbb{D}$, the geodesic lines are precisely the intersections with $\mathbb{D}$ of circles that cut the unit circle orthogonally, plus diameters of the boundary circle.

The Poincaré metric in the upper halfplane $\mathbb{U} = \{z = x + iy \in \mathbb{C} : y > 0\}$ is given infinitesimally at a point $z = x + iy \in \mathbb{U}$ by

$$ds_{\mathbb{U}}^2 = \frac{dx^2 + dy^2}{y^2}, \qquad ds_{\mathbb{U}} = \frac{|dz|}{y},$$

and so the hyperbolic area element is

$$dA_{\mathbb{U}} = \frac{4\,dx\,dy}{y^2}.$$

Given $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{U}$, the associated distance function satisfies

$$d_{\mathbb{U}}(z_1, z_2) = \log \frac{|z_1 - \overline{z_2}| + |z_1 - z_2|}{|z_1 - \overline{z_2}| - |z_1 - z_2|}, \quad \sinh^2 \frac{d_{\mathbb{U}}(z_1, z_2)}{2} = \frac{|z_1 - z_2|^2}{4\,y_1\,y_2}.$$

The geodesic lines are precisely the intersections with $\mathbb{U}$ of circles orthogonal to the real line, plus rays perpendicular to the real line.

Both Poincaré models preserve hyperbolic angles, and are thereby conformal. All isometries within these models are therefore Möbius transformations. The halfplane model is "identical" (isometric) to the Poincaré disc model.

The area of a triangle in the hyperbolic plane increases more slowly and the area of a disk increases quicker than in the Euclidean setting. Let us now say more about both of these.

There is a simple and remarkable relationship between angles and area of a triangle which can be obtained as a consequence of Gauss-Bonnet formula:

The hyperbolic area of a triangle with interior angles α, β, γ is $\pi - (\alpha + \beta + \gamma)$. This holds even if one or more vertices of the triangle are on the ideal boundary (in which case the associated angles are zero). It follows from the Gauss-Bonnet formula that if we rescale upwards the sidelengths of a hyperbolic triangle, its area increases, with a limiting area of π as the sidelengths tend to infinity.

Then, Euclidean triangles are "wider" than hyperbolic triangles, and one can think that the Euclidean plane is "wider" than the hyperbolic plane.

The area A_r of a hyperbolic disk of radius r is independent of the center, and is given by $4\pi \sinh^2(r/2)$. The length L_r of the hyperbolic circle of radius r is $2\pi \sinh r$. Therefore, A_r and L_r are very similar to the corresponding Euclidean quantities when r is small:

$$A_r \approx \pi r^2, \qquad L_r \approx 2\pi r, \qquad \text{as } r \to 0^+.$$

However they increase far faster than in the Euclidean setting when r is large:

$$A_r \approx L_r \approx \pi e^r, \qquad \text{as } r \to \infty.$$

Hence, the hyperbolic plane is "wider" than the Euclidean plane (although Euclidean triangles are "wider" than hyperbolic ones).

There are many excellent books about hyperbolic geometry, e.g., the books by Anderson [9], Beardon [15] and Krantz [77].

In complex analysis, the most important property of the Poincaré metric is that holomorphic mappings are contractions with respect to it. More precisely, we have (see [2, p.3]):

Theorem 6.1.1 (Schwarz-Pick Lemma)**.** *Every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ verifies*

$$d_{\mathbb{D}}\big(f(z_1), f(z_2)\big) \leqslant d_{\mathbb{D}}(z_1, z_2)$$

for every $z_1, z_2 \in \mathbb{D}$.

Furthermore, if the equality holds for some $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$, then f is an automorphism (i.e., a Möbius self-map of $\mathbb{D}$), and so $d_{\mathbb{D}}\big(f(z_1), f(z_2)\big) = d_{\mathbb{D}}(z_1, z_2)$ for every $z_1, z_2 \in \mathbb{D}$.

In fact, the Poincaré metric can be defined for any domain $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ has more than one point. If we denote by d_Ω the Poincaré distance in Ω, then we have the following generalization of Schwarz-Pick Lemma (see, e.g., the books [77] and [100]):

Theorem 6.1.2. *If Ω_1, Ω_2 are open connected subsets of $\mathbb{C}$, $\partial\Omega_1$ and $\partial\Omega_2$ have more than one point and $f : \Omega_1 \to \Omega_2$ is holomorphic, then*

$$d_{\Omega_2}\big(f(z_1), f(z_2)\big) \leqslant d_{\Omega_1}(z_1, z_2)$$

for every $z_1, z_2 \in \Omega_1$.

Furthermore, if the equality holds for some $z_1, z_2 \in \Omega_1$ with $z_1 \neq z_2$, then f is a conformal map of Ω_1 onto Ω_2, and so $d_{\Omega_2}\big(f(z_1), f(z_2)\big) = d_{\Omega_1}(z_1, z_2)$ for every $z_1, z_2 \in \Omega_1$.

The simplest particular case of Schwarz-Pick Lemma is the classical Schwarz's Lemma (see [2, p.3]):

Theorem 6.1.3. *Every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ verifies*

$$\big|f(z)\big| \leqslant |z|$$

for every $z \in \mathbb{D}$.

This is a bound for the growth of *every* holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ (with the normalization $f(0) = 0$).

One of the most famous applications of Theorem 6.1.2 is the following (see [1] or [2, p.19]):

Theorem 6.1.4 (Schottky's Theorem)**.** *If $f : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0, 1\}$ is holomorphic, then*

$$|f(z)| \leqslant \exp\Big(\frac{1+|z|}{1-|z|}\big(7 + \max\big\{0, \log|f(0)|\big\}\big)\Big),$$

for every $z \in \mathbb{D}$.

Note that this is a bound for the growth of *every* holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0, 1\}$.

6.2 Gromov hyperbolic spaces

Now, let us introduce the concept of Gromov hyperbolicity and the main results concerning this theory. For detailed expositions about Gromov hyperbolicity, see e.g. [6],[26, II.H], [40], [54] or [115].

Gromov hyperbolicity was introduced by Gromov in the setting of geometric group theory [40], [54], [57], [58], but has played an increasing role in analysis on general metric spaces [12], [22], [23], with applications to the Martin boundary, invariant metrics in several complex variables [12] and extendability of Lipschitz mappings [79].

Hyperbolic spaces play an important role in geometric group theory and in geometry of negatively curved spaces (see [6], [54], [57]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [6], [54], [57]).

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [57] and the references therein), where it was demonstrated to have a practical importance. This theory was applied initially to the study of automatic groups (see [87]), which play a role in the science of computation (indeed, hyperbolic groups are strongly geodesically automatic, that is, there is an automatic structure on the group, where the language accepted by the word acceptor is the set of all geodesic words [36]).

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. Another important application of these spaces is secure transmission of information on the internet (see [68], [69], [70]). Furthermore, the hyperbolicity plays an important role in the spread of viruses through the network (see [68], [70]). Ideas related to hyperbolicity have been applied in numerous other networks applications, e.g., to problems such as distance estimation, sensor networks, and traffic flow and congestion minimization [14], [73], [74], [85],[111], as well as large-scale data visualization [84]. The latter applications typically take important advantage of the idea that data are often hierarchical or tree-like and that there is "more room" in hyperbolic spaces than in Euclidean spaces. The hyperbolicity is also useful in the study of DNA data (see [27]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory (see, e.g., [14], [17], [18], [19], [27], [32], [33], [38], [39], [51], [68], [69], [70], [71], [72], [73], [74], [76], [80], [81], [84], [85], [88], [89], [90], [91], [97], [98], [101], [102], [103], [111], [113], [118]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [60]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [12], [22], [61], [62], [91], [92], [93], [102], [103]. In particular, in [91], [102], [103], [113] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a very simple graph; hence, it is useful to study hyperbolic graphs from this point of view.

Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. We define the *length* of γ as

$$L(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\} .$$

We say that γ is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. A *geodesic line* is a geodesic whose domain is $\mathbb{R}$.

We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. Recall that when the metric space X is a graph, we use the notation $[u, v]$ for the edge joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, identify (by an isometry) any edge with the real interval $[0, k]$, where k is a fixed constant with $L(e) = k$ for every $e \in E(G)$. Therefore, any point in the interior of an edge is a point of G. Then G is naturally equipped with a distance defined on its points, induced by taking the shortest paths in G, and we see G as a metric graph. Hence, any connected graph is a geodesic metric space.

Recall that if X is a metric space, $x \in X$ and $E \subseteq X$, the distance $d(x, E)$ is defined as $d(x, E) := \inf \big\{ d(x, y) |\, y \in E \big\}$.

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon with sides $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leqslant \delta$. In other words, a polygon is δ-thin if each of its sides is contained in the δ-neighborhood of the union of the other sides. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geqslant 0 |\, J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) |\, T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geqslant 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geqslant 0 |\, X \text{ is } \delta\text{-hyperbolic}\}$. If we have a triangle with two identical vertices, we call it a *bigon*; note that since this is a special case of the definition, every geodesic bigon in a δ-hyperbolic space is δ-thin.

One can check that if X is a δ-hyperbolic geodesic metric space, then every geodesic polygon with $n \geqslant 3$ sides is $(n - 2)\delta$-thin.

