Question 1

We are interested in developing a model of how deep a ball of mass m and radius r will penetrate into sand if dropped from some height above the sand. We make the following assumption:

- Sand is like a fluid, flowing freely, and obeying Pascal's Law for pressure as a function of depth, with a pressure of 0 on the surface.
- Any buoyant forces of the sand on the ball are negligible.
- The force of gravity on the ball is negligible compared to the primary resistive force.
- The depth d the ball will penetrate is much smaller than the height H from which the ball was dropped. However, $d \gg r$, the radius of the ball.

We then consider the following three possibilities for a frictional resistive force that is

a. proportional to the static pressure of the sand on the ball, $F \propto P$, much like kinetic sliding friction; or

b. proportional to the speed of the ball v as it moves through the sand, $F \propto v$, much like a viscous retarding force; or

c. a hybrid model proportional to the speed of the ball as it moves through the sand but also proportional to the pressure of the sand on the ball.

For each case, develop a formula to predict the depth d that a ball will sink if it is dropped from a height H above the surface of the sand. Your answer only needs to indicate the functional relationship between d and H.

An experiment is performed; that experiment yields the following data points:

H (meters)	0.7	1.4	2.9	4.6	5.8	5.9	6.2
d (meters)	0.27	0.31	0.39	0.44	0.48	0.46	0.47

d. By sketching an appropriate graph determine which model (or models) is most likely.

Solution

The pressure of the sand at depth y will be given by $\rho g y$, with ρ the density of the sand. Positive y is directed down. The kinetic energy of the ball just before it enters the sand will be given by mgH; the velocity of the ball just before it enters the sand will be given by $v_0 = \sqrt{2gH}$.

a. We assume $F = k_1 P$, where k_1 is a constant. Then

$$F = k_1 \rho g y$$

It is easiest to balance energy. The work done by the sand is then going to be

$$W = \int F\, dy = \frac{1}{2} k_1 \rho g d^2$$

Equating this to the potential energy of the ball, we arrive at

$$d \propto H^{1/2}$$

b. We assume that $F = k_2 v$, where k is a constant. It is now easiest to consider a simple differential equation for velocity,

$$m\frac{dv}{dt} = -k_2 v,$$

where the negative reflects the fact that friction opposes the motion. This expression is easily integrated;

$$m\ dv = -k_2 dy$$

so that

$$v = v_0 - \frac{k_2}{m}y.$$

The maximum depth is when $v = 0$, or $v_0 = k_2 d/m$. Combining with above energy expressions,

$$d \propto H^{1/2}$$

c. In the last case we have

$$F = -k_3 \rho g y v$$

which can be written as

$$m\frac{dv}{dt} = -k_3 \rho g y v$$

or

$$m\frac{dv}{dt} = -k_3 \rho g y \frac{dy}{dt},$$

an expression that can be easily integrated to yield

$$mv - mv_0 = -k_3 \rho g y^2$$

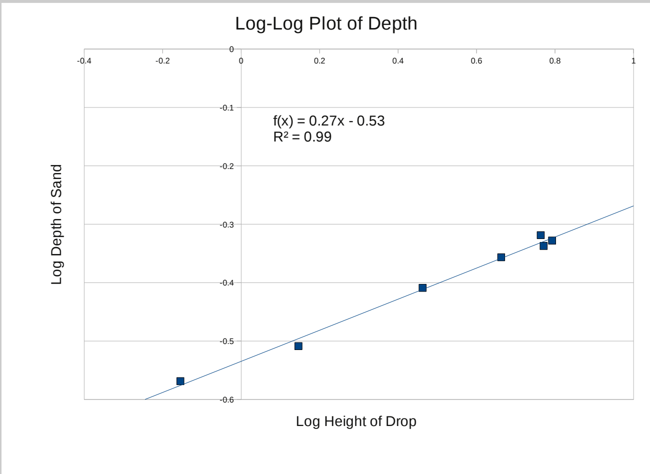
Maximum depth is when $v_0 = 0$, so

$$mv_0 = k_3 \rho g y^2$$

Combining with the energy relation for height, H, we get

$$d \propto H^{1/4}$$

d. A log-log plot is the best choice.