Examples:

1. Every bounded metric space X is $((\operatorname{diam} X)/2)$-hyperbolic.

2. The real line $\mathbb{R}$ is 0-hyperbolic: In fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore any geodesic triangle in $\mathbb{R}$ is 0-thin.

3. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic, since the midpoint of a side on a large equilateral triangle is far from all points on the other two sides.

These arguments can be applied to higher dimensions:

4. A normed real linear space is hyperbolic if and only if it has dimension 1.

5. Every metric tree with arbitrary length edges is 0-hyperbolic, by the same reason that the real line.

6. The unit disk $\mathbb{D}$ (with its Poincaré metric) is $\log(1+\sqrt{2}\,)$-hyperbolic: Consider any geodesic triangle T in $\mathbb{D}$. It is clear that T is contained in an ideal triangle T', all of whose sides are of infinite length, with $\delta(T) \leqslant \delta(T')$. Since all ideal triangles are isometric, we can consider just one fixed T'. Then, a computation gives $\delta(T') = \log(1+\sqrt{2}\,)$.

7. Every simply connected complete Riemannian manifold with sectional curvatures verifying $K \le -c^2 < 0$, for some constant c, is hyperbolic (see, e.g., [54, p.52]).

8. The graph Γ of the routing infrastructure of the Internet is also empirically shown to be hyperbolic (see [13]). One can think that this is a trivial (and then a non-useful) fact, since every bounded metric space X is $((\operatorname{diam} X)/2)$-hyperbolic. The point is that the quotient

$$\frac{\delta(\Gamma)}{\operatorname{diam}\Gamma}$$

is very small, and this makes the tools of hyperbolic spaces applicable to Γ (see, e.g., [39]).

As a remark, the main examples of hyperbolic graphs are trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [37]).

It is worth pointing out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Note that, first of all, one needs to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then, to take the supremum over all the possible choices for P and then over all the possible choices for T. Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that the location of geodesics in the space is not usually known.

If X is a metric space, we define the *Gromov product* of $x, y \in X$ with base point $w \in X$ by

$$(x,y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big).$$

A geometric interpretation of the Gromov product is obtained by mapping the triple (x,y,w) isometrically onto a triple (x',y',w') in the Euclidean plane. The circle inscribed to the triangle x', y', w' meets the sides $[w'x']$ and $[w'y']$ at points x^* and y^*, respectively, and we have $(x,y)_w = |x^* - w'| = |y^* - w'|$.

If X is a Gromov hyperbolic space, it holds

$$(x,z)_w \geqslant \min\big\{(x,y)_w,(y,z)_w\big\} - \delta \tag{6.2.1}$$

for every $x,y,z,w \in X$ and some constant $\delta \geqslant 0$ (see, e.g., [6, Proposition 2.1], [54, p.41] or [115, 2.34 and 2.35]). Let us denote by $\delta^*(X)$ the sharp constant for this inequality, i.e.,

$$\delta^*(X) := \sup\big\{\min\big\{(x,y)_w,(y,z)_w\big\} - (x,z)_w :\, x,y,z,w \in X\big\}.$$

Remark 6.2.1. *If X is a geodesic metric space, it is known that* (6.2.1) *is, in fact, equivalent to our definition of Gromov hyperbolicity; furthermore,$\delta^*(X) \leqslant 3\,\delta(X)$ and $\delta(X) \leqslant 3\,\delta^*(X)$ (see, e.g., [115, 2.34 and 2.35]).*

Then (6.2.1) extends the definition of Gromov hyperbolicity to the context of (non-necessarily geodesic) metric spaces. The disadvantage of the Gromov product definition is that its geometric meaning is unclear at first sight, whereas the thin triangles definition is very easy to understand geometrically.

The following useful estimate is the key to understand the geometric meaning of the Gromov product definition (6.2.1):

Proposition 6.2.2. (*[54, Lemma 1.7, p.38]*) *If X is a δ-hyperbolic geodesic metric space, then for every $x,y,w \in X$ and every geodesic $[xy]$, we have*

$$d(w,[xy]) - 4\delta \leqslant (x,y)_w \leqslant d(w,[xy]).$$

Indeed only the lower bound requires hyperbolicity.

We would like to recall the surprising geometric interpretation of the Gromov product in the hyperbolic plane. Like in the Euclidean setting, there exists a relation among the three sides of a right-angled triangle in the hyperbolic plane (the hyperbolic Pythagorean theorem):

$$\cosh c = \cosh a\, \cosh b.$$

We also have the hyperbolic Cosine rule for any hyperbolic triangle:

$$\cosh c = \cosh a\, \cosh b - \sinh a\, \sinh b\, \cos\theta.$$

If a,b,c are large, then this latter formula is asymptotically equivalent to

$$\frac{1}{2}e^c \approx \frac{1}{4}e^{a+b}(1-\cos\theta),$$

and we deduce

$$e^c \approx e^{a+b}\sin^2(\theta/2) \qquad \Longrightarrow \qquad \frac{1}{2}(a+b-c) \approx \log\frac{1}{\sin(\theta/2)}\,.$$

If x, y, w are the vertices of this hyperbolic triangle and θ is the angle at w, then we have

$$(x, y)_w \approx \log \frac{1}{\sin(\theta/2)} ,$$

and we can determine (approximately) the angle θ in terms of the Gromov product.

Then we could expect that the Gromov product allows to estimate "something like angles" in hyperbolic spaces (for instance, in hyperbolic graphs!); this is the case and, consequently, the hyperbolic spaces have richer structure than the general metric spaces.

In the setting of linear spaces with inner product, the main angle is $\pi/2$; however, in hyperbolic spaces the main angle is π: One can check that, in a geodesic metric space, $(x, y)_w = 0$ if and only if w belongs to a geodesic joining x and y.

The following result is a good example of "angle estimation" in hyperbolic spaces:

Theorem 6.2.3. *([54, Theorem 16, p.92]) Let us consider constants $\delta \geqslant 0, r, \ell > 0$, a δ-hyperbolic geodesic metric space X and a finite sequence $\{x_j\}_{0\leqslant j\leqslant n}$ in X with*

$$d(x_{j-1}, x_{j+1}) \geqslant \max\{d(x_{j-1}, x_j), d(x_j, x_{j+1})\} + 18\delta + r, \qquad \text{for every } 0 < j < n,$$
$$d(x_{j-1}, x_j) \leqslant \ell, \qquad \text{for every } 0 < j \leqslant n.$$

Then $[x_0x_1] \cup [x_1x_2] \cup \cdots \cup [x_{n-1}x_n]$ is an $(\alpha, 0)$-quasigeodesic, with $\alpha := \max\{\ell, 1/r\}$.

Quasigeodesics are a generalization of geodesics; we present now the precise definition. Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \geq 1, \beta \geq 0$ if, for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function f is ε-*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \geq 1, \beta, \varepsilon \geq 0$ such that f is an ε-full (α, β)-quasi-isometric embedding.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \longrightarrow Y$. It is quite easy to see that being quasi-isometric is an equivalence relation.

An (α, β)-*quasigeodesic* of a metric space X is an (α, β)-quasi-isometric embedding $\gamma : I \longrightarrow X$, where I is an interval of $\mathbb{R}$. Note that a $(1, 0)$-quasigeodesic is a geodesic.

Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call ε-*neighborhood* of Y in X, denoted by $V_\varepsilon(Y)$ to the set $\{x \in X : d_X(x, Y) \leq \varepsilon\}$. The *Hausdorff distance* between two subsets Y and Z of X, denoted by $\mathcal{H}(Y, Z)$, is the number defined by:

$$\inf\{\varepsilon > 0 : Y \subset V_\varepsilon(Z) \text{ and } Z \subset V_\varepsilon(Y)\}.$$

6.3 Gromov hyperbolicity, Mathematical Analysis and Geometry

The ideal boundary of a metric space is a type of boundary at infinity which is a very useful concept when dealing with negatively curved spaces. We want to talk about some subjects in which this boundary is useful.

A main problem in the study of Partial Differential Equations in Riemannian Manifolds is whether or not there exist nonconstant bounded harmonic functions. A way to approach this problem is to study whether the so-called Dirichlet problem at infinity (or the asymptotic Dirichlet problem) is solvable on a complete Riemannian manifold M. That is to say, raising the question as to whether every continuous function on the boundary ∂M has a (unique) harmonic extension to M. Of course, the answer, in general, is no, since the simplest manifold $\mathbb{R}^n$ admits no positive harmonic functions other than constants. However, the answer is positive for the unit disk $\mathbb{D}$.

In [7] Ancona studied the asymptotic Dirichlet problem on Gromov hyperbolic graphs and in [8] on Gromov hyperbolic Riemannian manifolds with bounded geometry and a positive lower bound $\lambda_1(M) > 0$ for Dirichlet eigenvalues. In the papers [30] and [75] conditions on Gromov hyperbolic manifolds M that imply the positivity of $\lambda_1(M)$ are given and, consequently, the Dirichlet problem is solvable for many Gromov hyperbolic manifolds.

One of the most important features of the transition from a Gromov hyperbolic space to its Gromov boundary is that it is functorial. If $f : X \longrightarrow Y$ is in a certain class of maps between two Gromov hyperbolic spaces X and Y, then there is a boundary map $\partial f : \partial X \longrightarrow \partial Y$ which is in some other class of maps. In particular, if f is a quasi-isometry, then ∂f is a bihölder map (with respect to the Gromov metric on the boundary).

It is well known that biholomorphic maps between domains (with smooth boundaries) in $\mathbb{C}$ can be extended as a homeomorphism between their boundaries. If we consider domains in $\mathbb{C}^n$ ($n > 1$) instead in $\mathbb{C}$, then the problem is very difficult. C. Fefferman showed in Inventiones Mathematicae (see [46]), with a very long and technical proof, that biholomorphic maps between bounded strictly pseudoconvex domains with smooth boundaries can be extended as a homeomorphism between their boundaries. It is possible to give a "more elementary" proof of this extension result using the functoriality of Gromov hyperbolic spaces: If we consider the Carathéodory metric on a bounded smooth strictly pseudoconvex domain in $\mathbb{C}^n$, then it is Gromov hyperbolic, and the Gromov boundary is homeomorphic to the topological boundary (see [11]). Since any biholomorphic map f between such two domains is an isometry for the Carathéodory metrics, the boundary map ∂f is essentially a boundary extension of f that is a homeomorphism between the boundaries (in fact, it is bihölder with respect to the Carnot-Carathéodory metrics in the boundaries). Fefferman's result gives much more precise information, but this last proof is simpler and gives information about a class of maps that is much more general than biholomorphic maps: the quasi-isometries for the Carathéodory metrics.

In applications to several areas of mathematics, the Gromov boundary can be similarly be proved (under appropriate conditions) to coincide with other "finite" boundaries, such as the Euclidean or inner Euclidean boundary, or the Martin boundary, so we obtain a variety of boundary extension results as above.

For instance, isometries (and quasi-isometries) in a hyperbolic space X can be extended (as an homeomorphism) to the Gromov boundary ∂X of the space. This fact allows to classify the isometries as *hyperbolic*, *parabolic* and *elliptic*, like the Möbius maps in $\mathbb{D}$, in terms of their fixed points in $X \cup \partial X$.

It can be proved that there are just three possibilities:

- There are exactly two fixed points in $X \cup \partial X$ and both are in ∂X (hyperbolic isometry).
- There is a single fixed point in $X \cup \partial X$ and it is in ∂X (parabolic isometry).
- There is a single fixed point in $X \cup \partial X$ and it is in X (elliptic isometry).

A main ingredient in the proof of this result in the unit disk $\mathbb{D}$ is that the isometries are holomorphic functions. Surprisingly, the tools in hyperbolic spaces provide a new and general proof just in terms of distances!

6.4 Main results on hyperbolic spaces

We state now some of the main facts about hyperbolic spaces.

In the study of any mathematical property, the class of maps which preserve that property plays a central role in the theory. The following result shows that quasi-isometries preserve hyperbolicity.

Theorem 6.4.1. *(Invariance of hyperbolicity, [54, p.88]) Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic.*

Besides, if f is ε-full for some $\varepsilon \geqslant 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

We next discuss the connection between hyperbolicity and *geodesic stability*. In the complex plane (with its Euclidean distance), there is only one optimal way of joining two points: a straight line segment. However if we allow "limited suboptimality", the set of "reasonably efficient paths" (quasigeodesics) are well spread. For instance, if we split the circle $\partial D(0, R) \subset \mathbb{C}$ into its two semicircles between the points R and $-R$, then we have two reasonably efficient paths (two $(\pi/2, 0)$-quasigeodesics) between these endpoints such that the point Ri on one of the semicircles is far from all points on the other semicircle provided that R is large. Even an additive suboptimality can lead to paths that fail to stay close together. For instance, the union of the two line segments in $\mathbb{C}$ given by $[0, R + i\sqrt{R}\,]$ and

$[R + i\sqrt{R}\,, 2R]$ gives a path of length less than $2R + 1$ (since $2\sqrt{R^2 + R} \leqslant 2R + 1$), and so it is "additively inefficient" by less than 1 (a $(1, 1)$-quasigeodesic). However, its corner point is very far from all points on the line segment $[0, 2R]$ when R is very large.

The situation in Gromov hyperbolic spaces is very different, since all such reasonably efficient paths ((α, β)-quasigeodesics for fixed α, β) stay within a bounded distance of each other:

Theorem 6.4.2. *(Geodesic stability, [54, p.87]) For any constants $\alpha \geqslant 1$ and $\beta, \delta \geqslant 0$ there exists a constant $H = H(\delta, \alpha, \beta)$ such that for every δ-hyperbolic geodesic metric space and for every pair of (α, β)-quasigeodesics g, h with the same endpoints, $\mathcal{H}(g, h) \leq H$.*

The geodesic stability is not just a useful property of hyperbolic spaces; in fact, M. Bonk proves in [24] that the geodesic stability is equivalent to the hyperbolicity:

Theorem 6.4.3. ([24, p.286]) *Let X be a geodesic metric space with the following property: For each $a \geqslant 1$ there exists a constant H such that for every $x, y \in X$ and any $(a, 0)$-quasigeodesic g in X starting in x and finishing in y there exists a geodesic γ joining x and y satisfy $\mathcal{H}(g, \gamma) \leqslant H$. Then X is hyperbolic.*

Theorem 6.4.1 can be easily deduced from Theorem 6.4.2:

Proof of Theorem 6.4.1. By hypothesis there exists $\delta \geqslant 0$ such that Y is δ-hyperbolic.

Let T be a geodesic triangle in X with sides g_1, g_2 y g_3, and T_Y the triangle with (α, β)-quasigeodesic sides $f(g_1)$, $f(g_2)$ and $f(g_3)$ in Y. Let γ_j be a geodesic joining the endpoints of $f(g_j)$, for $j = 1, 2, 3$, and T' the geodesic triangle in Y with sides $\gamma_1, \gamma_2, \gamma_3$.

Let p be any point in $f(g_1)$. We are going to prove that there exists a point $q \in f(g_2) \cup f(g_3)$ with $d_Y(p, q) \leqslant K$, where $K := \delta + 2H(\delta, \alpha, \beta)$. By Theorem 6.4.2, there exists a point $p' \in \gamma_1$ with $d_Y(p, p') \leqslant H(\delta, \alpha, \beta)$. Since T' is a geodesic triangle, it is δ-thin and there exists $q' \in \gamma_2 \cup \gamma_3$ with $d_Y(p', q') \leqslant \delta$. Using again Theorem 6.4.2, there exists a point $q \in f(g_2) \cup f(g_3)$ con $d_Y(q, q') \leqslant H(\delta, \alpha, \beta)$. Therefore,

$$\begin{aligned} d_Y(p, f(g_2) \cup f(g_3)) \leqslant d_Y(p, q) &\leqslant d_Y(p, p') + d_Y(p', q') + d_Y(q', q) \\ &\leqslant H(\delta, \alpha, \beta) + \delta + H(\delta, \alpha, \beta) = K. \end{aligned}$$

Let $z \in T$; without loss of generality we can assume that $z \in g_1$. We have seen that there exists a point $q \in f(g_2) \cup f(g_3)$ with $d_Y(f(z), q) \leqslant K$. If $w \in g_2 \cup g_3$ satisfies $f(w) = q$, then

$$d_X(z, g_2 \cup g_3) \leqslant d_X(z, w) \leqslant \alpha d_Y(f(z), q) + \alpha\beta \leqslant \alpha K + \alpha\beta.$$

Hence, T is $(\alpha K+\alpha\beta)$-thin. Since T is an arbitrary geodesic triangle, X is $(\alpha\delta+2\alpha H(\delta, \alpha, \beta)+\alpha\beta)$-hyperbolic.

Assume now that f is ε-full. One can check that a quasi-isometry $f^- : Y \longrightarrow X$ can be constructed as follows: for $y \in Y$ choose $x \in X$ with $d_Y(f(x), y) \leqslant \varepsilon$ and define $f^-(y) := x$. Then the first part of the Theorem gives the result. $\square$

Chapter 7

Hyperbolicity constant of cubic graphs

Along this chapter by a graph we mean a connected graph such that every edge has length k, for some fixed constant k.

In this chapter we obtain information about the hyperbolicity constant of cubic graphs. They are a very interesting class of graphs with many applications (see, e.g., [25, 29, 42, 88]); furthermore, they are also very important in the study of Gromov hyperbolicity, since for any graph G with bounded maximum degree there exists a cubic graph G^* such that G is hyperbolic if and only if G^* is hyperbolic (see [18, Section 4] and [88, Theorem 2.2]). We find some characterizations for the cubic graphs which have small hyperbolicity constants, *i.e.*, the graphs which are like trees (in the Gromov sense); in [88, Theorem 3.13] also appears a result on this topic, but our results are different and stronger. Besides, we obtain bounds for the hyperbolicity constant of the complement graph of a cubic graph; our main result of this kind says that for any finite cubic graph G which is not isomorphic either to K_4 or to $K_{3,3}$, the inequalities $5k/4 \le \delta(\overline{G}) \le 3k/2$ hold, improving the previous result [88, Theorem 4.9]. This is a very precise result, since it implies that $\delta(\overline{G})$ is either $5k/4$ or $3k/2$, by [17, Theorem 2.6].