Question 2

A student constructs a simple heat engine that consists of a cylinder and a piston. She designs a cycle that has four processes: (A) isothermal expansion, (B) adiabatic expansion, (C) isothermal compression, and (D) adiabatic compression. The isothermal processes are done in contact with large heat reservoirs; at temperature T_H for the expansion and at temperature T_C for the compression.

Instead of filling the cylinder with an ideal gas, she first evacuates the cylinder, and then adds a small amount of liquid. Since $T_C < T_H < T_{critical}$, the critical temperature, *some* of the liquid evaporates and fills the cylinder with vapor. At all points in the cycle there is some liquid present; there is also vapor present, and except for the fact that the vapor can condense into liquid, we shall assume the vapor is an ideal gas. Finally, the liquid and vapor are always in thermal equilibrium, but assume that the volume of the liquid is at all times negligible compared to the volume of the vapor.

a. Sketch a PV diagram for this process.

b. Calculate the efficiency for this cycle, in terms of T_H and T_C.

c. The condition for thermal equilibrium between the liquid and the vapor is given by

$$\frac{1}{n_l}\left(V_l dP - S_l dT\right) = \frac{1}{n_g}\left(V_g dP - S_g dT\right)$$

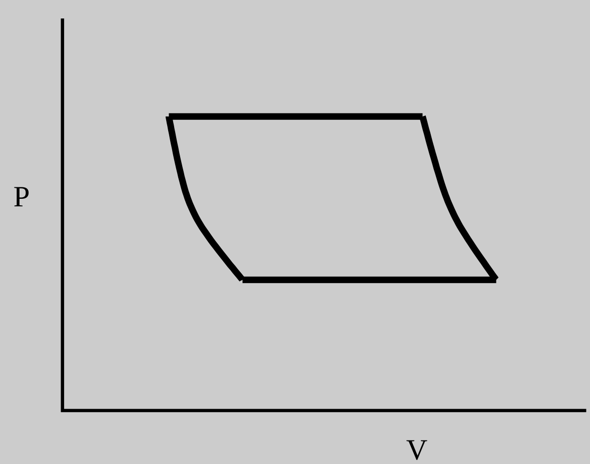
where the subscript l refers to liquid, and g refers to the vapor state; n is moles, S entropy, V volume, P pressure, and T temperature. Continuing with the assumption that $V_l \approx 0$, and that the vapor is an ideal gas, derive an expression for

i. dP/dT in terms of the latent heat of vaporization per mole L_v, the volume of the cylinder V, the temperature T, the number of moles of vapor n, and any relevant fundamental constants; and

ii. P in terms of T, L_v, any relevant fundamental constants, and a reference pressure and temperature P_0, T_0.

Solution

a. The process is shown below. Since the vapor coexists with the liquid, any isothermal line is also a isobaric line. There is no easy way to determine the exact shape of the adiabatic line.



b. No heat is transferred during the adiabats; and the het transfer during the isotherms is given by $Q = T\Delta S$, where T depends on the temperature. But ΔS is the same for the two processes, since it is a reversible cycle, and therefore

$$e = 1 - \frac{Q_C}{Q_H} = 1 - \frac{T_C}{T_H}$$

c. Two parts.

i. Rearrange,

$$\frac{dP}{dT} = \frac{S_g/n_g - S_l/n_l}{V_g/n_g - V_l/n_l}$$

We are allowed to assume $V_l/n_l \approx 0$.