7.1 Small values of the hyperbolicity constant

We recall that by *cycle* in a graph we mean a simple closed curve, *i.e.*, a path with different vertices, except for the last one, which is equal to the first vertex. If C is any path in a graph G and $w \in V(C)$, we denote by $\deg_C(w)$ the degree of the vertex w in the subgraph induced by $V(C)$.

We denote by $\operatorname{diam}_G V(G)$ or $\operatorname{diam} V(G)$ the standard diameter of the graph G, *i.e.*, $\operatorname{diam} V(G) := \sup\{d_G(x,y) : x,y \in V(G)\}$. By $\operatorname{diam}_G G$ or $\operatorname{diam} G$ we denote the diameter of the geodesic metric space G, *i.e.*, $\operatorname{diam} G := \sup\{d_G(x,y) : x,y \in G\}$. A subgraph Γ of G

is said *isometric* if $d_\Gamma(x,y) = d_G(x,y)$ for every $x, y \in \Gamma$.

In [16, Theorem 4.10] appear the following result which characterizes the hyperbolic graphs G with hyperbolicity constant verifying $\delta(G) = k$. The case $\delta(G) < k$ will be treated later in Lemma 7.1.21. Also, we would like to comment that $\delta(G)$ must be a multiple of $k/4$ (see Theorem 7.1.18).

Theorem 7.1.1. *Let G be any graph. Then $\delta(G) = k$ if and only if the following conditions hold:*

(1) *There exists a cycle isomorphic to the cycle graph C_4.*

(2) *For every cycle γ with $L(\gamma) \geq 5k$ we have $\deg_\gamma(w) \geq 3$ for every vertex $w \in \gamma$.*

In this section we study the hyperbolic cubic graphs G with hyperbolicity constant verifying $\delta(G) \leqslant 3k/2$.

We start with a consequence of Theorem 7.1.1. Recall that a *Hamiltonian cycle* in a graph G is a cycle that visits each vertex of G exactly once (except for the vertex that is both the start and end, which is visited twice); a graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

Proposition 7.1.2. *Let G be any cubic graph. Then $\delta(G) = k$ if and only if the following conditions hold:*

(1) *There exists a cycle isomorphic to C_4.*

(2) *Every cycle in G with length greater than $4k$ is Hamiltonian.*

Proof. If γ is a cycle in G with $\deg_\gamma(w) \geq 3$ for every vertex $w \in \gamma$, then $\deg_\gamma(w) = 3$ for every vertex $w \in \gamma$, since G is a cubic graph. Hence, if $L(\gamma) > 4k$, then every vertex of G belongs to γ, and γ is a Hamiltonian cycle. □

Corollary 7.1.3. *Let G be any cubic graph. If $\delta(G) = k$ and there exists a cycle γ with $L(\gamma) > 4k$, then G is a Hamiltonian graph and γ is a Hamiltonian cycle.*

Theorem 7.1.4. *Let G be any cubic graph. Then $\delta(G) = k$ if and only if we have either:*

(1) *G is isomorphic to K_4 or $K_{3,3}$.*

(2) *G is an infinite graph such that every cycle γ in G has length $L(\gamma) \leqslant 4k$ and there exists a cycle with length $4k$.*

Proof. If we have either (1) or (2), then Proposition 7.1.2 gives $\delta(G) = k$.

Assume that $\delta(G) = k$.

If G is an infinite graph, then Proposition 7.1.2 gives that every cycle γ in G has length $L(\gamma) \leqslant 4k$ and that there exists a cycle with length $4k$.

Assume that G is a graph of order n.

If $n \leqslant 6$, then one can check that G is isomorphic to K_4 or $K_{3,3}$.

We are going to finish the proof by showing that it is not possible to have $n > 6$. Seeking for a contradiction, assume that $n > 6$; then we have $n \geqslant 8$, since G is a cubic graph.

Assume that there exists a cycle C in G with length $L(C) > 4k$. By Proposition 7.1.2, C has length $L(C) = kn$; since G is a cubic graph, given any vertex $v \in V(C) = V(G)$ there exists an edge $[v, w] \in E(G) \setminus E(C)$. Let us consider the two cycles $C_1, C_2 \subset C \cup [v, w]$ with $L(C_1) \leqslant L(C_2) < L(C) = kn$; since $n \geqslant 8$, we have $4k < L(C_2) < kn$, which contradicts Proposition 7.1.2.

Assume that every cycle γ in G has length $L(\gamma) \leqslant 4k$. Let $\gamma_1, \dots, \gamma_r$ be cycles such that $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$, and if γ is any cycle in G then $V(\gamma) \subseteq V(\gamma_j)$ for some $j \in \{1, \dots, r\}$ (note that the set $\gamma_i \cap \gamma_j$ cannot be a single vertex, since it would have a degree of at least four; if $\gamma_i \cap \gamma_j$ were a single edge e, then γ_i and γ_j would have length $3k$ and we could take the new cycle $\gamma_i \cup \gamma_j \setminus \{e\}$, with length $4k$, instead of γ_i and γ_j). We define now a graph T in the following way: $V(T) = \{v_1, \dots, v_r\}$ (each v_j represents to γ_j) and $[v_i, v_j] \in E(T)$ if and only if there exists a path η in G joining γ_i and γ_j such that every edge $e \in E(\eta)$ verifies that $G \setminus \{e\}$ is not connected. One can check that T is a (finite) tree; hence, there exists a vertex v_s with degree one. Therefore, there exists a vertex in the cycle γ_s joined with an infinite tree, since G is a cubic graph; but this is a contradiction, since G is a finite graph.

We conclude that it is not possible to have a finite cubic graph with $\delta(G) = k$ and $n > 6$. This finishes the proof. □

In [16, Proposition 4.12] appears the following result.

Proposition 7.1.5. *Let G be any graph. If $\delta(G) \geq 3k/2$, then there exists a cycle g in G with* $\operatorname{diam} g \geq 3k$.

The converse of Proposition 7.1.5 does not hold, even for cubic graphs. For instance, in the cubic graph H, see Figure 7.1, there is a cycle C with $\operatorname{diam} C = 3k$, but $\delta(H) = 5k/4 < 3k/2$.

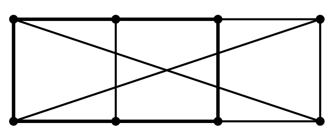


Figure 7.1: Cubic graph H with $\delta(H) = 5k/4$ and a cycle C such that $\operatorname{diam} C = 3k$.

We also have a kind of converse of Proposition 7.1.5 for cubic graphs.

Proposition 7.1.6. *Let G be any cubic graph. If there exists a cycle g in G such that* $\operatorname{diam} V(g) \geq 3k$, *then* $\delta(G) \geq 3k/2$.

Proof. Note that $L(g) \geq 6k$ since $\operatorname{diam} V(g) \geq 3k$.

Define Γ as the set of geodesics γ joining two vertices in g at distance $3k$, and for each $\gamma \in \Gamma$ let A_γ as the set of cycles $A_\gamma := \{\rho / \, \rho$ is a cycle in G containing $\gamma\}$.

We claim that $A_\gamma \neq \emptyset$ for some $\gamma \in \Gamma$.

For the moment, we will assume this claim; we will show a proof for it at the end of this proof.

Fix $\gamma \in \Gamma$ with $A_\gamma \neq \emptyset$; then γ joins two vertices $u, v \in g$ with $d(u, v) = 3k$. Let us choose $\sigma \in A_\gamma$ with $L(\sigma) \leqslant L(\rho)$ for every $\rho \in A_\gamma$.

Let us consider the midpoints p of γ and q of $\sigma \setminus \gamma$; consider the curves γ_1, γ_2 joining, respectively, u and q, and v and q, with $\gamma_1 \cup \gamma_2 = (\sigma \setminus \gamma) \cup \{u, v\}$.

Assume first that $L(\sigma) \leqslant 8k$ or σ is an isometric cycle. Then γ_1 and γ_2 are geodesics. If T is the geodesic triangle $T = \{\gamma, \gamma_1, \gamma_2\}$, then $\delta(G) \geqslant d_G(p, \gamma_1 \cup \gamma_2) = d_G(p, \{u, v\}) = 3k/2$.

Assume now that $L(\sigma) \geqslant 9k$ and σ is not an isometric cycle. A *shortcut* in the cycle σ is a geodesic $[pq]$ in G joining two vertices $p, q \in \sigma \cap V(G)$ such that $L([pq]) = d_G(p, q) < d_\sigma(p, q)$, and $[pq] \cap \sigma = \{p, q\}$. Since σ is not an isometric cycle, there exists a shortcut $[xy]$ in σ. Since γ is a geodesic, it is not possible to have $x, y \in \gamma$. Denote by γ' the curve $\gamma' := \gamma_1 \cup \gamma_2$; since $L(\sigma) \leqslant L(\rho)$ for every $\rho \in A$, it is not possible to have $x, y \in \gamma'$ (however, it is possible to have $d_G(x_0, y_0) < d_\sigma(x_0, y_0)$ for some $x_0, y_0 \in \gamma'$ if the geodesic joining x_0 and y_0 intersects $\gamma \setminus \{u, v\}$). Without loss of generality we can assume that $x \in \gamma \setminus \{u, v\}$, $y \in \sigma \setminus \gamma$ and $d_G(x, u) = k$. Define the set of geodesics $B := \{[xz] / \, z \in \sigma \setminus \gamma$ and $[xz]$ is a shortcut in $\sigma\}$; note that $B \neq \emptyset$ since $[xy] \in B$. Let us choose $[xz_0] \in B$ with $d_{\gamma'}(z_0, v) \leqslant d_{\gamma'}(z, v)$ for every $[xz] \in B$. Denote by η_u and η_v the curves contained in γ' joining u and z_0, and v and z_0, respectively.