The quantity S_g/n_g is a measure of entropy per mole; $S_g/n_g - S_l/n_l$ is then the measure of a change in entropy per mole as liquid changes to gas. But this is the same as $\Delta Q/n/T$, where $\Delta Q/n = L_v$ is heat per mole of vaporization, and T is temperature of vaporization. Then

$$\frac{dP}{dT} = \frac{L_v}{TV/n}$$

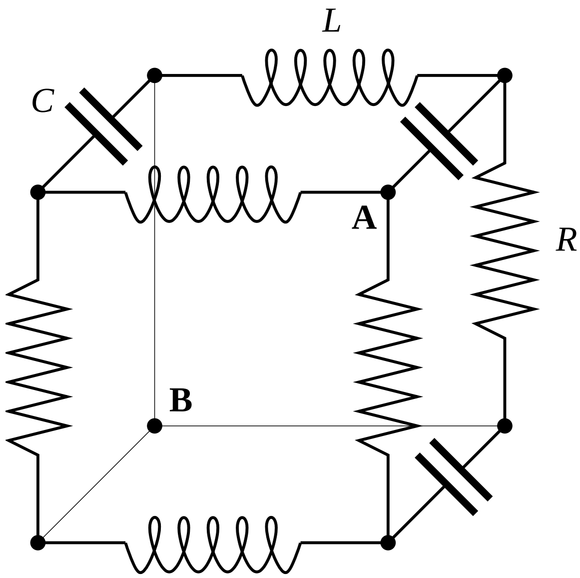
where n is n_g.

ii. This expression is not hard to integrate, if we let $n/V = P/R_gT$;

$$\begin{aligned}
\frac{dP}{dT} &= \frac{L_v n}{TV} \\
&= \frac{L_v}{R_g}\frac{P}{T^2} \\
\frac{dP}{P} &= \frac{L_v}{R_g}\frac{dT}{T^2} \\
\ln\frac{P}{P_0} &= \frac{L_v}{R_g}\left(\frac{1}{T_0} - \frac{1}{T}\right) \\
P &= P_0 e^{\frac{L_v}{R_g T_0}} e^{-\frac{L_v}{R_g T}}
\end{aligned}$$

Question 3

A circuit is wired in the shape of a cube. There are twelve circuit elements: three identical inductances L, three identical capacitances c, three identical resistances R, and three identical resistanceless wires.



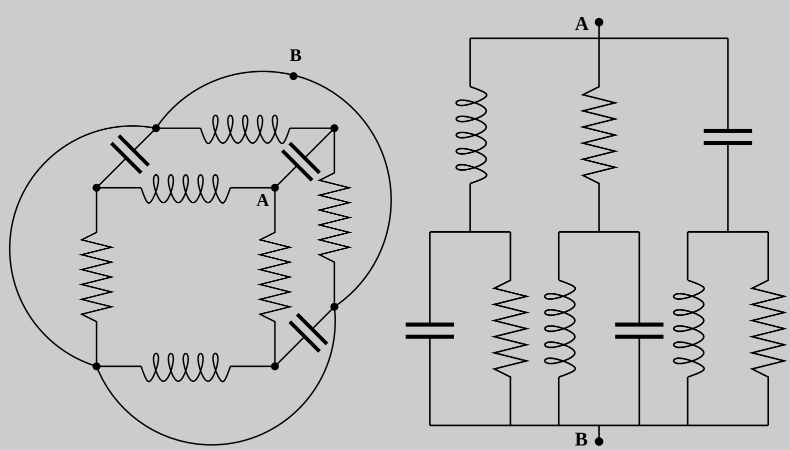
The time constant of a series circuits containing L and R is measured to be τ_L. The time constant of a series circuit containing C and R is measured to be τ_R.

a. Derive an expression for the impedance of the cube as measured between corners A and B in terms of L, R, C and an applied sinusoidal potential with an angular frequency ω. You may use complex number notation.

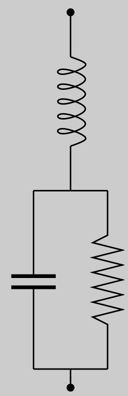
b. Assuming that $\omega = 1/\sqrt{\tau_L \tau_R}$, find the impedance between A and B in terms of τ_L, τ_R, and R.

Solution

a. Start by flattening the circuit! Redrawn, it looks like



Consider a fairly simple circuit element like the one below. We wish to compute the impedance.



For the parallel section,

$$\frac{1}{Z_{23}} = \frac{1}{Z_2} + \frac{1}{Z_3}$$

or

$$Z_{23} = \frac{Z_2 Z_3}{Z_2 + Z_3}.$$

Adding this to the series portion gives

$$Z_{123} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_2 + Z_3}$$

Ever so symmetric and aesthetically pleasing!

There are three of these elements in the full cube, except that we rotate items 1, 2, and 3. The only thing that changes is the denominator of the above expression, but since the three elements are in parallel, we add reciprocals, and the final impedance is going to be given by

$$\frac{1}{Z} = 2\frac{Z_1 + Z_2 + Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

Let $Z_1 = R$, $Z_2 = 1/i\omega C$, and $Z_3 = i\omega L$, then

$$Z = \frac{iR(\omega L - 1/\omega C) + L/C}{2(R + i(\omega L - 1/\omega C))}$$

This is of the form

$$Z = \frac{R}{2}\frac{i\alpha + L/RC}{R + i\alpha}$$

which has magnitude

$$|Z| = \frac{R}{2}\sqrt{\frac{\alpha^2 + (L/RC)^2}{\alpha^2 + R^2}}$$

b. If $\omega = 1/\sqrt{\tau_L \tau_C}$, the good things happen, since

$$\omega = \frac{1}{\sqrt{(L/R)(RC)}} = \frac{1}{\sqrt{LC}}.$$

Then $\alpha = 0$, or

$$\omega L - 1/\omega C = \omega L \left(1 - \frac{1}{\omega^2 LC}\right) = 0$$

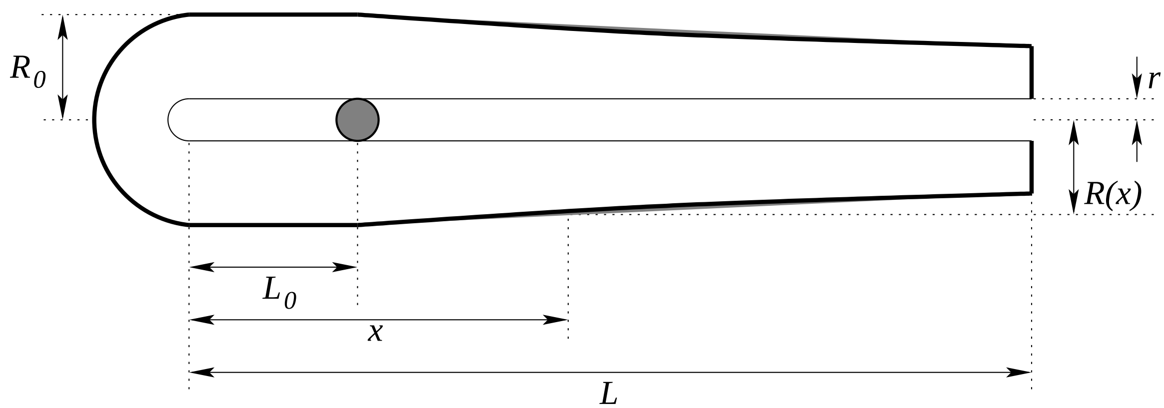
and the impedance simplifies to

$$Z = \frac{L}{2RC} = \frac{L}{2R^2C} R = \frac{1}{2}\frac{\tau_L}{\tau_C} R.$$

Question 4

A cannon is effectively a gently tapered cylindrical tube with a cylindrical bore. We can model the cannon as having an inner radius r, and outer radius $R(x)$ that depends on the distance from the end of the cannon bore. We choose $R(x)$ to be as small as possible so that the cannon does not explode from the pressure of internal gases.