If $L(\eta_v) \geqslant 3k$, then $d_G(z_0, v) \geqslant 3k$ (since G is a cubic graph) and $g_1 := [xz_0] \cup \eta_v \cup [vx]$ is a cycle with $L(g_1) < L(g)$ and $\operatorname{diam} V(g_1) \geq 3k$.

If $L(\eta_v) \leqslant 2k$, then the cycle $g_1 := [xz_0] \cup \eta_u \cup [u, x]$ is a cycle with $L(g_1) \geqslant 6k$. Denote by u' the point in η_u with $d_{\gamma'}(u, u') = 2k$; by the minimizing property of σ, it follows that $d_G(u, u') = 2k$; then $d_G(z_0, u') \geqslant 3k$ (since G is a cubic graph) and $g_1 := [xz_0] \cup \eta_v \cup [u, x]$ is a cycle with $L(g_1) < L(g)$ and $\operatorname{diam} V(g_1) \geq 3k$.

Iterating this process we obtain a cycle g_r verifying the properties of g and such that $L(g_r) \leqslant 8k$ or g_r is an isometric cycle. Therefore, we conclude $\delta(G) \geqslant 3k/2$ if the claim holds.

We will finish the proof by showing the claim. Seeking for a contradiction, assume that $A_\gamma = \emptyset$ for every geodesic $\gamma \in \Gamma$. Let us fix any $\gamma \in \Gamma$; then γ joins two vertices $u, v \in g$ with $d(u, v) = 3k$. Let g_1, g_2 be the subcurves of g with $g_1 \cup g_2 = g$ and $g_1 \cap g_2 = \{u, v\}$. Since $A_\gamma = \emptyset$, g_1 and g_2 contain some interior vertex of γ; since G is a cubic graph, g_1 and g_2 contain some edge of γ; if $\gamma = [u, u_0] \cup [u_0, v_0] \cup [v_0, v]$, then without loss of generality we can assume that $[u, u_0] \subset g_1$ and $[v_0, v] \subset g_2$.

Let x be the vertex in g_1 with $d_{g_1}(x, u_0) = 2k$. If γ_0 is the subcurve of g_1 joining u and x, then $d_G(x, u) \leqslant L(\gamma_0) = 3k$.

If $d_G(x,u) = 3k$, then $\gamma_0 \in \Gamma$ and $g \in A_{\gamma_0} \neq \emptyset$, which is a contradiction.

If $d_G(x,u) = 2k$, then there exists a geodesic γ_1 joining u and x with $L(\gamma_1) = 2k$; if y is the interior vertex in γ_1, then $y \notin \{u_0, v_0\}$ since G is a cubic graph. Therefore, $\gamma_1 \cup (g_1 \setminus \gamma_0) \cup \gamma \in A_\gamma \neq \emptyset$, which is a contradiction.

If $d_G(x,u) = k$, then $[u,x] \in E(G)$ and $[u,x] \cup (g_1 \setminus \gamma_0) \cup \gamma \in A_\gamma \neq \emptyset$, which is a contradiction.

Hence, the claim and the proposition hold. □

The converse of Proposition 7.1.6 does not hold. For instance, the Petersen graph P satisfies $\operatorname{diam} V(P) = 2k$ and $\delta(P) = 3k/2$ (see [101, Theorem 11]).

In [16, Theorem 4.2 and Proposition 4.11] appear the following results.

Theorem 7.1.7. *Let G be any graph. Then $\delta(G) \geqslant 5k/4$ if and only if there exist a cycle g in G with length $L(g) \geqslant 5k$ and a vertex $w \in g$ such that $\deg_g(w) = 2$.*

Proposition 7.1.8. *Let G be any graph. Assume that the following conditions hold:*

(1) *There exist a cycle g in G such that $L(g) \geq 5k$ and a vertex $w \in g$ satisfying $\deg_g(w) = 2$.*

(2) *For every cycle γ in G, we have $\operatorname{diam} \gamma \leqslant 5k/2$.*

Then we have $\delta(G) = 5k/4$.

Example after Proposition 7.1.5 shows that the converse of Proposition 7.1.8 does not hold, since it provides a graph H with $\delta(H) = 5k/4$ and a cycle C in H with $\operatorname{diam} C = 3k$.

Theorem 7.1.7 and Proposition 7.1.6 give a kind of converse of Proposition 7.1.8 for cubic graphs.

Proposition 7.1.9. *Let G be any cubic graph. If $\delta(G) = 5k/4$, then the following conditions hold:*

(1) *There exist a cycle g in G such that $L(g) \geq 5k$ and a vertex $w \in g$ satisfying $\deg_g(w) = 2$.*

(2) *For every cycle γ in G we have $\operatorname{diam} V(\gamma) \leqslant 2k$.*

The converse of Proposition 7.1.9 does not hold. For instance, the Petersen graph P satisfies $\operatorname{diam} V(P) = 2k$ and $\delta(P) = 3k/2$ (see [101, Theorem 11]).

From [81, Proposition 5 and Theorem 7] we deduce the following result.

Lemma 7.1.10. *Let G be any graph with a cycle g. If $L(g) \geqslant 3k$, then $\delta(G) \geqslant 3k/4$. If $L(g) \geqslant 4k$, then $\delta(G) \geqslant k$.*

Proposition 7.1.11. *Let G be a cubic graph. If there exists a cycle g with $L(g) = jk$, for some $j \in \{3, 4, 5\}$, then $\delta(G) \geq jk/4$.*

Proof. The results for $j = 3, 4$ are consequence of Lemma 7.1.10; the result for $j = 5$ follows from Theorem 7.1.7, since G is a cubic graph. ☐

The previous result does not hold for $j = 6$; in fact, condition $L(g) \geqslant 6k$ does not imply $\delta(G) \geqslant 5k/4$, since the complete bipartite graph $K_{3,3}$ has a cycle with length $6k$ and verifies $\delta(K_{3,3}) = k$ (see [101, Theorem 11]).

Let us define the *circumference* $c(G)$ of a graph G as the supremum of the lengths of its cycles if G is not a tree; we define $c(G) = 0$ for every tree G. The following result (see [33, Proposition 3.9]) will be useful.

Proposition 7.1.12. *For any graph G we have $\delta(G) \leqslant c(G)/4$.*

The following result provides a simple and explicit formula for the hyperbolicity constant of a large class of cubic graphs.

Proposition 7.1.13. *If G is any cubic graph with $c(G) \leqslant 5k$, then $\delta(G) = c(G)/4$.*

Proof. Since $c(G) \leqslant 5k$, Proposition 7.1.11 gives $\delta(G) \geqslant c(G)/4$. Proposition 7.1.12 gives the converse inequality. ☐

The previous result does not hold for graphs with $c(G) = 6k$, since $c(K_{3,3}) = 6k$ and $\delta(K_{3,3}) = k < 3k/2$.

In [102, Lemma 2.1] or [18, Corollary 4] we found the following result.

Lemma 7.1.14. *In any graph G we have*

$$\delta(G) = \sup\left\{\delta(T) : T \text{ is a geodesic triangle that is a cycle}\right\}.$$

We have a sufficient condition in order to obtain $\delta(G) = 3k/2$ for cubic graphs G.

Proposition 7.1.15. *Let G be any cubic graph. Assume that the following conditions hold:*

(1) *For every cycle γ in G we have* $\operatorname{diam} \gamma \leq 3k$.

(2) *There exists a cycle g such that* $\operatorname{diam} V(g) \geqslant 3k$.

Then we have $\delta(G) = 3k/2$.

Proof. By (2) and Proposition 7.1.6, we have $\delta(G) \geq 3k/2$. For every geodesic triangle T that is a cycle, using (1), we have $d_G(x, y) \leq 3k$ for every $x, y \in T$ and, in consequence, $\delta(T) \leq 3k/2$. By Lemma 7.1.14 we deduce $\delta(G) \leq 3k/2$. ☐

The converse implication in the last proposition does not hold. The Petersen graph with edges of length k has $\delta(G) = 3k/2$ (see [101, Theorem 11]), it satisfies (1) but it does not satisfy (2). Let G be the Cartesian product of an infinite path and a path with two vertices, with edges of length k; then $\delta(G) = 3k/2$, G satisfies (2) but it does not satisfy (1).

In order to prove our following result we need to introduce a useful concept.

Given a graph G, we say that a family of subgraphs $\{G_n\}_n$ of G is a *T-decomposition* of G if $\cup_n G_n = G$, $G_n \cap G_m$ is either a vertex or the empty set for each $n \neq m$, and if the graph R defined as follows is a tree: for each n let us define a point v_n (v_n is an abstract point, it is not contained in G); we have $V(R) := \{v_n\}_n$ and $[v_n, v_m] \in E(R)$ if and only if $G_n \cap G_m \neq \emptyset$.