We will consider a cannon barrel that has length $L \gg r$; and we will write the initial position of the cannon shot as L_0. For $x < L_0$, the outer radius of the cannon is R_0. This cannon is made of a metal with a tensile strength of σ, a measure of the maximum pulling force per unit area that the metal can support without tearing. The cannonball is made of a metal of density ρ.



When the explosive charge is set off, it instantaneously creates n_0 moles of an ideal monatomic gas at a temperature of T_0 in the space behind the cannonball. Ignore friction between the cannonball and the walls of the cannon and assume that the cannonball moves so quickly out of the cannon that no heat is exchanged between the expanding gas and the cannon or cannonball. For convenience, assume the cannon is fired in a vacuum; to avoid confusion, you ought write the ideal gas constant as R_g.

a. Find an expression for R_0 in terms of n_0, T_0, L_0, r, σ, and any relevant fundamental constants.

b. Find an expression for R_x in terms of x, R_0, r, L_0, and any relevant fundamental constants.

c. Find an expression for the kinetic energy given to the cannonball as a function of n_0, T_0, L_0, L, and any relevant fundamental constants.

d. A reasonable approximation to the cost of a cannon is to assume that it is proportional to $M = LR_0^2$. Ignoring tapering, find the maximum kinetic energy that can be given to the cannonball in terms of n_0, T_0, σ, M, and any relevant fundamental constants.

Solution

a. From the ideal gas law, we have the pressure behind the cannonball as

$$P_0V_0 = n_0R_gT_0$$

with the volume $V_0 = \pi r^2 L_0$. Imagining the cannon as being split in half lengthwise, and then being glued together with a substance with strength σ. The cross sectional area inside the bore is $2rL_0$, so the force from the gases is

$$F = 2rL_0P_0$$

and the force of the "glue" is

$$F = 2(R_0 - r)L_0\sigma.$$

Equating,

$$(R_0 - r)L_0\sigma = rL_0P_0 = rL_0\frac{n_0R_gT_0}{\pi r^2 L_0}$$

or

$$R_0 = r\left(\frac{n_0R_gT_0}{\pi\sigma r^2 L_0} + 1\right)$$

b. The gas expands adiabatically, so

$$PV^\gamma = P_0{V_0}^\gamma$$

In terms of position, we can write

$$P = P_0\left(\frac{L_0}{x}\right)^\gamma$$

and substitute this into the expression for R, so that

$$R(x) = r\left(\frac{n_0R_gT_0}{\pi\sigma r^2 L_0}\left(\frac{L_0}{x}\right)^\gamma + 1\right)$$

or

$$R(x) = (R_0 - r)\left(\frac{L_0}{x}\right)^\gamma + r$$

c. The work done for adiabatic expansion is

$$W = \int P\,dV$$

which, along the cannon bore, can be written as

$$W = \pi r^2 \int P\,dx$$

Using the answers from above, we get

$$\begin{aligned} W &= \pi r^2 \int_{L_0}^{L} P_0\left(\frac{L_0}{x}\right)^\gamma\,dx, \\ &= \pi r^2 P_0 \frac{1}{\gamma - 1} L_0\left(1 - \left(\frac{L_0}{x}\right)^{\gamma-1}\right), \\ &= \frac{1}{\gamma - 1} n_0R_gT_0\left(1 - \beta^{1-\gamma}\right), \end{aligned}$$

where $\beta = L/L_0$, the fraction of the total length of the cannon to the original length. $\gamma = C_P/C_V = 5/3$, and therefore

$$W = \frac{3}{2}n_0R_gT_0\left(1 - \beta^{-2/3}\right),$$

d. Cannon cost is given by $M = R_{02}L$. W depends only on L via β, so we would want to maximize L, or minimize R_0. The minimum R_0 depends on the strength of the material, so we start with

$$R_0 = \frac{n_0 R_g T_0}{\pi \sigma r L_0} + r.$$

Taking the derivative with respect to the only variable (r) yields

$$0 = \frac{dR_0}{dr} = -\frac{n_0 R_g T_0}{\pi \sigma r^2 L_0} + 1$$

with solution

$$r = \sqrt{\frac{n_0 R_g T_0}{\pi \sigma L_0}}$$

and then

$$R_{0,\mathrm{min}} = 2\sqrt{\frac{n_0 R_g T_0}{\pi \sigma L_0}}$$

Since M is fixed, we will shrink R_0 to make L as large as possible, and then

$$L_{\mathrm{max}} = \frac{M}{{R_0}^2} = \frac{\pi \sigma M L_0}{4 n_0 R_g T_0}$$

Then we have

$$W_{\mathrm{max}} = \frac{3}{2} n_0 R_g T_0 \left(1 - \left(\frac{\pi \sigma M}{4 n_0 R_g T_0} \right)^{-2/3} \right),$$

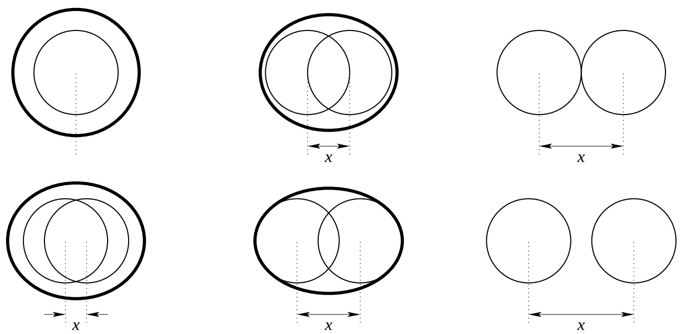
Note that $U_0 = \frac{3}{2} n_0 R_g T_0$ is a measure of the energy of the explosion. We could then define the cannon efficiency by

$$e = 1 - \left(\frac{3\pi \sigma M}{8 U_0} \right)^{-2/3}$$

For a "typical" cannon, $U_0 \approx 5$ MJ (per kilogram of gunpowder). A cannon might be configured to have $M = 2\ \mathrm{m}^3$ of bronze, and $\sigma = 200$ MPa. As such, we expect a maximum efficiency of about 95%.

Question 5

A simple model for nuclear fission is to treat the nucleus as a charged liquid drop that splits in two. The liquid has a constant density δ, a constant charge density ρ, and a constant surface tension γ. Originally the drop is spherical with a radius R and charge Q; by adding some energy (in the form of shape oscillations) the large drop splits into two identical smaller droplets that then move apart. The first four parts of this question expect *exact* result; the purpose of the last five parts is to develop a reasonable approximation to an otherwise very difficult problem.



The figure shows the large drop (darker curve that evolves into an ellipsoid) as the effective center of the two daughter droplets move apart. The daughter droplets can not actually overlap; the excess fluid must rearrange itself around the outline of the two droplets.

a. Exactly calculate the electrostatic potential self energy of a uniformly charged spherical drop of charge Q and radius R and any necessary fundamental constants. Call this energy U_e

b. Exactly calculate the electrostatic potential self energy of one of the daughter droplets in terms of U_e

c. Exactly calculate the surface tension energy of a spherical drop of surface tension γ and radius R and any necessary fundamental constants. Call this energy U_s

d. Exactly calculate the surface tension energy of one of the daughter droplets in terms of U_s

e. Sketch an approximate graph of the electrostatic potential energy of the system as a function of x, the distance between the centers of the two daughter droplets.

   For this graph, and the two following graphs, make sure to clearly indicate the functional form of any relevant straight lines or curves, values of intercepts or asymptotes, and the distance x_s when the two droplets are just touching.

f. Sketch an approximate graph of the surface tension energy of the system as a function of x.

g. Sketch an approximate graph of the total potential energy of the system as a function of x.

h. Estimate a minimum energy of required so that the single drop would split into the two smaller droplets.

i. Estimate the minimum ratio of U_s/U_e so that the drop is *stable* under spontaneous fission.