A T-decomposition of G always exists, as we will show now (although it can be trivial: if the graph is two-connected then any T-decomposition has just one element). We say that a vertex v of a graph G is a *cut-vertex* if $G \setminus \{v\}$ is not connected. A graph is *two-connected* if it is connected and it does not contain cut-vertices. Given any edge in G, let us consider the maximal two-connected subgraph containing it. It is clear that the set of these maximal two-connected subgraphs $\{G_n\}_n$ is a T-decomposition of G; we call it the *canonical T-decomposition* of G.

Lemma 7.1.16 below shows that T-decompositions are useful in the study of hyperbolic graphs. In particular, the canonical T-decomposition of a graph plays an interesting role, since it is the T-decomposition that minimizes $\sup_n \operatorname{diam} G_n$.

Note that every G_n in any T-decomposition of G is an isometric subgraph of G.

We will need the following result, which allows to obtain global information about the hyperbolicity of a graph from local information.

Lemma 7.1.16. *Let G be any graph and let $\{G_n\}_n$ be any T-decomposition of G. Then*

$$\delta(G) \leqslant \frac{1}{2} \sup_n \operatorname{diam} G_n.$$

Proof. Let us consider a geodesic triangle $T = \{x, y, z\}$ in G and $p \in [xy]$ with $\delta(T) = d_G(p, [xz] \cup [zy])$. By Lemma 7.1.14, without loss of generality we can assume that T is a cycle. Note that, since $\{G_n\}_n$ is a T-decomposition of G, there exists m such that $T \subseteq G_m$. Hence,

$$\begin{aligned}\delta(T) = d_G(p, [xz] \cup [zy]) &= d_{G_m}(p, [xz] \cup [zy]) \leqslant d_{G_m}(p, \{x, y\}) \\ &\leqslant \frac{1}{2} \operatorname{diam}_{G_m} G_m \leqslant \frac{1}{2} \sup_n \operatorname{diam}_G G_n.\end{aligned}$$

□

Lemma 7.1.17. *Let G be any cubic graph with canonical T-decomposition $\{G_n\}_n$. If we have $\sup_n \operatorname{diam} V(G_n) \geqslant 3k$, then $\delta(G) \geq 3k/2$.*

Proof. Let us choose m with $\operatorname{diam} V(G_m) \geqslant 3k$ and $x, y \in V(G_m)$ with $d_G(x, y) \geqslant 3k$. Since G_m is two-connected, Whitney's Theorem (see [116]) guarantees that there exists a cycle g in G with $x, y \in V(g)$. Since $\operatorname{diam} V(g) \geq 3k$, Proposition 7.1.6 gives $\delta(G) \geq 3k/2$. □

The following result appears in [17, Theorem 2.6].

Theorem 7.1.18. *For every hyperbolic graph G, $\delta(G)$ is an integer multiple of $k/4$.*

Lemma 7.1.17 and Theorem 7.1.18 have the following consequence.

Corollary 7.1.19. *Let G be any cubic graph with canonical T-decomposition $\{G_n\}_n$. If $\delta(G) \leq 5k/4$, then $\sup_n \operatorname{diam} V(G_n) \leqslant 2k$.*

Theorem 7.1.20. *Let G be any cubic graph with canonical T-decomposition $\{G_n\}_n$. If $\delta(G) = 5k/4$, then $\sup_n \operatorname{diam} V(G_n) = 2k$.*

Proof. Since $\delta(G) = 5k/4$, Lemma 7.1.16 gives $\sup_n \operatorname{diam} G_n \geqslant 5k/2$; hence, $\sup_n \operatorname{diam} V(G_n) \geqslant 2k$. Corollary 7.1.19 gives the converse inequality. □

The converse of Theorem 7.1.20 is not true (see Proposition 7.1.22 and Remark 7.1.23 below). In any case, we also have a kind of converse of Theorem 7.1.20. We need the following result which appears in [81, Theorem 11].

Lemma 7.1.21. *Let G be any graph.*

(1) *$\delta(G) = 0$ if and only if G is a tree.*

(2) *$\delta(G) = k/4, k/2$ is not satisfied for any graph G.*

(3) *$\delta(G) = 3k/4$ if and only if G is not a tree and every cycle in G has length $3k$.*

Proposition 7.1.22. *Let G be any cubic graph with canonical T-decomposition $\{G_n\}_n$.*

(1) *If $\sup_n \operatorname{diam} V(G_n) = k$, then $\delta(G) \leqslant k$.*

(2) *If $\sup_n \operatorname{diam} V(G_n) = 2k$, then $k \leqslant \delta(G) \leqslant 3k/2$.*

Proof. If $\sup_n \operatorname{diam} V(G_n) = k$, then we have $\sup_n \operatorname{diam} G_n \leqslant 2k$. Lemma 7.1.16 gives $\delta(G) \leqslant k$.

Assume now $\sup_n \operatorname{diam} V(G_n) = 2k$.

Then we have $\sup_n \operatorname{diam} G_n \leqslant 3k$. Lemma 7.1.16 gives $\delta(G) \leqslant 3k/2$.

If $\delta(G) < k$, then Lemma 7.1.21 gives that every G_n is either isomorphic to an edge or a cycle graph C_3; consequently, $\sup_n \operatorname{diam} V(G_n) = k$, which is a contradiction. □

Remark 7.1.23. *Note that every bound for $\delta(G)$ in Proposition 7.1.22 is attained: the bound in (1) is attained for the complete graph K_4, the lower one in (2) for the complete bipartite graph $K_{3,3}$ and the upper one for the Petersen graph (see [101, Theorem 11]).*

7.2 Complement of cubic graphs

The paper [19] studies the hyperbolicity of the complement of graphs. In this section, we obtain new results for this operation on cubic graphs.

In [101, Theorem 8] we find the following result (note that it can be deduced from Lemma 7.1.16).

Lemma 7.2.1. *In any graph G the inequality $\delta(G) \leqslant (\operatorname{diam} G)/2$ holds.*

Given any finite cubic graph G, we denote by $\overline{G}$ its complement graph. If $\overline{G}$ is not connected, with connected components $\overline{G}_1, \dots, \overline{G}_r$ ($r \geqslant 2$), we define

$$\delta\big(\,\overline{G}\,\big) := \max\big\{\delta\big(\,\overline{G}_1\big), \dots, \delta\big(\,\overline{G}_r\big)\big\}\,,$$
$$\operatorname{diam} \overline{G} := \max\big\{\operatorname{diam} \overline{G}_1, \dots, \operatorname{diam} \overline{G}_r\big\}\,;$$

hence, Lemma 7.2.1 also holds for non-connected $\overline{G}$.

The following lemma is a consequence of [43, Proposition 1.3.1].

Lemma 7.2.2. *If $m \geqslant 2$ is a natural number and G is any finite graph with edges of length k and $\deg v \geqslant m$ for every $v \in V(G)$, then there exists a cycle η in G with $L(\eta) \geqslant k(m+1)$.*

Lemmas 7.2.2 and 7.1.10 give the following proposition.

Proposition 7.2.3. *Let G be any cubic graph of order n.*

- *If $n = 6$, then $\delta\big(\,\overline{G}\,\big) \geqslant 3k/4$.*
- *If $n > 6$, then $\delta\big(\,\overline{G}\,\big) \geqslant k$.*

In [101, Theorem 11] and [81, Theorem 30] we find the following results.

Lemma 7.2.4. *For any cycle graph C_n with $n \geqslant 3$, we have $\delta(C_n) = L(C_n)/4 = nk/4$.*

Lemma 7.2.5. *Let G be any graph of order n. Then $\delta(G) \leqslant nk/4$.*

This inequality can be improved for cubic graphs (see [88, Theorem 3.7]).

Theorem 7.2.6. *Let G be any cubic graph of order n. Then $\delta(G) \leqslant k \min\left\{\dfrac{3n}{16} + 1, \dfrac{n}{4}\right\}$.*

The following lemma can be deduced from [19, Proposition 2.7]. However, we give a proof with a new and interesting argument.

Lemma 7.2.7. *Given any finite cubic graph G, we have $\delta\big(\,\overline{G}\,\big) \leqslant 3k/2$.*

Proof. It suffices to prove the result for $k = 1$. Let n be the order of G.

Assume first that $n \leqslant 6$. Denote by $\overline{G}_1, \dots, \overline{G}_r$ ($r \geqslant 1$), the connected components of G, with $n_1, \dots, n_r$ vertices, respectively. Then Lemma 7.2.5 gives $\delta\big(\overline{G}_j\big) \leqslant n_j/4 \leqslant n/4 \leqslant 3/2$ for $j = 1, \dots, r$, and we conclude $\delta\big(\overline{G}\big) \leqslant 3/2$.

Assume now that $n \geq 8$. Since $\overline{G}$ is a $(n-4)$-regular graph with n vertices, it is connected. Let $T := \{\gamma_1, \gamma_2, \gamma_3\}$ be a geodesic triangle in $\overline{G}$. In order to compute $\delta\big(\overline{G}\big)$, by Lemma 7.1.14, we can assume that T is a cycle. Let us consider $p \in T$; without loss of generality we can assume that $p \in \gamma_1 := [xy]$.

If $L(T) \leqslant 6$, then $L(\gamma_1) \leq 3$ and $d_{\overline{G}}(p, \gamma_2 \cup \gamma_3) \leq d_{\overline{G}}(p, \{x, y\}) \leq L(\gamma_1)/2 \leq 3/2$.

If $L(T) \geqslant 7$, then there are $n - L(T)$ vertices in $V\big(\overline{G}\big) \setminus V(T)$. Hence, since $\overline{G}$ is a $(n-4)$-regular graph, every vertex in $V(T)$ has at least $n - 4 - (n - L(T)) = L(T) - 4 \geqslant 3$ neighbors in $V(T)$ and, therefore, at least one neighbor in a different side of T. Now, there exists a vertex $v \in V(T)$ with $d_{\overline{G}}(p, v) \leqslant 1/2$. If $v \in \gamma_2 \cup \gamma_3$, then $d_{\overline{G}}(p, \gamma_2 \cup \gamma_3) \leq 1/2$. If $v \in \gamma_1$, then $d_{\overline{G}}(p, \gamma_2 \cup \gamma_3) \leq d_{\overline{G}}(p, v) + d_{\overline{G}}(v, \gamma_2 \cup \gamma_3) \leq 3/2$.

Then $\delta\big(\overline{G}\big) \leqslant 3/2$ and this finishes the proof. □

The following result provides precise bounds for the hyperbolicity constant of $(n-4)$-regular graphs.

Proposition 7.2.8. *Let G be any $(n-4)$-regular graph with order $n \geq 6$. Then we have $k \leqslant \delta(G) \leqslant 3k/2$.*

Proof. For $n = 6$, we have that G is isomorphic to C_6 and Lemma 7.2.4 gives $\delta(C_6) = 3k/2$. Assume now that $n \geq 8$. Since G is the complement of a cubic graph, Proposition 7.2.3 gives the lower bound. Finally, Lemma 7.2.7 gives the upper bound in this case. □

The lower bound in Proposition 7.2.8 is attained for the complete bipartite graph $K_{4,4}$ and the upper one for the cycle graph C_6.

Proposition 7.2.9. *Let G be any $(n-4)$-regular graph with order $n \geq 6$. Then, $\delta(G) = k$ if and only if* $\operatorname{diam} G = 2k$.

Proof. If $\operatorname{diam} G = 2k$, then by Proposition 7.2.8 and Lemma 7.2.1 we have $\delta(G) = k$.

Assume that $\delta(G) = k$. By Lemma 7.2.1, $\operatorname{diam} G \geqslant 2k$.

If $n = 6$, then G is isomorphic to C_6; so, we have $\operatorname{diam} C_6 = 3k$ and Lemma 7.2.4 gives $\delta(C_6) = 3k/2$. Assume that $n \geq 8$.

We prove now the following statement: *for any couple of vertices $v, w \in V(G)$ such that $[v, w] \notin E(G)$, there are two vertices that are neighbors of v and w.* Since $[v, w] \notin E(G)$ and $\deg(v) + \deg(w) = 2n - 8 \geq (n-2) + 2$, there exist at least two vertices that are neighbors of v and w.

Hence, we have $\operatorname{diam} V(G) = 2k$; thus, $\operatorname{diam} G \leq 3k$. If $\operatorname{diam} G > 2k$, then we have either $\operatorname{diam} G = 5k/2$ or $\operatorname{diam} G = 3k$.

If $\operatorname{diam} G = 5k/2$, there exist $v \in V(G)$ and a midpoint p in G such that $d_G(v,p) = 5k/2$. Let v_1, v_2 be vertices in G such that $p \in [v_1, v_2]$. By the previous statement, there is a cycle C in G such that $L(C) = 5k$ and $v, p \in C$. Let γ_1 and γ_2 be two geodesics joining v and p such that $\gamma_1 \cup \gamma_2 = C$ and $\gamma_1 \cap \gamma_2 = \{v, p\}$. This implies that $B = \{\gamma_1, \gamma_2\}$ is a geodesic bigon. Since $d_G(w, \gamma_2) = 5k/4$ if w is the midpoint of γ_1, we deduce that $\delta(B) = 5k/4$; hence, we obtain $\delta(G) \geq 5k/4$. Therefore, if $\operatorname{diam} G = 5k/2$, then $\delta(G) \geq 5k/2$.

If $\operatorname{diam} G = 3k$, there exist two midpoints p_1 and p_2 in G such that $d_G(p_1, p_2) = 3k$. Let v_1, v_2, w_1, w_2 be vertices in G such that $p_1 \in [v_1, w_1]$ and $p_2 \in [v_2, w_2]$. By the previous statement, there is a cycle C in G such that $L(C) = 6k$ and $p_1, p_2 \in C$. Let γ_1, γ_2 be two geodesics joining p_1 and p_2 such that $\gamma_1 \cup \gamma_2 = C$ and $\gamma_1 \cap \gamma_2 = \{p_1, p_2\}$. This implies that $B = \{\gamma_1, \gamma_2\}$ is a geodesic bigon. Since $d_G(p, \gamma_2) = 5k/4$ if p is the point in γ_1 with $d_G(p, p_1) = 5k/4$, we deduce that $\delta(B) \geqslant 5k/4$; hence, $\delta(G) \geq 5k/4$. Therefore, if $\operatorname{diam} G = 3k$, then $\delta(G) \geq 5k/4$. This finishes the proof. $\square$

Proposition 7.2.9 has the following consequence.

Corollary 7.2.10. *Let G be any finite cubic graph which is not isomorphic either to K_4 or to $K_{3,3}$. Then, $\delta\big(\overline{G}\big) = k$ if and only if $\operatorname{diam} \overline{G} = 2k$.*

Theorem 7.2.11. *Given any finite cubic graph G which is not isomorphic to $P_2 \times C_3$, we have $\operatorname{diam}_{\overline{G}} V(\overline{G}) \leq 2k$. Furthermore, if G has order $n \geq 8$, then $\operatorname{diam}_{\overline{G}} V(\overline{G}) = 2k$.*

Proof. Let n be the order of G. If $n = 4$, then G is isomorphic to the complete graph K_4 and $0 = \operatorname{diam}_{\overline{G}} V(\overline{G}) \leq 2k$. If $n = 6$, then G is isomorphic either to $K_{3,3}$ or to $P_2 \times C_3$. If G is isomorphic to $K_{3,3}$, then $\overline{G}$ is isomorphic to the union of two graphs C_3 and $k = \operatorname{diam}_{\overline{G}} V(\overline{G}) \leq 2k$; if G is isomorphic to $P_2 \times C_3$, then $\overline{G}$ is isomorphic to C_6 and $2k < \operatorname{diam}_{\overline{G}} V(\overline{G}) = 3k$.

The complement of any cubic graph G with order $n \geq 8$ is an $(n-4)$-regular connected graph. If u, v are two vertices in $\overline{G}$, then the number of common neighbors of u and v in $\overline{G}$ is at least $n - 4 + n - 4 - (n-2) = n - 6 \geqslant 2$. Hence, $d_{\overline{G}}(u,v) \leq 2k$, and $\operatorname{diam}_{\overline{G}} V(\overline{G}) \leq 2k$.

On the other hand, if $n \geq 8$, then $\operatorname{diam}_{\overline{G}} V(\overline{G}) > k$ and we conclude $\operatorname{diam}_{\overline{G}} V(\overline{G}) = 2k$. $\square$

Theorem 7.2.12. *For any finite cubic graph G which is not isomorphic either to K_4 or to $K_{3,3}$, we have $\operatorname{diam} \overline{G} \geq 5k/2$.*

Proof. Let n be the order of G. We just consider the case $n \geqslant 6$, since if $n = 4$, then G is isomorphic to the complete graph K_4. If $n = 6$, then G is isomorphic to $P_2 \times C_3$, $\overline{G}$ is isomorphic to C_6 and $\operatorname{diam} \overline{G} = 3k$.

Assume now $n \geq 8$. Since G is a cubic graph, $\operatorname{diam}_G V(G) \geq 2k$. Therefore, there is an induced subgraph isomorphic to P_3 with vertices u, v and w (in this order); hence, $[u,v], [v,w] \in E(G)$ and $[u,w] \notin E(G)$. Since $[u,v], [v,w] \notin E(\overline{G})$, we have $d_{\overline{G}}(v, \{u,w\}) \geqslant 2k$. Thus, if t is the midpoint of $[u,w] \in E(\overline{G})$, then $d_{\overline{G}}(v,t) \geqslant 5k/2$ and we conclude $\operatorname{diam} \overline{G} \geq 5k/2$. $\square$

The following is one of the main results in this chapter; it states that the hyperbolicity constant of the complement of (almost) every finite cubic graph can take only two values, either $5k/4$ or $3k/2$.

Theorem 7.2.13. *For any finite cubic graph G which is not isomorphic either to K_4 or to $K_{3,3}$, we have $5k/4 \leqslant \delta(\overline{G}) \leq 3k/2$.*

Proof. Let n be the order of G. We just consider the case $n \geqslant 6$, since if $n = 4$, then G is isomorphic to the complete graph K_4. If $n = 6$, then G is isomorphic to $P_2 \times C_3$, $\overline{G}$ is isomorphic to C_6 and $\delta(\overline{G}) = 3k/2$.