Solution

a. Use Gauss's law to find the electric field in and around a charged drop, a distance r from the center of the drop:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

Outside the droplet ($r > R$), the integral is trivial:

$$\begin{aligned} \oint \vec{E} \cdot d\vec{A} &= \frac{1}{\epsilon_0} \int \rho \, dV \\ 4\pi r^2 E &= \frac{1}{\epsilon_0} \frac{4}{3} \pi R^3 \rho \\ E &= \frac{\rho R^3}{3\epsilon_0 r^2} \end{aligned}$$

Inside the droplet ($r < R$), the integral is only a little more difficult

$$\begin{aligned} \oint \vec{E} \cdot d\vec{A} &= \frac{1}{\epsilon_0} \int \rho \, dV \\ 4\pi r^2 E &= \frac{1}{\epsilon_0} \frac{4}{3} \pi r^3 \rho \\ E &= \frac{\rho r}{3\epsilon_0} \end{aligned}$$

There are several ways to find the self energy. We will use the energy stored in the electric field, where the energy density is given by

$$u = \frac{\epsilon_0}{2} E^2$$

and then integrate over all space. For $r > R$ we have

$$\begin{aligned} \int u \, dV &= 2\pi\epsilon_0 \int_R^\infty \left(\frac{\rho R^3}{3\epsilon_0 r^2} \right)^2 r^2 \, dr, \\ &= \frac{2\pi}{9\epsilon_0} \rho^2 R^6 \int_R^\infty \frac{dr}{r^2}, \\ &= \frac{2\pi}{9\epsilon_0} \rho^2 R^5. \end{aligned}$$

For $r < R$ we have

$$\begin{aligned} \int u \, dV &= 2\pi\epsilon_0 \int_0^R \left(\frac{\rho r}{3\epsilon_0} \right)^2 r^2 \, dr, \\ &= \frac{2\pi}{9\epsilon_0} \rho^2 \int_0^R r^4 \, dr, \\ &= \frac{2\pi}{9\epsilon_0} \rho^2 \frac{1}{5} R^5. \end{aligned}$$

Add the two terms, and

$$U = \frac{4\pi}{15\epsilon} \rho^2 R^5.$$

Writing this in terms of $Q = \frac{4}{3}\pi R^3 \rho$ gives

$$U_e = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R}$$

It is convenient to write things in terms of

$$U_0 = \frac{Q^2}{4\pi\epsilon_0 R}$$

so

$$U_e = \frac{3}{5}U_0$$

b. Since the volume of the daughter droplet is half that of the parent drop, we have $R_d = R/\sqrt[3]{2} \approx 4R/5$ for the radius of the daughter droplet. The charge is also half that of the full drop, so

$$U_{daughter} = \frac{3\sqrt[3]{2}}{20}\frac{Q^2}{4\pi\epsilon_0 R} = \frac{3\sqrt[3]{2}}{20}U_0 \approx \frac{1}{5}U_0$$

c. Surface tension energy is given by γA, where A is the surface area. Then

$$U_s = 4\pi\gamma R^2$$

d. Using the smaller radius of the droplet yields

$$U_{daughter} = \frac{4}{\sqrt[3]{4}}\pi\gamma R^2 \approx \frac{5}{8}U_s$$

e. As one big drop, $x = 0$, we use the answer above,

$$U(x = 0) = \frac{3}{5}U_0$$

As two drops infinitely far apart, we double the smaller droplet energy, and

$$U(x = \infty) \approx \frac{2}{5}U_0$$

In between, we have to consider the electrostatic energy of two droplets. Approximate them as point charges, and we get