Assume now that $n \geqslant 8$. Proposition 7.2.3 and Lemma 7.2.7 give $k \leqslant \delta(\overline{G}) \leq 3k/2$. By Theorem 7.2.12 and Corollary 7.2.10 we have $\delta(\overline{G}) \neq k$. Finally, by Theorem 7.1.18, we conclude $5k/4 \leqslant \delta(\overline{G}) \leq 3k/2$. □

Proposition 7.2.9 and Theorem 7.2.13 have the following consequence, which improves Proposition 7.2.8.

Corollary 7.2.14. *Let G be any $(n-4)$-regular graph with order $n > 8$. Then we have $5k/4 \leqslant \delta(G) \leqslant 3k/2$.*

We have found as well sufficient conditions in order to guarantee that the hyperbolicity constant is equal to $5k/4$ or $3k/2$.

Theorem 7.2.15. *If G is a finite cubic graph with order $n > 4$ and it does not contain an induced subgraph isomorphic to a cycle C_4, then $\operatorname{diam} \overline{G} = 5k/2$ and $\delta(\overline{G}) = 5k/4$.*

Proof. If $n = 6$, then G is isomorphic either to $K_{3,3}$ or to $P_2 \times C_3$. Note that both graphs contain an induced subgraph isomorphic to a cycle C_4.

If G is a cubic graph with order $n \geq 8$, then $\operatorname{diam}_{\overline{G}} V(\overline{G}) = 2k$ by Theorem 7.2.11, and $\operatorname{diam} \overline{G} \leqslant \operatorname{diam}_{\overline{G}} V(\overline{G}) + k = 3k$. By Theorem 7.2.12, $\operatorname{diam} \overline{G} \geq 5k/2$, and we have either $\operatorname{diam} \overline{G} = 5k/2$ or $\operatorname{diam} \overline{G} = 3k$ by Theorem 7.1.18.

Now, we prove $\operatorname{diam} \overline{G} = 5k/2$. Seeking for a contradiction, assume that $\operatorname{diam} \overline{G} = 3k$. Since $\operatorname{diam}_{\overline{G}} V(\overline{G}) = 2k$, there exists a closed path in $\overline{G}$ with vertices $u_1, u_2, u_3, u_4, u_5, u_6$ (in this order, i.e., $[u_i, u_{i+1}] \in E(\overline{G})$ for $i = 1, \dots, 5$ and $[u_6, u_1] \in E(\overline{G})$) with $d_{\overline{G}}(x, y) = 3k$, where x is the midpoint of $[u_1, u_2]$ and y is the midpoint of $[u_4, u_5]$ (note that it is possible to have $u_3 = u_6$). Since $d_{\overline{G}}(x, y) = 3k$, $[u_2, u_4], [u_1, u_5], [u_1, u_4], [u_2, u_5] \notin E(\overline{G})$; hence, $[u_2, u_4], [u_1, u_5], [u_1, u_4], [u_2, u_5] \in E(G)$. Then u_1, u_4, u_2, u_5 (in this order) is a cycle in G; furthermore, it is an induced subgraph, since $[u_1, u_2], [u_4, u_5] \notin E(G)$. This is a contradiction and we conclude that $\operatorname{diam} \overline{G} = 5k/2$.

Now, Lemma 7.2.1 gives that $\delta(\overline{G}) \leqslant 5k/4$. Finally, $\delta(\overline{G}) = 5k/4$ by Theorem 7.2.13. □

Theorem 7.2.16. *Let G be any finite cubic graph. If there exists an induced subgraph C in $\overline{G}$ isomorphic to a cycle C_6 with $\operatorname{diam}_{\overline{G}} C = 3k$, then $\delta(\overline{G}) = 3k/2$.*

Proof. Let $x, y \in C$ with $d_{\overline{G}}(x, y) = 3k$; we can choose x and y such that they are either vertices or midpoints of edges in C. Since $L(C) = 6k$ and $\operatorname{diam}_{\overline{G}} C = 3k$, there are two geodesics g_1, g_2 with $g_1 \cup g_2 = C$ and $g_1 \cap g_2 = \{x, y\}$. Let p be the midpoint of g_1.

Assume first that $x, y \in V(\overline{G})$. Then p is the midpoint of an edge in C and $d_{\overline{G}}(p, g_2) \geqslant 3k/2$. Since $L(g_1) = 3k$, we have $d_{\overline{G}}(p, g_2) = 3k/2$.

Assume now that x and y are midpoints of edges in C. Hence, $p \in V(\overline{G})$. If $v \in V(\overline{G}) \cap g_2$, then $d_{\overline{G}}(p, v) \geqslant 2k$ since C is an induced subgraph. Therefore, $d_{\overline{G}}(p, g_2) = d_{\overline{G}}(p, \{x, y\}) = 3k/2$.

Then we have in any case $d_{\overline{G}}(p, g_2) = 3k/2$. Since $B = \{g_1, g_2\}$ is a geodesic bigon in $\overline{G}$, we deduce $\delta\big(\overline{G}\big) \geqslant \delta(B) \geqslant d_{\overline{G}}(p, g_2) = 3k/2$. Finally, $\delta\big(\overline{G}\big) = 3k/2$ by Theorem 7.2.13. ☐

Finally, we bound $\delta(G) + \delta\big(\overline{G}\big)$.

Theorem 7.2.17. *Let G be any cubic graph with n vertices.*

- *If $n = 4$, then $\delta(G) + \delta\big(\overline{G}\big) = k$.*
- *If $n = 6$, then $\delta(G) + \delta\big(\overline{G}\big) \in \{7k/4\ ,\ 11k/4\}$.*
- *If $8 \leqslant n \leqslant 16$, then $9k/4 \leqslant \delta(G) + \delta\big(\overline{G}\big) \leqslant (n + 6)k/4$.*
- *If $n \geqslant 18$, then $9k/4 \leqslant \delta(G) + \delta\big(\overline{G}\big) \leqslant (3n + 40)k/16$.*

Proof. If $n = 4$, then G is isomorphic to K_4, $\delta(G) = k$ and $\delta\big(\overline{G}\big) = 0$. If $n = 6$, then G is isomorphic either to $K_{3,3}$ or $P_2 \times C_3$. Since $\overline{K_{3,3}}$ is isomorphic to the disjoint union of two cycle graphs C_3 and $\overline{P_2 \times C_3}$ is an isomorphic graph to C_6, we obtain the result since $\delta(K_{3,3}) = k$, $\delta\big(\overline{K_{3,3}}\big) = 3k/4$, $\delta(P_2 \times C_3) = 5k/4$ and $\delta(C_6) = 3k/2$. If $n \geq 8$, then Lemmas 7.2.2 and 7.1.10 give directly that $\delta(G) \geqslant k$; besides, by Theorem 7.2.13 we have $5k/4 \leqslant \delta\big(\overline{G}\big) \leqslant 3k/2$. Finally, Theorem 7.2.6 gives that $\delta(G) \leqslant k \min\left\{\dfrac{3n}{16} + 1, \dfrac{n}{4}\right\}$. ☐

Conclusions

The main results obtained in this PhD Thesis are the following:

- We introduce the alliance polynomial of a graph and we develop and implement an algorithm that computes it in an efficient way. We compute the alliance polynomial for some graphs and we study its coefficients. We investigate the alliance polynomials of path, cycle, complete and complete bipartite graphs. Also we prove that the path, cycle, complete and star graphs are characterized by their alliance polynomials.

- We obtain further results about the alliance polynomial of cubic graphs. In particular, we prove that the family of alliance polynomials of cubic graphs is a very special one, since it does not contain alliance polynomials of graphs which are not cubic. Furthermore, we obtain (computationally) the alliance polynomials of cubic graphs with small order and we prove that they satisfy uniqueness.

- We prove that the family of alliance polynomials of connected Δ-regular graphs with small degree is a very special one, since it does not contain alliance polynomials of graphs which are not connected Δ-regular.

- We find some characterizations for the cubic graphs which have small hyperbolicity constants. Besides, we obtain bounds for the hyperbolicity constant of the complement graph of a cubic graph; our main result of this kind says that for any finite cubic graph G which is not isomorphic either to K_4 or to $K_{3,3}$, the inequalities $5k/4 \le \delta\big(\overline{G}\big) \le 3k/2$ hold, if k is the length of every edge in G. This is a very precise result, since it implies that $\delta\big(\overline{G}\big)$ is either $5k/4$ or $3k/2$.

Future work

At the light of the results in this PhD Thesis we asked the following question: can the alliance polynomials characterize the graphs? (i.e., do non-isomorphic graphs have different alliance polynomials?). This is an interesting open problem. However, since this is a very ambitions goal, it is reasonable to ask the question for several classes of graphs (regular graphs, planar graphs, chordal graphs, bridged graphs, ...).

Another problem is to obtain further properties of alliance polynomials and their coefficients. This problem is interesting by itself and, furthermore, it can help to solve the previous one.

Also, we would like to study the relation between $A(G;x)$ and other graph polynomials.

Another open problem is whether there exist linear recurrence relations for $A(G;x)$ with respect to elementary edge and vertex operations.

A natural problem is to generalize the results on cubic graphs in Chapter 7 to Δ-regular graphs (graphs such that the degree of every vertex is Δ), for any fixed integer $\Delta \geq 4$.

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