$$U(x) \approx \frac{(Q/2)^2}{4\pi\epsilon_0 x} + \frac{2}{5}U_0,$$

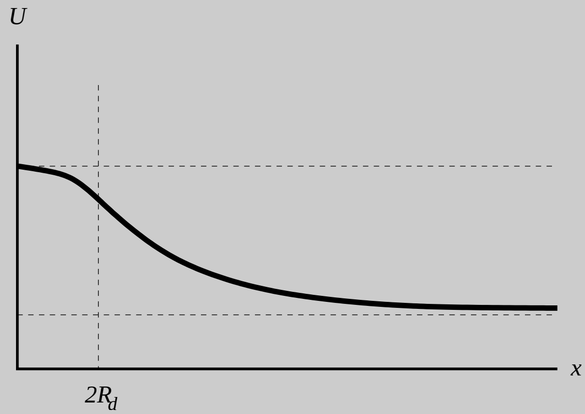
or

$$U(x) \approx \frac{R}{x}\frac{1}{4}U_0 + \frac{2}{5}U_0,$$

Certainly valid for $x >> R$, but not so reasonable for $x \approx 2R_d = 2R/\sqrt[3]{2}$, the distance at which the droplets would "touch". Note that, at that distance,

$$U(2R_d) \approx \left(\frac{\sqrt[3]{2}}{8} + \frac{2}{5}\right)U_0 \approx \frac{5}{9}U_0$$

We will assume that it is a "smooth" transition to $x = 0$, and always increasing.



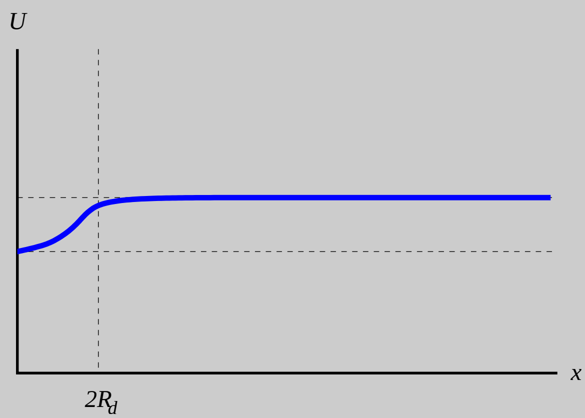
f. For $x > 2R_d$ the two droplets don't touch, and the energy is

$$U(x) \approx \frac{5}{4}U_s$$

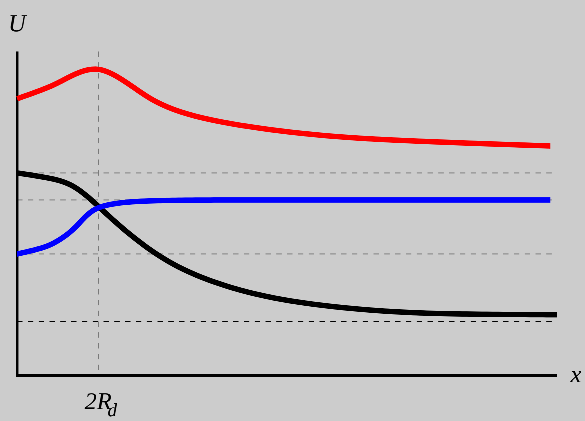
For one drop, $x = 0$, and the energy is

$$U(0) = U_s$$

We assume that the surface energy *decrease* smoothly (but fairly rapidly) to the minimum value as x decreases from R_d.



g. Add the potential energies, and



h. Assume the maximum total potential energy of the system is right when the droplets are barely touching. Then

$$U_{\mathrm{max}} \approx \frac{5}{4}U_s + \frac{5}{9}U_0$$

The minimum is one drop, or

$$U_{\mathrm{min}} \approx U_s + \frac{3}{5}U_0$$

The difference is the required energy, or

$$\Delta E \approx \frac{1}{4}U_s - \frac{2}{45}U_0 \approx \frac{1}{4}U_s - \frac{2}{27}U_e$$

i. The system is only stable if ΔE is positive, otherwise it will spontaneously fizz. So

$$\frac{1}{4}U_s > \frac{2}{27}U_e$$

or

$$\frac{U_s}{U_e} > \frac{8}{27} \approx \frac{1}{3}$$